ALGEBRA MADE EASY

FOR

INTERMEDIATE AND F. A. STUDENTS OF
THE INDIAN UNIVERSITIES

*(With an Appendix completing the syllabus
prescribed under the new regulations
of the Calcutta University)*

BY

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"ELEMENTARY MODERN GEOMETRY, PARTS I. & II"., &c.

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ELEMENTARY

MODERN GEOMETRY

FOR

(*Schools and Colleges*)

Meeting the requirements of the Matriculation and Intermediate Examinations prescribed under the New Regulations of the Calcutta University.

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PREFACE.

The present work has grown out of my attempts to make Algebra attractive and easy to my pupils. It is meant to afford the F. A. Student all the information and exercise in Algebra that he needs for the First Examination in Arts of the Calcutta University. I have endeavoured to reason out in a clear and accurate manner the leading propositions in the different parts of the subject and to illustrate and apply these propositions in practice so as to enable the average student to grasp the principles without the aid of a teacher. Besides the illustrative examples there is a much larger number of others which are meant as exercises for the learner. Particular care has been taken in the choice and arrangement of these exercises. Occasional hints have been furnished in cases where students of ordinary mathematical ability are likely to be puzzled, but not to such an extent as to do away with all stimulus to exertion. There is ample scope left for the exercise of ingenuity on the part of the more zealous mathematical workers. I have thus tried to meet the requirements of different degrees of intelligence and ardour among the students.

The F. A. Examination Papers of the Calcutta University for the last 28 years are reprinted in chronological order at the end of the book and references are given to the pages in the body of the work, where the more interesting problems from these papers are to be found either actually worked out or left to be solved with occasional hints.

There are several other features of the present work which seem to call for special mention :—

(1) By introducing suitable variety in the types used, I have tried to enable the student to take in the contents of a chapter at a glance for the purpose of recapitulation.

(2) Equations and Examples are brought under separate heads according to the methods employed in their solution.

(3) Interesting examples in exceptionally large numbers are brought together in the chapters on the *Progressions* and

on *Permutations and Combinations*. This latter is a highly entertaining and instructive subject, and has been, it is believed, more thoroughly and copiously treated here than in any other work accessible to students.

(4) A number of miscellaneous examples is appended in the form of separate examination papers, and the student is recommended to set himself systematically to answer each of these groups at a sitting of about two hours.

(5) All the latest improvements in this branch of Analytical Science have, so far as has been judged suitable for F. A. students, been incorporated into the work. Most of the recent treatises on Algebra have been consulted for the purpose.

(6) While many of the examples have been devised especially for this work, I have drawn largely from existing collections, as well as from published papers of Indian and English Universities.

Though the book has been written with great care, it has had, I am sorry to say, to be hurried through the press, and has not, therefore, been subjected to a sufficiently critical revision. Several errors and short-comings may thus have escaped observation.

Any corrections or suggestions for increasing the usefulness of the book will be thankfully received.

K. P. BASU.

Calcutta : *June*, 1888.

PREFACE TO THE SECOND EDITION.

The favour with which this work has been received—the first edition having been exhausted in a few months—seems to show that it has been found useful to those for whom it was intended. During the last session it has also, I understand, been in use in several colleges as a text-book. Owing, however, to certain pressing engagements, I was not able to bring out the present edition early enough to prevent disappointment on the part of many students.

No material changes have been made in this edition. I have thought it, indeed, desirable to add a chapter on Logarithms and to enlarge the chapter on the Exponential Theorem with an article on Logarithmic series. The questions on Algebra in the last F. A Examination of the Calcutta University have also been appended, hints for the solution of most of which will be found in their proper places. The book has, moreover, been subjected to a careful revision, so as to eliminate several errors which the last impression contained.

In the preface to the first edition I spoke in general terms of my having consulted with profit many recent treatises and collections of examples in Algebra. I take this opportunity, however, to express in particular my obligations to my friend Babu Saradaranjan Ray, M. A, for some useful suggestions derived from his "Algebraical Artifices" and also for some beautiful methods of solution which, so far as I am aware, were first made known to the public in that admirable little book.

K. P. BASU.

Calcutta : *June*, 1889.

PREFACE TO THE FOURTH EDITION.

In this edition the subject of Identities, to which great importance is generally attached, has been introduced, and the article on Elimination has been considerably improved. A chapter on Ratio and Proportion has been added, and some prominence has been given to the chapter on Variation by shifting it to an earlier position. The most important articles on Variation, as well as a few others which seemed wanting in clearness or otherwise defective, have been re-written. Besides these, several other important additions and alterations have been occasionally made. It is therefore hoped that the work in its present form will be found considerably more useful than the previous editions, to which a gratifying reception has already been accorded.

K. P. BASU.

Dacca : *September*, 1891.

PREFACE TO THE FIFTH EDITION.

In this Edition the bulk of the work has increased by about 80 pages. Several additions have been made, of which the following may be specially mentioned :—(1) a Resume of important theorems and examples from the 1st volume of the work ; (2) a chapter on Square and Cube Roots, with a clear exposition of the methods of extracting the square and cube roots of numbers ; (3) a chapter on Surds ; (4) a chapter on Mathematical Induction ; and (5) a collection, in chronological order, of the F. A. Papers of the later years of the Universities of Madras, as well as of the corresponding Papers of the Universities of Bombay, Punjab and Allahabad. On the other hand, as a partial compensation for the increase of bulk due to these additions, the Calcutta University F. A. Papers of some of the earlier years, which are comparatively of much smaller importance, have been removed. Illustrative examples, as well as examples for exercise, with occasional hints, have now been chosen not only from the Papers of the University of Calcutta but also from those of the other Indian Universities. These additions and alterations have been made chiefly with a view to the requirements of the Universities of Madras and Bombay, and it is hoped that the present edition will be received in those provinces with greater favour than the previous one. For the improvements effected in this edition I am largely indebted to the kind suggestions of S. Padmanabha Aiyar, Esq., B. A., Mathematical Lecturer, Hindu College, Vizagapatam.

K. P. BASU.

Calcutta : *June*, 1893.

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ALGEBRA MADE EASY.

INTRODUCTORY CHAPTER.

Resume OF IMPORTANT THEOREMS AND EXAMPLES FROM THE EARLIER PORTIONS OF THE SUBJECT.

1. To prove that $a \times b = b \times a$, *i.e.*, b multiplied by a gives the same result as a multiplied by b.

(i) First let a and b be any two positive integers.

Place b units in a horizontal row and write down a such rows in such a manner that units in similar positions in the different rows may be in the same vertical column; thus:—

$$\begin{array}{cccccccl}
1 & 1 & 1 & 1 & 1 & . & . & b \text{ times} \\
1 & 1 & 1 & 1 & 1 & . & . & b \text{ times} \\
1 & 1 & 1 & 1 & 1 & . & . & b \text{ times} \\
. & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & .
\end{array}$$

to a lines.

This being done, evidently it may also be said that we have written down b columns each containing a units.

Now let us count up the total number of units thus written down.

Since we have got a rows each containing b units, the total number of units = (the number in the first row) + (the number in the second row) + ... + (the number in the a^{th} row) $= b + b + b + \ldots\ldots$ to a terms $= a \times b$ ... ... (1)

Also, since we have b columns each containing a units, the total number of units = (the number in the 1st. column) + (the number in the 2nd. column) + (the number in the 3rd. column) + ... + (the number in the b^{th} column) $= a + a + a + \ldots\ldots$ to b terms $= b \times a$ ... ... (2)

Hence, from (1) and (2), we have $a \times b = b \times a$, *i.e.*, b taken a times = a taken b times.

(ii) Next let a and b be two positive fractions; suppose $a = \frac{m}{n}$ and $b = \frac{p}{q}$, where m, n, p, q are positive integers.

$$\text{Then } a \times b = \frac{m}{n} \times \frac{p}{q} = m \times \left\{\left(\frac{p}{q}\right) \div n\right\}$$

$$= m \times \frac{p}{nq} = \frac{mp}{nq} \quad \ldots \quad \ldots \quad \text{(I)}$$

$$\text{and } b \times a = \frac{p}{q} \times \frac{m}{n} = p \times \left\{\left(\frac{m}{n}\right) \div q\right\}$$

$$= p \times \frac{m}{qn} = \frac{pm}{qn} \quad \ldots \quad \ldots \quad \text{(II)}$$

But m and p are positive integers, therefore $mp = pm$, and similarly $nq = qn$.

Hence, from (I) and (II) we have $a \times b = b \times a$.

Thus it is established that for *all positive values* of a and b we must have $a \times b = b \times a$. ... ... (A)

Cor. 1. We have $x \times (-y) = -(xy)$, and $(-y) \times x = -(yx)$; but $xy = yx$, $\therefore x \times (-y) = (-y) \times x$ ... ... (B)

Cor. 2. $(-x) \times (-y) = +xy$, and $(-y) \times (-x) = +yx$; but $xy = yx$, $\therefore (-x) \times (-y) = (-y) \times (-x)$ ... (C)

Hence, from (A), (B) and (C) we conclude that for *all values* of a and b, $a \times b = b \times a$.

2. To prove that $(ab) \times c = a \times (bc)$ *or* $= b \times (ac)$, *i.e.*, to multiply c by the product of a and b is the same as to multiply c first by either of them and then that result by the other.

Place b brackets in a horizontal row each containing c units and write down a such rows in such a manner that the brackets in similar positions in the different rows may be in the same vertical column; thus :—

[c] [c] [c] [c] . . . . b times
[c] [c] [c] [c] . . . . b times
[c] [c] [c] [c] . . . . b times
. . . .
. . . .
to a rows.

This being done, it may also be said that we have written down b columns each containing a brackets.

As we have got altogether $a \times b$ brackets and as each bracket contains c units, the total number of units $= (ab) \times c \ldots (\alpha)$

Again, since we have got b brackets in a row each containing c units, the number of units in a row $= bc$, and as there are a rows altogether, therefore the total number of units $= a \times (bc)$ ... ... (β)

Again, since we have got a brackets in a column each containing c units, the number of units in a column $= ac$, and as there are b columns altogether, therefore the total number of units $= b \times (ac)$ ... ... (γ)

Hence, from (α), (β) and (γ) we have

$$(ab) \times c = a \times (bc) = b \times (ac).$$

3. Formulæ to be committed to memory.

1. $(a+b)^2 = a^2 + 2ab + b^2.$

2. $(a-b)^2 = a^2 - 2ab + b^2.$

3. $(a+b)(a-b) = a^2 - b^2.$

4. $$(a+b)^3 \left.\begin{aligned} &= a^3 + 3a^2b + 3ab^2 + b^3 \\ &= a^3 + b^3 + 3ab(a+b). \end{aligned}\right\}$$

5. $$\begin{aligned}(a+b+c)^3 &= a^3 + b^3 + c^3 + 3a^2(b+c) + 3b^2(a+c) \\ &\quad + 3c^2(a+b) + 6abc.\end{aligned}$$

6. $$(a-b)^3 \left.\begin{aligned} &= a^3 - 3a^2b + 3ab^2 - b^3 \\ &= a^3 - b^3 - 3ab(a-b). \end{aligned}\right\}$$

7. $$a^3 + b^3 \left.\begin{aligned} &= (a+b)^3 - 3ab(a+b) \\ &= (a+b)(a^2 - ab + b^2). \end{aligned}\right\}$$

8. $$a^3 - b^3 \left.\begin{aligned} &= (a-b)^3 + 3ab(a-b) \\ &= (a-b)(a^2 + ab + b^2). \end{aligned}\right\}$$

9. $(x+a)(x+b) = x^2 + (a+b)x + ab.$

10. $(x-a)(x+b) = x^2 + (b-a)x - ab.$

11. $(x-a)(x-b) = x^2 - (a+b)x + ab.$

12. $$\begin{aligned}(x+a)(x+b)(x+c) &= x^3 + (a+b+c)x^2 \\ &\quad + (ab+ac+bc)x + abc.\end{aligned}$$

13. $$\begin{aligned}(x-a)(x-b)(x-c) &= x^3 - (a+b+c)x^2 \\ &\quad + (ab+ac+bc)x - abc.\end{aligned}$$

14. $(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc.$

15. $(a+b+c+d)^2 = a^2+b^2+c^2+d^2+2a(b+c+d)$
$\qquad\qquad\qquad\qquad + 2b(c+d) + 2cd.$

16. $(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4.$

17. $(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5.$

18. $a^3+b^3+c^3-3abc = (a+b+c)(a^2+b^2+c^2-ab-ac-bc).$

19. $$\left.\begin{aligned} a^2(b-c)+b^2(c-a)+c^2(a-b) &= (a-b)(a-c)(b-c) \\ &= -(a-b)(b-c)(c-a). \end{aligned}\right\}$$

20. $$\left.\begin{aligned} ab(a-b)+bc(b-c)+ca(c-a) &= (a-b)(a-c)(b-c) \\ &= -(a-b)(b-c)(c-a) \end{aligned}\right\}$$

4. The expression $x^n - a^n$ is divisible by $x - a$ for *all* positive integral values of n.

Since
$$x^n - a^n = x^n - ax^{n-1} + ax^{n-1} - a^n$$
$$= x^{n-1}(x-a) + a(x^{n-1} - a^{n-1}),$$
we have $\dfrac{x^n - a^n}{x-a} = x^{n-1} + a.\dfrac{x^{n-1} - a^{n-1}}{x-a}$,

which shews that $x-a$ will divide $x^n - a^n$, *if* it divides $x^{n-1} - a^{n-1}$.

Hence, $x-a$ will divide $x^4 - a^4$, because we know that it divides $x^3 - a^3$ (*i.e*, $x^{4-1} - a^{4-1}$); and since $x-a$ divides $x^4 - a^4$ (*i.e.*, $x^{5-1} - a^{5-1}$), therefore it will divide $x^5 - a^5$; and so on.

Thus for all positive integral values of n, $x^n - a^n$ is divisible by $x-a$.

Again, since $x^n + a^n = (x^n - a^n) + 2a^n$, of which $x^n - a^n$ is divisible by $x-a$ and $2a^n$ is not, $\therefore$ $x^n + a^n$ is *not* divisible by $x-a$.

Thus when n is a positive integer,

$$\left.\begin{array}{ll} x-a & \textit{always} \text{ divides } x^n - a^n \\ \text{but} & \textit{never} \text{ divides } x^n + a^n \end{array}\right\} \quad \cdots \quad \cdots \quad \text{(A)}$$

Cor. 1. $(x+a)$ divides $x^n - a^n$ *only* when n is an *even* integer.

For, when n is even, $(-a)^n = a^n$, and $\therefore$ $x^n - a^n = x^n - (-a)^n$;
when n is odd, $(-a)^n = -a^n$, and $\therefore$ $x^n - a^n = x^n + (-a)^n$;
also $\quad x+a = x-(-a)$.

Now, from (A), we know that $x-(-a)$ divides $x^n-(-a)^n$ but not $x^n+(-a)^n$. Hence $x+a$ divides x^n-a^n when n is even, but not when n is odd; *i.e.*, $x+a$ divides x^n-a^n *only* when n is an *even* integer.

Cor. 2. $x+a$ divides x^n+a^n *only* when n is an odd integer.

$$\left.\begin{aligned}\text{For, when } n \text{ is odd, } (-a)^n &= -a^n, \text{ and } \therefore\ x^n+a^n=x^n-(-a)^n;\\ \text{when } n \text{ is even, } (-a)^n &= a^n, \text{ and } \therefore\ x^n+a^n=x^n+(-a)^n;\end{aligned}\right\}$$

$$\text{also} \quad x+a=x-(-a).$$

Now, from (A), we know that $x-(-a)$ divides $x^n-(-a)^n$ but not $x^n+(-a)^n$ Hence, $x+a$ divides x^n+a^n when n is odd, but not when n is even; *i.e.*, $x+a$ divides x^n+a^n *only* when n is an *odd* integer.

Thus we have obtained the following results :—

$$x-a \text{ divides } \left.\begin{aligned}&x^n-a^n \ \textit{always},\\ &x^n+a^n \ \textit{never}.\end{aligned}\right\}$$

$$x+a \text{ divides } \left.\begin{aligned}&x^n-a^n \ \textit{only} \text{ when } n \text{ is } \textit{even},\\ &x^n+a^n \ \textit{only} \text{ when } n \text{ is } \textit{odd}.\end{aligned}\right\}$$

5. Examples in Factorisation.

Example 1. Resolve into factors $a^4+a^2b^2+b^4$.

$$\begin{aligned}a^4+a^2b^2+b^4 &= (a^4+2a^2b^2+b^4)-a^2b^2\\ &= (a^2+b^2)^2-(ab)^2\\ &= \{(a^2+b^2)+ab\}\{(a^2+b^2)-ab\}\\ &= (a^2+ab+b^2)(a^2-ab+b^2).\end{aligned}$$

Example 2. Resolve into factors x^4+4.

$$\begin{aligned}x^4+4 &= (x^4+4x^2+4)-4x^2\\ &= (x^2+2)^2-(2x)^2\\ &= \{(x^2+2)+2x\}\{(x^2+2)-2x\}\\ &= (x^2+2x+2)(x^2-2x+2).\end{aligned}$$

Example 3. Resolve into factors $x^2+2xy-8y^2-4z^2+12yz$.

The given expression

$$\begin{aligned}&= (x^2+2xy+y^2)-(9y^2+4z^2-12yz)\\ &= (x+y)^2-(3y-2z)^2\\ &= \{(x+y)+(3y-2z)\}\{(x+y)-(3y-2z)\}\\ &= (x+4y-2z)(x-2y+2z).\end{aligned}$$

Example 4. Resolve into factors $2a^2+5ab-12b^2$.

$$2a^2+5ab-12b^2 = 2\left(a^2+\frac{5}{2}ab-6b^2\right)$$

$$= 2\left\{a^2+\frac{5}{2}ab+\left(\frac{5b}{4}\right)^2-\left(\frac{25b^2}{16}+6b^2\right)\right\}$$

$$= 2\left\{\left(a+\frac{5}{4}b\right)^2-\frac{121}{16}b^2\right\}$$

$$= 2\left\{\left(a+\frac{5}{4}b\right)+\frac{11}{4}b\right\}\left\{\left(a+\frac{5}{4}b\right)-\frac{11}{4}b\right\}$$

$$= 2(a+4b)\left(a-\frac{3}{2}b\right)$$

$$= (a+4b)(2a-3b).$$

Example 5. Resolve into factors

$$4(x^2+2x+5)^2+17(x^2+2x+5)(x^2+6x)+4(x^2+6x)^2.$$

Putting a for x^2+2x+5 and b for x^2+6x, the given expression becomes $4a^2+17ab+4b^2$; and it is easy to see that $4a^2+17ab+4b^2 = (a+4b)(4a+b)$.

Hence, the given expression

$$= \{(x^2+2x+5)+4(x^2+6x)\}\{4(x^2+2x+5)+(x^2+6x)\}$$

$$= (5x^2+26x+5)(5x^2+14x+20)$$

$$= (x+5)(5x+1)(5x^2+14x+20).$$

Example 6. Resolve into factors

$$6x^2+17xy+7y^2-2x-23y-20.$$

The given expression arranged according to descending powers of x

$$= 6x^2+(17y-2)x+(7y^2-23y-20)$$

$$= 6\left\{x^2+\frac{1}{6}\left(17y-2\right)x+\frac{1}{6}\left(7y^2-23y-20\right)\right\}$$

$$= 6\left\{x^2+\frac{1}{6}\left(17y-2\right)x+\frac{1}{144}\left(17y-2\right)^2 -\frac{1}{144}\left(17y-2\right)^2+\frac{1}{6}\left(7y^2-23y-20\right)\right\}$$

$$= 6\left[\left\{x+\frac{1}{12}\left(17y-2\right)\right\}^2-\frac{1}{144}\left\{\left(289y^2-68y+4\right) -24(7y^2-23y-20)\right\}\right]$$

$$= 6\left[\left\{x+\frac{1}{12}\left(17y-2\right)\right\}^2-\frac{121}{144}\left(y+2\right)^2\right]$$

$$= 6\left[\left\{x+\frac{1}{12}\left(17y-2\right)\right\}+\frac{11}{12}\left(y+2\right)\right]$$

$$\times\left[\left\{x+\frac{1}{12}\left(17y-2\right)\right\}-\frac{11}{12}\left(y+2\right)\right]$$

$$= 6\left(x+\frac{7}{3}y+\frac{5}{3}\right)\left(x+\frac{1}{2}y-2\right)$$

$$= 3\left(x+\frac{7}{3}y+\frac{5}{3}\right)\times 2\left(x+\frac{1}{2}y-2\right)$$

$$= (3x+7y+5)(2x+y-4).$$

Example 7. Resolve into factors $x^3+7x^2-21x-27$.

The given expression

$$= (x^3-27)+(7x^2-21x)$$

$$= (x-3)(x^2+3x+9)+7x(x-3)$$

$$= (x-3)\{(x^2+3x+9)+7x\}$$

$$= (x-3)(x^2+10x+9)$$

$$= (x-3)(x+9)(x+1).$$

Example 8. Resolve into factors

$$(a+b+c)\,(ab+bc+ca)-abc.$$

Arranging the expression within brackets according to powers of a we have the given expression

$$= \{a+(b+c)\}\{a(b+c)+bc\}-abc$$

$$= a^2(b+c)+a(b+c)^2+bc(b+c)$$

$$= (b+c)\{a^2+a(b+c)+bc\}$$

$$= (b+c)(a+b)(a+c).$$

Example 9. Resolve into factors

$$a^3(b-c)+b^3(c-a)+c^3(a-b).$$

$$a^3(b-c)+b^3(c-a)+c^3(a-b)$$

$$= a^3(b-c)-a(b^3-c^3)+bc(b^2-c^2)$$

[arranged according to powers of a]

$$= (b-c)\{a^3-a(b^2+bc+c^2)+bc(b+c)\}$$

$$= (b-c)\{-b^2(a-c)-bc(a-c)+a(a^2-c^2)\}$$

[arranged according to powers of b]

$$= (b-c)(a-c)\{-b^2-bc+a(a+c)\}$$

$$= (b-c)(a-c)\{c(a-b)+(a^2-b^2)\}$$

[arranged according to powers of c]

$$= (b-c)(a-c)(a-b)(c+a+b).$$

Example 10. Resolve into factors

$$a^3(b^2-c^2)+b^3(c^2-a^2)+c^3(a^2-b^2).$$

In this expression also the letters occur in cyclic order and we can at once proceed as in the last example.

$$a^3(b^2-c^2)+b^3(c^2-a^2)+c^3(a^2-b^2)$$

$$= a^3(b^2-c^2)-a^2(b^3-c^3)+b^2c^2(b-c)$$

[arranged according to powers of a]

$$= (b-c)\{a^3(b+c)-a^2(b^2+bc+c^2)+b^2c^2\}$$

$$= (b-c)\{-b^2(a^2-c^2)+ba^2(a-c)+a^2c(a-c)\}$$

[arranged according to powers of b]

$$= (b-c)(a-c)\{-b^2(a+c)+ba^2+a^2c\}$$

$$= (b-c)(a-c)\{c(a^2-b^2)+ab(a-b)\}$$

[arranged according to powers of c]

$$= (b-c)(a-c)(a-b)\{c(a+b)+ab\}$$

$$= (b-c)(a-c)(a-b)(ab+bc+ca).$$

Example 11. Find the quotient of $a^3+b^3+c^3-3abc$ by $a+b+c$.

Since $b^3+c^3=(b+c)^3-3bc(b+c)$, we have

$$a^3+b^3+c^3-3abc$$

$$= a^3+(b+c)^3-3bc(b+c)-3abc$$

$$= \{a^3+(b+c)^3\}-3bc\{a+(b+c)\}$$

$$= \{a+(b+c)\}\{a^2-a(b+c)+(b+c)^2\}-3bc(a+b+c)$$

$$= (a+b+c)\{a^2-a(b+c)+(b+c)^2-3bc\}$$

$$= (a+b+c)\{a^2-a(b+c)+(b^2+2bc+c^2)-3bc\}$$

$$= (a+b+c)(a^2+b^2+c^2-ab-ac-bc).$$

Hence, the required quotient $=a^2+b^2+c^2-ab-ac-bc$.

Example 12. Resolve into factors $x^3+7x^2+14x+8$.

On inspection it is observed that we can split up the given expression into parts each of which is divisible by $x+2$ in either of the two following ways :—

(i) $(x^3+8)+7x(x+2)$;

(ii) $x^2(x+2)+5x(x+2)+4(x+2)$.

Hence, choosing the latter way, we have

$$x^3+7x^2+14x+8$$

$$= (x+2)(x^2+5x+4)$$

$$= (x+2)(x+1)(x+4).$$

Example 13. Resolve into factors $8x^3 + 16x - 9$.

We find that the given expression can be split up into parts divisible by $2x - 1$ in either of the two following ways :—

(i) $(8x^3 - 1) + 8(2x - 1)$;

(ii) $2x(4x^2 - 1) + 9(2x - 1)$.

Hence, choosing the former way, we have

$$\begin{aligned} 8x^3 + 16x - 9 &= (8x^3 - 1) + 8(2x - 1) \\ &= (2x - 1)\{(4x^2 + 2x + 1) + 8\} \\ &= (2x - 1)(4x^2 + 2x + 9). \end{aligned}$$

Example 14. Resolve into factors $a^3 + 7ab^2 - 22b^3$.

We find that the expression can be split up into parts each of which is divisible by $a - 2b$ in either of the two following ways :—

(i) $(a^3 - 8b^3) + 7b^2(a - 2b)$;

(ii) $a(a^2 - 4b^2) + 11b^2(a - 2b)$.

Hence, choosing the former way, we have

$$\begin{aligned} a^3 + 7ab^2 - 22b^3 &= (a^3 - 8b^3) + 7b^2(a - 2b) \\ &= (a - 2b)\{(a^2 + 2ab + 4b^2) + 7b^2\} \\ &= (a - 2b)(a^2 + 2ab + 11b^2). \end{aligned}$$

Example 15. Resolve into factors $(a^2 - b^2)(x^2 - y^2) + 4abxy$.

The given expression

$$\begin{aligned} &= a^2x^2 - a^2y^2 - b^2x^2 + b^2y^2 + 4abxy \\ &= (a^2x^2 + b^2y^2 + 2abxy) - (a^2y^2 + b^2x^2 - 2abxy) \\ &= (ax + by)^2 - (ay - bx)^2 \\ &= \{(ax + by) + (ay - bx)\}\{(ax + by) - (ay - bx)\} \\ &= \{(a - b)x + (a + b)y\}\{(a + b)x - (a - b)y\}. \end{aligned}$$

Example 16. Resolve into factors

$$x^4 + 2x^3y + 3x^2y^2 + 2xy^3 + y^4.$$

The given expression

$$\begin{aligned} &= (x^4 + 2x^2y^2 + y^4) + x^2y^2 + (2x^3y + 2xy^3) \\ &= (x^2 + y^2)^2 + (xy)^2 + 2(xy)(x^2 + y^2) \\ &= \{(x^2 + y^2) + xy\}^2 \\ &= (x^2 + xy + y^2)^2. \end{aligned}$$

Example 17. Resolve into factors

$$(x-1)(x-2)(x+3)(x+4)+4.$$

$$\begin{aligned}(x-1)(x-2)(x+3)(x+4) &= \{(x-1)(x+3)\}\{(x-2)(x+4)\}\\ &= (x^2+2x-3)(x^2+2x-8).\end{aligned}$$

Hence, putting z for x^2+2x, the given expression

$$\begin{aligned}&= (z-3)(z-8)+4\\ &= z^2-11z+28\\ &= (z-4)(z-7)\\ &= (x^2+2x-4)(x^2+2x-7).\end{aligned}$$

Example 18. If $x+y = a$ and $xy = b^2$, find the value of (i) x^4+y^4 and (ii) $x^3-x^2y-xy^2+y^3$ in terms of a and b.

$$\begin{aligned}\text{(i) } x^4+y^4 &= (x^2+y^2)^2-2x^2y^2\\ &= \{(x+y)^2-2xy\}^2-2x^2y^2,\end{aligned}$$

and $\therefore$ the required value

$$\begin{aligned}&= (a^2-2b^2)^2-2b^4\\ &= a^4-4a^2b^2+2b^4.\end{aligned}$$

$$\begin{aligned}\text{(ii) } & x^3-x^2y-xy^2+y^3\\ &= x^2(x-y)-y^2(x-y)\\ &= (x-y)(x^2-y^2)\\ &= (x-y)^2(x+y)\\ &= \{(x+y)^2-4xy\}(x+y)\\ &= (a^2-4b^2)a.\end{aligned}$$

Example 19. Find the value of $x^4-x^3+x^2+2$, when $x^2+2 = 2x$.

$$\begin{aligned}x^4-x^3+x^2+2 &= (x^4+x^3+x^2)-2(x^3-1)\\ &= x^2(x^2+x+1)-2(x-1)(x^2+x+1)\\ &= (x^2+x+1)\{x^2-2(x-1)\}\\ &= (x^2+x+1)(x^2-2x+2),\end{aligned}$$

and $\therefore$ the required value

$$= (x^2+x+1)\times 0 = 0.$$

Example 20. Find the value of

$$a^4+b^4+c^4-2b^2c^2-2c^2a^2-2a^2b^2, \text{ when } a+b = c.$$

The given expression

$$\begin{aligned} &= a^4 - 2(b^2+c^2)a^2 + b^4 + c^4 - 2b^2c^2 \\ &= \{a^4 - 2(b^2+c^2)a^2 + (b^2+c^2)^2\} \\ &\qquad\qquad - (b^2+c^2)^2 + b^4 + c^4 - 2b^2c^2 \\ &= \{a^2 - (b^2+c^2)\}^2 - 4b^2c^2 \\ &= \{(a^2-b^2-c^2)+2bc\}\{(a^2-b^2-c^2)-2bc\} \\ &= \{a^2-(b-c)^2\}\{a^2-(b+c)^2\} \\ &= (a+b-c)(a-b+c)(a+b+c)(a-b-c), \\ \text{and } \therefore\ &= 0, \text{ when } a+b = c. \end{aligned}$$

6. The ordinary method of finding the H. C. F. of two multinomial expressions which have no monomial factors.

Let A and B stand for two such expressions both arranged according to descending powers of some common letter, and let the index of the highest power of that letter in A be not less than the index of the highest power of that letter in B.

Divide A by B, and let Q be the quotient and C the remainder.

$$\left.\begin{aligned} &\text{Then we must have } C = A - BQ \quad \ldots \quad \ldots \quad (1) \\ &\text{or,} \qquad\qquad\qquad\quad A = BQ + C \quad \ldots \quad \ldots \quad (2) \end{aligned}\right\}$$

From (1) it is clear that *every* common factor of A and B is a factor of C [for if $A = pa$ and $B = pb$, we have $C = p(a - bQ)$]. Hence, if H denote the H. C F. of A and B, H also is a factor of C, and is therefore a *common factor* of B *and* C.

It is clear therefore that the H. C F. of B and C is either H or an expression of higher dimensions than H ... ... (a)

Now, from (2) it is evident that *every* common factor of B and C is a factor of A and is therefore a common factor of B and A. Hence, the H. C. F. of B and C also is a common factor of B and A, and therefore cannot be of higher dimensions than H.

Hence, from (a), the H. C. F. of B and C is H.

Thus the H. C. F. of B and C is the H. C. F. required.

Similarly, if B be divided by C, and D be the new remainder, the H. C. F. of C and D is the same as the H. C. F. of B and C and is therefore the H. C. F required.

Now, divide C by D and let there be no remainder. Then D is the H. C. F. of C and D and is therefore the H. C. F. required.

***Cor.* 1.** As the H. C. F. of any divisor and the corresponding dividend is the H. C. F. required, it is clear that, for the sake of convenience, either of them may be multiplied or divided by any monomial expression *which is not a factor of the other.*

***Cor.* 2.** In dividing A by B if we stop before the complete quotient is obtained so that q is the partial quotient and C' the corresponding remainder, then the H. C. F. of B and C', just as the H. C. F. of B and C, is the H. C. F. required. Hence, by Cor. 1. in dividing C' by B (or if convenient, B by C' when C' is *not* of higher degree than B) we can multiply or divide either of them, if necessary, by any monomial expression *which is not a factor of the other.*

Hence, we have the following rule :—

Arrange the two expressions according to descending powers of some common letter ; divide the expression which is of higher degree in that letter by the other, or if they be of the same degree, either of them by the other ; if there be any remainder take it for a new divisor and the preceding divisor for the dividend, and continue the process till there is no remainder The last divisor will be the H. C. F. required. Of any divisor and the corresponding dividend either may be multiplied or divided by any number which is not a factor of the other.

Example. Find the H. C. F. of

$4x^4+11x^3+27x^2+17x+5$ and $6x^4+14x^3+36x^2+14x+10$

The 2nd expression $=2(3x^4+7x^3+18x^2+7x+5)$, but 2 is not a factor of the 1st expression. Hence, the H. C. F. required is the H. C. F. of the 1st. expression and $3x^4+7x^3+18x^2+7x+5$.

$$\begin{array}{r|l|l}
 & 4x^4+11x^3+27x^2+17x+5 & \\
 & 3 & \\
\hline
3x^4+7x^3+18x^2+7x+5 & 12x^4+33x^3+81x^2+51x+15 & 4 \\
 & 12x^4+28x^3+72x^2+28x+20 & \\
\hline
 & \quad 5x^3+\ 9x^2+23x-\ 5 &
\end{array}$$

$$\begin{array}{rl} & 3x^4 + 7x^3 + 18x^2 + 7x + 5 \\ & 5 \\ \hline 5x^3 + 9x^2 + 23x - 5) & 15x^4 + 35x^3 + 90x^2 + 35x + 25\,(3x \\ & 15x^4 + 27x^3 + 69x^2 - 15x \\ \hline & 8x^3 + 21x^2 + 50x + 25 \\ & 5 \\ \hline & 40x^3 + 105x^2 + 250x + 125\,(8 \\ & 40x^3 + 72x^2 + 184x - 40 \\ \hline & 33)\,33x^2 + 66x + 165 \\ \hline & x^2 + 2x + 5 \end{array}$$

$$\begin{array}{rl} x^2 + 2x + 5) & 5x^3 + 9x^2 + 23x - 5\,(5x - 1 \\ & 5x^3 + 10x^2 + 25x \\ \hline & -x^2 - 2x - 5 \\ & -x^2 - 2x - 5 \\ \hline \end{array}$$

Thus the required H. C. F. $= x^2 + 2x + 5$.

7. An important principle.

If A and B denote two expressions having no monomial factors and if m, n, p, q be any four *numerical* quantities such that $mq - np$ is *not* equal to zero, then the H. C. F. of A and B is the same as the H. C. F. of $mA + nB$ and $pA + qB$, *numerical common factors*, if any, being left out. This may be proved as follows :—

Let H denote the H. C. F. of A and B, and H′ the H. C. F. of $mA + nB$ and $pA + qB$ after removal from them of any numerical common factor that may occur.

Now, since *every* common factor of A and B is a factor of $mA + nB$ and also of $pA + qB$, therefore H is a common factor of $mA + nB$ and $pA + qB$.

Hence, H′ is either equal to H or is an expression of higher dimensions than H ... ... (a)

Again, since $q(mA + nB) - n(pA + qB) = (mq - np)A$, and $m(pA + qB) - p(mA + nB) = (mq - np)B$, it is clear that *every* common factor of $mA + nB$ and $pA + qB$ is a factor of $(mq - np)A$, and also of $(mq - np)B$. Hence, as $mq - np$ is only a numerical quantity, *every* common factor of those expressions *other than numerical* must be a factor of A as well as of B. Hence H′ is a common factor of A and B and therefore cannot be of higher dimensions than H.

Hence, by (a), H′ = H, which proves the proposition.

Cor. 1. The H. C. F. of A and B is the same as the H. C. F. of A + B and A − B. Here $m = 1$, $n = 1$, $p = 1$ and $q = -1$.

Cor. 2. The H. C. F. of A and B is the same as the H. C. F. of A ± B and B; here $m = 1$, $n = \pm 1$, $p = 0$ and $q = 1$. Similarly, it is the same as the H. C. F. of A ± B and A.

Example 1. Find the H C. F. of

$$x^4 + x^3 - 5x^2 - 3x + 2 \text{ and } x^4 - 3x^3 + x^2 + 3x - 2.$$

$$\begin{aligned}
\text{Let}\quad A &= x^4 + x^3 - 5x^2 - 3x + 2,\\
\text{and}\quad B &= x^4 - 3x^3 + x^2 + 3x - 2.\\
\text{Then}\quad A + B &= 2x^4 - 2x^3 - 4x^2\\
&= 2x^2(x^2 - x - 2),\\
\text{and}\quad A - B &= 4x^3 - 6x^2 - 6x + 4\\
&= 2(2x^3 - 3x^2 - 3x + 2).
\end{aligned}$$

Hence, by Cor. 1, the required H. C. F. is the H. C. F. of $x^2(x^2 - x - 2)$ and $2x^3 - 3x^2 - 3x + 2$, and therefore of $x^2 - x - 2$ and $2x^3 - 3x^2 - 3x + 2$.

$$\begin{aligned}
\text{Let}\quad A' &= x^2 - x - 2,\\
\text{and}\quad B' &= 2x^3 - 3x^2 - 3x + 2.\\
\text{Then}\quad A' + B' &= 2x^3 - 2x^2 - 4x\\
&= 2x(x^2 - x - 2).
\end{aligned}$$

Hence, the required H. C. F.

= the H. C. F. of A′ and A′ + B′ [Cor. 2.]

$= x^2 - x - 2.$

Example 2. Find the H. C. F. of

$$4x^4 + 11x^3 + 27x^2 + 17x + 5 \text{ and } 3x^4 + 7x^3 + 18x^2 + 7x + 5.$$

$$\begin{aligned}
\text{Let}\quad A &= 4x^4 + 11x^3 + 27x^2 + 17x + 5,\\
\text{and}\quad B &= 3x^4 + 7x^3 + 18x^2 + 7x + 5.\\
\text{Then}\quad A - B &= x^4 + 4x^3 + 9x^2 + 10x\\
&= x(x^3 + 4x^2 + 9x + 10),
\end{aligned}$$

and $3A - 4B = 5x^3 + 9x^2 + 23x - 5$.

Hence, the H. C. F. of $x^3 + 4x^2 + 9x + 10$ and $5x^3 + 9x^2 + 23x - 5$ is the H. C. F. required.

$$\begin{aligned} \text{Let} \quad A' &= x^3 + 4x^2 + 9x + 10, \\ \text{and} \quad B' &= 5x^3 + 9x^2 + 23x - 5. \\ \text{Then } A' + 2B' &= 11x^3 + 22x^2 + 55x \\ &= 11x(x^2 + 2x + 5), \\ \text{and } 5A' - B' &= 11x^2 + 22x + 55 \\ &= 11(x^2 + 2x + 5). \end{aligned}$$

Hence, the H. C. F. required is the H. C. F. of $x(x^2 + 2x + 5)$ and $x^2 + 2x + 5$, and is therefore $= x^2 + 2x + 5$.

8. The H. C. F. of three or more expressions whose factors cannot be easily found.

Let A, B, C stand for any three expressions of which the H. C. F. is to be found.

Let G denote the H. C. F. of A and B and H that of G and C.

Then G being the product of *all* the elementary common factors of A and B, every factor of G is a common factor of A and B, and therefore every common factor of G and C is a common factor of A, B and C.

Hence, H also is a common factor of A, B and C. Therefore the H. C. F. required is either H or an expression of higher dimensions than H ... . (β)

But, since every common factor of A and B is a factor of G, *every* common factor of A, B and C is a common factor of G and C. Hence, the H. C. F. required is a common factor of G and C and therefore cannot be of higher dimensions than H.

Hence, by (β), the H. C. F. required = H.

By a similar reasoning it follows that if D be a fourth expression, then the H. C. F. of H and D is the H. C. F. of A, B, C and D.

Thus we have the following rule :—*To find the H. C. F. of any number of expressions A, B, C, D, &c., first find the H. C. F. of A and B, then the H. C. F. of this result and C, and so on ; the result obtained last of all is the H. C. F. required.*

9. L. C. M. of two expressions whose factors are not obvious by inspection.

Let A and B stand for two such expressions, and suppose their H. C. F. is found to be H.

Divide A and B by H and let the respective quotients be a and b. Then we have

$$\left.\begin{array}{l} A = aH \\ B = bH \end{array}\right\}$$

Hence, since a and b have no common factor, *every* common multiple of A and B must *necessarily* contain $a \times H \times b$ as a factor.

Hence, the L C. M. required $= aHb$.

$$\text{But} \qquad \left.\begin{array}{ll} aHb = a(Hb) & = \frac{A}{H} \times B \\ \text{or} = (aH)b & = A \times \frac{B}{H} \end{array}\right\}$$

Hence, the required L. C. M. $= \frac{A}{H} \times B$, or $= A \times \frac{B}{H}$.

Thus, *to find the L. C. M. of any two expressions we have to divide one of them by their H. C. F. and multiply the quotient by the other.*

Cor. If L denote the L. C. M. of A and B we have $L \times H = A \times B$; that is, *the product of the L. C. M. and H. C. F. of any two expressions is equal to the product of those expressions.*

NOTE. If any two expressions have no common factor, their L. C. M. is evidently equal to their product.

10. L. C. M. of three or more expressions whose factors are not obvious by inspection.

Let A, B, C stand for three such expressions; to find their L. C. M.

Let L denote the L. C. M. of A and B, and M that of L and C.

Then evidently *every* common multiple of L and C is a common multiple of A, B, C; ... ... (1)

also *every* common multiple of A, B, C is a common multiple of L and C. ... ... (2)

From (1) M is a common multiple of A, B, C. Hence, either M or an expression of a lower degree than M is the L. C. M. of A, B, C.

But an expression of a lower degree than M cannot be the L. C. M. of A, B, C; because, from (2), the L. C. M. of A, B, C is a common multiple of L and C.

Hence, the required L. C. M. = M.

Thus. *to find the L. C. M. of any number of expressions A, B, C, D, &c., we have first to find the L. C. M. of A and B, then the L. C. M. of this result and C, and so on; the last result thus obtained is the L. C. M. required.*

Example. If h_1, h_2, h_3 be the highest common factors, and l_1, l_2, l_3 the lowest common multiples of B and C, C and A, A and B respectively; and if H be the highest common factor, and L the lowest common multiple of A, B and C; prove that

$$\text{(i)} \qquad h_1h_2h_3l_1l_2l_3 = (\text{ABC})^2,$$

$$\text{(ii)} \qquad \frac{\text{L}}{\text{H}} = \frac{\text{ABC}}{h_1h_2h_3}.$$

Since h_1 is the H. C. F., and l_1 the L. C. M., of B and C, we must have

$$h_1l_1 = \text{BC}. \qquad \text{[Cor. Art. 9.]}$$

Similarly $\quad h_2l_2 = \text{CA},$

and $\quad h_3l_3 = \text{AB}.$

Hence, $h_1h_2h_3l_1l_2l_3 = (\text{ABC})^2$.

Again, since H is the H. C. F. of A, B and C, it is a factor common to B and C and is therefore a factor of h_1. Hence we must have

$$h_1 = \text{H}.f_1 \quad \ldots\ldots\ldots \quad (1)$$

where f_1 is the quotient of h_1 by H.

Similarly we have

$$h_2 = \text{H}.f_2 \quad \ldots\ldots\ldots \quad (2)$$

and $$h_3 = \text{H}.f_3 \quad \ldots\ldots\ldots \quad (3)$$

It is clear that no two of the three quantities f_1, f_2, f_3 have a common factor; for instance if f_2 and f_3 had a common factor, say k, then Hk would be a factor of h_2 as well as of h_3 and would therefore be a common factor of C, A, B, which is impossible.

Now, from hypothesis, it is evident that both h_1 and h_2 are factors of C; hence from (1) and (2) we must have

$$C = Hf_1f_2.p \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where p is the quotient of C by Hf_1f_2.

Similarly we must have

$$A = Hf_2f_3.q, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$\text{and } B = Hf_3f_1.r \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

From these relations it is easy to see that the following pairs of quantities cannot have a common factor :—(p, q), (p, r), (q, r), (p,f_3), (q,f_1) and (r,f_2). For instance if p and f_3 had a common factor, say k', then Hf_2k', *i.e.*, h_2k' would be, as evident from (4) and (5), a common factor of C and A, which is impossible.

Now, from (4), (5) and (6) it is clear that

$$L = Hf_1f_2f_3pqr;$$

and we have also

$$ABC = H^3.(f_1f_2f_3)^2.pqr.$$

$$\text{Hence,} \quad \frac{L}{ABC} = \frac{Hf_1f_2f_3pqr}{H^3.(f_1f_2f_3)^2.pqr}$$

$$= \frac{H}{H^3.f_1f_2f_3}$$

$$= \frac{H}{(Hf_1).(Hf_2).(Hf_3)}$$

$$= \frac{H}{h_1h_2h_3},$$

$$\text{and} \quad \therefore \quad \frac{L}{H} = \frac{ABC}{h_1h_2h_3}.$$

11. A few examples in fractions.

Example 1. Reduce to its simplest form

$$bc\cdot\frac{a+d}{(a-b)(a-c)} + ac\cdot\frac{b+d}{(b-a)(b-c)} + ab\cdot\frac{c+d}{(c-a)(c-b)}.$$

Since $\quad b - a = -(a-b)$,

and $(c-a)(c-b) = [-(a-c)]\times[-(b-c)] = (a-c)(b-c)$, the given expression

$$= bc\cdot\frac{a+d}{(a-b)(a-c)} + ac\cdot\frac{-(b+d)}{(a-b)(b-c)} + ab\cdot\frac{c+d}{(a-c)(b-c)}$$

$$= \frac{bc(a+d)(b-c) - ac(b+d)(a-c) + ab(c+d)(a-b)}{(a-b)(a-c)(b-c)}.$$

Now, the numerator
$$\begin{aligned}
&= abc\{(b-c)-(a-c)+(a-b)\} \\
&\qquad + d\{bc(b-c)-ac(a-c)+ab(a-b)\} \\
&= d\{bc(b-c)-ac(a-c)+ab(a-b)\} \\
&= d\{a^2(b-c)+b^2(c-a)+c^2(a-b)\} \\
&= d(a-b)(a-c)(b-c).
\end{aligned}$$

[Formula 19, page 4.]

Hence, the given expression $= d$.

Example 2. Simplify

$$\frac{a^2}{(a-b)(a-c)(x+a)}+\frac{b^2}{(b-a)(b-c)(x+b)}+\frac{c^2}{(c-a)(c-b)(x+c)}.$$

The given expression

$$= \frac{a^2}{(a-b)(a-c)(x+a)}+\frac{-b^2}{(a-b)(b-c)(x+b)}+\frac{c^2}{(a-c)(b-c)(x+c)}$$

$$= \frac{a^2(b-c)(x+b)(x+c)-b^2(a-c)(x+c)(x+a)+c^2(a-b)(x+a)(x+b)}{(a-b)(a-c)(b-c)(x+a)(x+b)(x+c)}.$$

Now, the numerator

$$\begin{aligned}
&= a^2(b-c)\{x^2+x(b+c)+bc\}+b^2(c-a)\{x^2+x(c+a)+ca\} \\
&\qquad + c^2(a-b)\{x^2+x(a+b)+ab)\} \\
&= x^2\{a^2(b-c)+b^2(c-a)+c^2(a-b)\} \\
&\qquad + x\{a^2(b^2-c^2)+b^2(c^2-a^2)+c^2(a^2-b^2)\} \\
&\qquad + abc\{a(b-c)+b(c-a)+c(a-b)\} \\
&= x^2\{a^2(b-c)+b^2(c-a)+c^2(a-b)\} \\
&= x^2(a-b)(a-c)(b-c).
\end{aligned}$$

[Formula 19, Page 4.]

Hence, the given expression $= \dfrac{x^2}{(x+a)(x+b)(x+c)}$.

Example 3. Simplify

$$\frac{a^3}{(a-b)(a-c)}+\frac{b^3}{(b-c)(b-a)}+\frac{c^3}{(c-a)(c-b)}.$$

The given expression

$$= \frac{a^3}{(a-b)(a-c)}+\frac{-b^3}{(b-c)(a-b)}+\frac{c^3}{(a-c)(b-c)}$$

$$= \frac{a^3(b-c)-b^3(a-c)+c^3(a-b)}{(a-b)(a-c)(b-c)}.$$

Now, the numerator

$$= a^3(b-c)+b^3(c-a)+c^3(a-b)$$

$$= (a-b)(a-c)(b-c)(a+b+c).$$

[See example 9, Page 7.]

Hence, the given expression $= a+b+c$.

12. Solutions of some equations.

Example 1. Solve $\dfrac{3}{x-2}+\dfrac{5}{x-6}=\dfrac{8}{x+3}$.

Since $$\frac{8}{x+3}=\frac{3}{x+3}+\frac{5}{x+3}$$

we have $$\frac{3}{x-2}+\frac{5}{x-6}=\frac{3}{x+3}+\frac{5}{x+3}$$

Hence, by transposition,

$$\frac{3}{x-2}-\frac{3}{x+3}=\frac{5}{x+3}-\frac{5}{x-6}$$

or, $$\frac{15}{(x-2)(x+3)}=\frac{-45}{(x+3)(x-6)}.$$

Multiplying both sides by $x+3$, and dividing by 15,

we have $$\frac{1}{x-2}=\frac{-3}{x-6}.$$

Hence, $$x-6=-3(x-2);$$

$$\therefore\ 4x=12,$$

$$\therefore\ x=3.$$

Example 2. Solve $\dfrac{8}{2x-1}+\dfrac{9}{3x-1}=\dfrac{7}{x+1}$.

We have $$\frac{8}{2x-1}+\frac{9}{3x-1}=\frac{4}{x+1}+\frac{3}{x+1}.$$

Hence, $$\left\{\frac{8}{2x-1}-\frac{4}{x+1}\right\}+\left\{\frac{9}{3x-1}-\frac{3}{x+1}\right\}=0$$

or, $$\frac{12}{(2x-1)(x+1)}+\frac{12}{(3x-1)(x+1)}=0.$$

Hence, $$\frac{1}{2x-1}+\frac{1}{3x-1}=0.$$

Multiplying both sides by $(2x-1)(3x-1)$,

we have $$(3x-1)+(2x-1)=0.$$

Therefore $5x=2$, or, $x=\frac{2}{5}$.

Example 3. Solve $\dfrac{a-c}{2b+x}+\dfrac{b-c}{2a+x}=\dfrac{a+b-2c}{a+b+x}$.

We have
$$\begin{aligned}\frac{a-c}{2b+x}+\frac{b-c}{2a+x} &= \frac{(a-c)+(b-c)}{a+b+x}\\ &= \frac{a-c}{a+b+x}+\frac{b-c}{a+b+x}.\end{aligned}$$

Hence, by transposition,
$$(a-c)\left\{\frac{1}{2b+x}-\frac{1}{a+b+x}\right\} = (b-c)\left\{\frac{1}{a+b+x}-\frac{1}{2a+x}\right\}$$
$$\text{or, } (a-c).\frac{a-b}{(2b+x)(a+b+x)} = (b-c).\frac{a-b}{(a+b+x)(2a+x)}.$$

Hence,
$$\frac{a-c}{2b+x} = \frac{b-c}{2a+x};$$
$$\begin{aligned}\therefore\quad (a-c)(2a+x) &= (b-c)(2b+x),\\ \therefore\quad x\{(a-c)-(b-c)\} &= 2b(b-c)-2a(a-c)\\ \text{or,}\quad x(a-b) &= 2(b^2-a^2)-2c(b-a)\\ &= 2(b-a)(b+a-c)\\ &= 2(a-b)(c-a-b),\\ \therefore\quad x &= 2(c-a-b).\end{aligned}$$

Example 4. Solve $\dfrac{7x-55}{x-8}+\dfrac{2x-17}{x-9} = \dfrac{6x-71}{x-12}+\dfrac{3x-14}{x-5}$.

We have
$$\frac{7(x-8)+1}{x-8}+\frac{2(x-9)+1}{x-9} = \frac{6(x-12)+1}{x-12}+\frac{3(x-5)+1}{x-5}$$
$$\text{or, } \left(7+\frac{1}{x-8}\right)+\left(2+\frac{1}{x-9}\right)=\left(6+\frac{1}{x-12}\right)+\left(3+\frac{1}{x-5}\right);$$
$$\therefore\quad \frac{1}{x-8}+\frac{1}{x-9} = \frac{1}{x-12}+\frac{1}{x-5}.$$

Hence, by transposition,
$$\frac{1}{x-8}-\frac{1}{x-5} = \frac{1}{x-12}-\frac{1}{x-9},$$
$$\text{or, } \frac{3}{(x-8)(x-5)} = \frac{3}{(x-12)(x-9)}$$
$$\begin{aligned}\therefore\quad (x-8)(x-5) &= (x-12)(x-9)\\ \text{or,}\quad x^2-13x+40 &= x^2-21x+108,\end{aligned}$$
$$\therefore\ 8x = 68,\ \therefore\ x = 8\tfrac{1}{2}.$$

Example 5. Solve $(x-2a)^3+(x-2b)^3 = 2(x-a-b)^3$.

By transposition, we have

$$(x-2a)^3-(x-a-b)^3 = (x-a-b)^3-(x-2b)^3.$$

Putting X for $x-2a$, Y for $x-2b$, and Z for $x-a-b$,

we have $$X^3-Z^3 = Z^3-Y^3,$$

or, $$(X-Z)(X^2+XZ+Z^2) = (Z-Y)(Z^2+ZY+Y^2).$$

But $X-Z = Z-Y$, because each of them $= b-a$.

$$\therefore \quad X^2+XZ+Z^2 = Z^2+ZY+Y^2.$$

Hence, by transposition,

$$X^2-Y^2 = Z(Y-X).$$

Removing the common factor $X-Y$, which $= 2b-2a$,

we have $$X+Y = -Z,$$

i.e., $$(x-2a)+(x-2b) = -(x-a-b);$$

$$\therefore \quad 3x = 3(a+b), \text{ and } \quad \therefore \quad x = a+b.$$

Example 6. Solve

$$\left.\begin{array}{rl} 4x-3y+2z = 40 & \dots \quad (1) \\ 5x+9y-7z = 47 & \dots \quad (2) \\ 9x+8y-3z = 97 & \dots \quad (3) \end{array}\right\}$$

Multiplying (1) by 7, and (2) by 2, we have

$$\left.\begin{array}{rl} & 28x-21y+14z = 280 \\ \text{and} & 10x+18y-14z = 94 \end{array}\right\}$$

Hence, by addition, $38x-3y = 374$ ... (4)

Again, multiplying (1) by 3, and (3) by 2, we have

$$\left.\begin{array}{rl} & 12x-\ 9y+6z = 120 \\ \text{and} & 18x+16y-6z = 194 \end{array}\right\}$$

Hence, by addition, $30x+7y = 314$ ... (5)

Now, from (4) and (5), we have

$$\left.\begin{array}{rl} & 38x-3y-374 = 0 \\ \text{and} & 30x+7y-314 = 0 \end{array}\right\}$$

Hence, by cross multiplication,

$$\frac{x}{3\times 314-7(-374)} = \frac{y}{(-374).30-(-314).38}$$

$$= \frac{1}{38\times 7-30(-3)}$$

$$\text{or,}\quad \frac{x}{942+2618} = \frac{y}{-11220+11932} = \frac{1}{266+90}$$

$$\text{or,}\quad \frac{x}{3560} = \frac{y}{712} = \frac{1}{356}.$$

Therefore $x = 10$, and $y = 2$.

Substituting these values of x and y in (1), we have

$$40 - 6 + 2z = 40, \text{ whence } z = 3.$$

Thus we have $x = 10$, $y = 2$, $z = 3$.

Example 7. Solve

$$\left.\begin{aligned} x+y+z &= 0 \\ (b+c)x+(c+a)y+(a+b)z &= 0 \\ bcx+cay+abz &= 1 \end{aligned}\right\}$$

$$\left.\begin{aligned} \text{Since } (b+c)x+(c+a)y+(a+b)z &= 0 \\ \text{and}\qquad x+y+z &= 0 \end{aligned}\right\}$$

therefore, by cross multiplication,

$$\frac{x}{(c+a)-(a+b)} = \frac{y}{(a+b)-(b+c)} = \frac{z}{(b+c)-(c+a)}$$

$$\text{or,}\quad \frac{x}{c-b} = \frac{y}{a-c} = \frac{z}{b-a}.$$

Supposing each of these fractions $= k$, we have

$$x = k(c-b),\ y = k(a-c),\ z = k(b-a).$$

Substituting these values of x, y, z in the 3rd equation, we have $k\{bc(c-b)+ca(a-c)+ab(b-a)\} = 1$.

$$\begin{aligned} \text{But}\quad & bc(c-b)+ca(a-c)+ab(b-a) \\ &= bc(c-b)+a^2(c-b)-a(c^2-b^2) \\ &= (c-b)\{bc+a^2-a(c+b)\} \\ &= (c-b)(a-c)(a-b). \end{aligned}$$

Thus, $k(c-b)(a-c)(a-b) = 1$;

$$\therefore\ k = \frac{1}{(c-b)(a-c)(a-b)}.$$

$$\begin{aligned} \text{Hence,}\ x &= k(c-b) = \frac{1}{(a-c)(a-b)}; \\ y &= k(a-c) = \frac{1}{(c-b)(a-b)}; \\ z &= k(b-a) = \frac{1}{(c-b)(c-a)}. \end{aligned}$$

CHAPTER I.

SQUARE AND CUBE ROOTS.

1. Extraction of square roots by the application of the formula $a^2 \pm 2ab + b^2 = (a \pm b)^2$.

Example 1. Find the square root of

$$4 - 4c + 2b + c^2 - bc + \frac{b^2}{4}.$$

The given expression, arranged according to powers of b,

$$= \frac{b^2}{4} - b(c-2) + (c^2 - 4c + 4)$$

$$= \left(\frac{b}{2}\right)^2 - 2\left\{\frac{b}{2}\left(c-2\right)\right\} + \left(c-2\right)^2$$

$$= \left\{\frac{b}{2} - \left(c-2\right)\right\}^2 = \left(\frac{b}{2} - c + 2\right)^2.$$

Therefore the required root $= \frac{b}{2} - c + 2$.

Example 2. Extract the square root of

$$(ab + ac + bc)^2 - 4abc(a + c).$$

The given expression

$$= \{b(a+c) + ac\}^2 - 4abc(a+c)$$

$$= b^2(a+c)^2 + a^2c^2 - 2abc(a+c)$$

$$= \{b(a+c) - ac\}^2 = (ab - ac + bc)^2.$$

Therefore the required root $= ab - ac + bc$.

Example 3. Extract the square root of

$$a^4 + b^4 + c^4 + d^4 - 2(a^2 + c^2)(b^2 + d^2) + 2a^2c^2 + 2b^2d^2.$$

Arranging the given expression according to descending powers of a, we have

$$a^4 - 2a^2(b^2 + d^2 - c^2) + \{b^4 + c^4 + d^4 - 2c^2(b^2 + d^2) + 2b^2d^2\},$$

and the expression within the braces, arranged according to descending powers of b,

$$\begin{aligned} &= b^4 - 2b^2(c^2 - d^2) + (c^4 + d^4 - 2c^2d^2) \\ &= b^4 - 2b^2(c^2 - d^2) + (c^2 - d^2)^2 \\ &= \{b^2 - (c^2 - d^2)\}^2. \end{aligned}$$

Hence, the given expression

$$\begin{aligned} &= a^4 - 2a^2(b^2 - c^2 + d^2) + (b^2 - c^2 + d^2)^2 \\ &= \{a^2 - (b^2 - c^2 + d^2)\}^2 \\ &= (a^2 - b^2 + c^2 - d^2)^2. \end{aligned}$$

Therefore the required root $= a^2 - b^2 + c^2 - d^2$.

Example 4. Extract the square root of

$$\frac{(a^2 + b^2)^2}{a^4 + b^4 - 2a^2b^2} + 4\frac{a}{a+b} \times \frac{b}{a-b}.$$

The given expression

$$\begin{aligned} &= \frac{(a^2 + b^2)^2}{(a^2 - b^2)^2} + \frac{4ab}{a^2 - b^2} \\ &= \frac{(a^2 + b^2)^2 + 4ab(a^2 - b^2)}{(a^2 - b^2)^2}, \end{aligned}$$

of which the numerator

$$\begin{aligned} &= \{(a^2 - b^2)^2 + 4a^2b^2\} + 4ab(a^2 - b^2) \\ &= (a^2 - b^2)^2 + 4ab(a^2 - b^2) + 4a^2b^2 \\ &= \{(a^2 - b^2) + 2ab\}^2 ; \end{aligned}$$

$\therefore$ the given expression $= \dfrac{(a^2 + 2ab - b^2)^2}{(a^2 - b^2)^2}$.

Therefore the required root $= \dfrac{a^2 + 2ab - b^2}{a^2 - b^2}$.

Exercise (1).

Find the square root of :—

1. $49a^2x^4 - 42ab^2x^2 + 9b^4$.
2. $a^2 + b^2 + c^2 - 2ab + 2ac - 2bc$.
3. $4a^2 + b^2 + 9c^2 + 6bc - 12ac - 4ab$.
4. $a^4 + 4b^4 + 9c^4 + 4a^2b^2 - 6a^2c^2 - 12b^2c^2$.

5. $x^2+\frac{a^2}{9}-bx+\frac{b^2}{4}-\frac{ab}{3}+\frac{2ax}{3}$.

6. $\left(x+\frac{1}{x}\right)^2-4\left(x-\frac{1}{x}\right)$. 7. $x^4+\frac{1}{x^4}+2\left(x^2+\frac{1}{x^2}\right)+3$.

8. $\frac{x^2}{y^2}+\frac{y^2}{x^2}-\left(\frac{x}{y}+\frac{y}{x}\right)\sqrt{2}+2\frac{1}{2}$.

9. $a^2+b^2+c^2+d^2-2a(b-c+d)-2b(c-d)-2cd$.

10. $(a-b)^4-2(a^2+b^2)(a-b)^2+2(a^4+b^4)$.

11. $a^4+b^4+c^4+d^4-2a^2(b^2+d^2)-2b^2(c^2-d^2)$
$+2c^2(a^2-d^2)$.

12. $2a^2(b+c)^2+2b^2(c+a)^2+2c^2(a+b)^2+4abc(a+b+c)$.

2. The ordinary method of finding the square root of a compound algebraical expression.

From our previous knowledge of formulæ the following results are obvious :—

$$(a+b)^2 = a^2+(2a+b)b\ ;$$
$$(a+b+c)^2 = a^2+(2a+b)b+(2a+2b+c)c\ ;$$
$$(a+b+c+d)^2 = a^2+(2a+b)b+(2a+2b+c)c + (2a+2b+2c+d)d\ ;$$

and so on.

Clearly therefore we must have

$$(ax^2+bx+c)^2 = a^2x^4+(2ax^2+bx)bx+(2ax^2+2bx+c)c,$$

and this latter, when arranged according to descending powers of x, $=a^2x^4+2abx^3+(b^2+2ac)x^2+2bcx+c^2$.

Now, if it is proposed to find the square root of the above expression, let us see what means we have of discovering successively the several terms of the root :—

The first term of the root, *viz.*, ax^2, is evidently the square root of the first term of the given expression, which is a^2x^4 ;

if we subtract a^2x^4 from the given expression, the remainder is $\{(2ax^2+bx)bx+(2ax^2+2bx+c)c\}$, in which the term containing the highest power of x = $2ax^2\times bx$, *i. e.*, = twice the first term of the root *into* the second term ; this enables us to get the 2nd term after having obtained the first ;

if now from the above remainder we subtract $(2ax^2+bx)bx$, the second remainder is $(2ax^2+2ax+c)c$, in which the term containing the highest power of $x = 2ax^2 \times c$, *i.e.*, $=$ twice the first term of the root *into* the third ; this shows how to get the 3rd term after having obtained the 1st and the 2nd.

Thus we are furnished with a clue for successively discovering the terms of the expression ax^2+bx+c when its square is given.

The operation may be performed as follows :—

$$\begin{array}{rl}
 & a^2x^4+2abx^3+(b^2+2ac)x^2+2bcx+c^2\,(ax^2+bx+c \\
 & a^2x^4 \\ \hline
2ax^2+bx\,) & 2abx^3+(b^2+2ac)x^2+2bcx+c^2 \\
 & 2abx^3+b^2x^2 \\ \hline
2ax^2+2bx+c\,) & 2acx^2+2bcx+c^2 \\
 & 2acx^2+2bcx+c^2 \\ \hline
\end{array}$$

(1) Find the square root of a^2x^4, the first term of the proposed expression, and set it down as the first term of the required root ;

(2) Subtract a^2x^4 from the given expression and bring down the remainder $2abx^3+(b^2+2ac)x^2+2bcx+c^2$;

(3) Set down $2ax^2$, *i.e.*, twice the 1st term of the root, on the left of the above remainder as the first term of a divisor ;

(4) Divide the first term of the remainder by $2ax^2$, and set down the quotient, bx, as the second term of the root and also as the second term of the divisor ;

(5) Multiply the divisor thus obtained by the second term of the root, and subtract the product from the first remainder ;

(6) Bring down the second remainder $2acx^2+2bcx+c^2$ and put $2ax^2+2bx$(*i e.*, twice the sum of the two terms of the root already obtained) on the left of the remainder for the first two terms of a divisor ;

(7) Divide the first term of the new remainder by the first term of the new divisor, and set down the quotient, c, as the third term of the root and also as the third term of the divisor ;

(8) Multiply the complete divisor thus obtained by the third term of the root, and subtract the product from the second remainder.

After this nothing remains, and we obtain ax^2+bx+c for the required root.

NOTE. The expression considered above stands arranged according to descending powers of x. Similarly every expression of which the square root is sought must be arranged according to the descending or ascending order of the powers of some letter.

Example 1. Extract the square root of

$$\frac{x^4}{4y^4}+\frac{4y^4}{x^4}+\frac{x^2}{y^2}+\frac{4y^2}{x^2}+3.$$

The expression when arranged according to descending powers of x stands thus :—

$$\frac{x^4}{4y^4}+\frac{x^2}{y^2}+3+\frac{4y^2}{x^2}+\frac{4y^4}{x^4},$$

for, *now* the indices of the powers of x in the successive terms are respectively 4, 2, 0, -2 and -4, which numbers evidently are in descending order of magnitude. Hence we proceed as follows :—

$$\begin{array}{ll}
 & \frac{x^4}{4y^4}+\frac{x^2}{y^2}+3+\frac{4y^2}{x^2}+\frac{4y^4}{x^4}\Big(\frac{x^2}{2y^2}+1+\frac{2y^2}{x^2} \\
 & \frac{x^4}{4y^4} \\
\hline
\frac{x^2}{y^2}+1 & \Big)\frac{x^2}{y^2}+3 \\
 & \frac{x^2}{y^2}+1 \\
\hline
\frac{x^2}{y^2}+2+\frac{2y^2}{x^2} & \Big)\,2+\frac{4y^2}{x^2}+\frac{4y^4}{x^4} \\
 & 2+\frac{4y^2}{x^2}+\frac{4y^4}{x^4} \\
\hline
\end{array}$$

Thus the required root $=\frac{x^2}{2y^2}+1+\frac{2y^2}{x^2}$.

Example 2. Extract the square root of

$$x^{\frac{8}{5}}-2a^{-\frac{3}{5}}x^{\frac{11}{5}}+2a^{\frac{4}{5}}x^{\frac{4}{5}}+a^{-\frac{6}{5}}x^{\frac{14}{5}}-2a^{\frac{1}{5}}x^{\frac{7}{5}}+a^{\frac{8}{5}}.$$

Let us proceed by arranging the expression according to descending powers of x, thus:—

$$\begin{array}{l} a^{-\frac{6}{5}}x^{\frac{14}{5}}-2a^{-\frac{3}{5}}x^{\frac{11}{5}}+x^{\frac{8}{5}}-2a^{\frac{1}{5}}x^{\frac{7}{5}}+2a^{\frac{4}{5}}x^{\frac{4}{5}}+a^{\frac{8}{5}}\Big(a^{-\frac{3}{5}}x^{\frac{7}{5}}-x^{\frac{4}{5}}-a^{\frac{4}{5}} \\ a^{-\frac{6}{5}}x^{\frac{14}{5}} \\ \hline 2a^{-\frac{3}{5}}x^{\frac{7}{5}}-x^{\frac{4}{5}}\Big)\ -2a^{-\frac{3}{5}}x^{\frac{11}{5}}+x^{\frac{8}{5}} \\ \qquad\qquad\qquad\ -2a^{-\frac{3}{5}}x^{\frac{11}{5}}+x^{\frac{8}{5}} \\ \hline \qquad 2a^{-\frac{3}{5}}x^{\frac{7}{5}}-2x^{\frac{4}{5}}-a^{\frac{4}{5}}\Big)\ -2a^{\frac{1}{5}}x^{\frac{7}{5}}+2a^{\frac{4}{5}}x^{\frac{4}{5}}+a^{\frac{8}{5}} \\ \qquad\qquad\qquad\qquad\qquad\ -2a^{\frac{1}{5}}x^{\frac{7}{5}}+2a^{\frac{4}{5}}x^{\frac{4}{5}}+a^{\frac{8}{5}} \\ \hline \end{array}$$

Thus the required root $=a^{-\frac{3}{5}}x^{\frac{7}{5}}-x^{\frac{4}{5}}-a^{\frac{4}{5}}$.

Exercise (2).

Find the square root of :—

1. $x^4-4x^3+10x^2-12x+9$.
2. $x^6-2x^4+2x^3+x^2-2x+1$.
3. $4x^4+8ax^3+4a^2x^2+16b^2x^2+16ab^2x+16b^4$.
4. $\dfrac{1051x^2}{25}-\dfrac{6x}{5}-\dfrac{14x^3}{5}+49x^4+9$.
5. $\dfrac{x^4}{4}+4x^2+\dfrac{ax^2}{3}+\dfrac{a^2}{9}-2x^3-\dfrac{4ax}{3}$.
6. $\dfrac{x^2}{y^2}+\dfrac{y^2}{x^2}-\dfrac{x}{y}+\dfrac{y}{x}-1\dfrac{3}{4}$.
7. $25\dfrac{3}{7}-\dfrac{20x}{7y}+\dfrac{9y^2}{16x^2}-\dfrac{15y}{2x}+\dfrac{4x^2}{49y^2}$.
8. $x^{\frac{5}{3}}-4x^{\frac{4}{3}}+2x+4x^{\frac{2}{3}}+x^{\frac{1}{3}}$.

9. $a^2x^{-2} + 2ax^{-1} + a^{-2}x^2 + 3 + 2a^{-1}x.$

10. $x^{\frac{3}{2}} + xy^{-\frac{1}{2}} - 2x^{\frac{5}{4}}y^{-\frac{1}{4}} - 2x^{\frac{1}{2}}y^{\frac{1}{4}} + 2x^{\frac{3}{4}}y^{\frac{1}{2}} + y.$

11. $\frac{9x^3}{4} - 5x^{\frac{5}{2}}y^{\frac{1}{2}} + \frac{179x^2y}{45} - \frac{4x^{\frac{3}{2}}y^{\frac{3}{2}}}{3} + \frac{4xy^2}{25}.$

12. $x^4 + 2(y+z)x^3 + (3y^2 + 2yz + 3z^2)x^2$
$\qquad + 2(y^3 + y^2z + yz^2 + z^3)x + y^4 + 2y^2z^2 + z^4.$

3. Square roots of numbers.

Example 1. Given that the greatest integer whose square is contained in 140925 is 375, find the square root of 140925,16.

$$\begin{array}{ll} \text{Since} & 140925 > (375)^2 \\ & \quad\text{and} < (376)^2, \\ \therefore & 140925{,}16 > (3750)^2 \\ & \quad\text{and} < (3760)^2. \end{array}$$

Evidently therefore the required root must lie between 3750 and 3760; let it therefore be represented by $3750 + x$, where x is some integer less than 10.

Hence, we must have

$$\begin{aligned} 140925{,}16 &= (3750 + x)^2 \\ &= 14062500 + 7500x + x^2, \\ \text{or,}\quad 30016 &= (7500 + x)x. \end{aligned}$$

By trial we find that $x = 4$ satisfies this equation.

Hence, the required root $= 3750 + 4 = 3754$.

NOTE. The operation may be performed briefly, as in finding the square root of a compound algebraical expression as follows :—

$$\begin{array}{rr|l} & 14092516 & 3750+4 \\ (3750)^2 = & 14062500 & \\ \hline 7500+4 & 30016 & \\ & 30016 & \\ \hline \end{array}$$

or, more briefly thus ;—

$$\begin{array}{rr|l} & 14092516 & 3754 \\ (375)^2 = & 140625 & \\ \hline 7504 & 30016 & \\ & 30016 & \\ \hline \end{array}$$

It may be observed that the figure 4 of the root is the same as the quotient obtained by dividing 3001 by 750.

Cor. Hence we can easily find the square roots of the following numbers :—784, 2116, and 5625, for the greatest integers whose squares are respectively contained in 7, 21 and 56 are known.

Example 2. Given that the greatest integer whose square is contained in 65739 is 256, find the greatest integer whose square is contained in 65739,82.

$$\begin{aligned} \text{Since}\qquad 65739 &> (256)^2 \\ \text{and} &< (257)^2, \\ \therefore\qquad 65739{,}82 &> (2560)^2 \\ \text{and} &< (2570)^2. \end{aligned}$$

Evidently therefore the required number must lie between 2560 and 2570 ; let it therefore be represented by $2560+x$, where x is some integer less than 10.

Hence we must have

$$\begin{aligned} 65739{,}82 &> (2560+x)^2 \\ \text{but} &< \{2560+(x+1)\}^2 \\ \textit{i.e.},\qquad &> 6553600 + (5120+x)x \\ \text{but} &< 6553600 + \{5120+(x+1)\}(x+1), \\ \text{and}\ \therefore\ 20382 &> (5120+x)x \\ \text{but} &< \{5120+(x+1)\}(x+1). \end{aligned}$$

By trial we find that $x = 3$ satisfies these conditions.

Hence the required number $= 2560+3 = 2563$.

NOTE. The operation may be performed briefly as follows :—

$$\begin{array}{rl} & 65739{,}82\,\big(\,2560+3 \\ (2560)^2 = & 65536{,}00\,\big(\\ \hline 5120+3\,\big) & 203\ 82 \\ & 153\ 69 \\ \hline & 5013 \end{array}$$

or, more briefly thus :—

$$\begin{array}{rl} & 65739{,}82\,\big(\,2563 \\ (256)^2 = & 65536\,\big(\\ \hline 5123\,\big) & 20382 \\ & 15369 \\ \hline & 5013 \end{array}$$

Thus the process is just the same in this example as in the last, with this difference only that there is a remainder in this example whilst there was none in the last.

Here also it may be observed that the figure 3 obtained by trial is the same as the quotient of 2038 by 512.

Cor. We are thus furnished with a clue for determining the square root of any given number. For instance, the determination of the square root of 14175225 depends, as evident from example 1, upon finding the greatest integer whose square is contained in 141752 ;

and this latter, by the present example, depends upon finding the greatest integer whose square is contained in 1417 ;

which again, in like manner, depends upon finding the greatest integer whose square is contained in 14, which is known. Hence it is easy to see that the successive stages in arriving at the required root are :—

(1) To find the greatest integer whose square is contained in 14 ;

(2) Thence to find the greatest integer whose square is contained in 1417 ;

(3) Thence to find the greatest integer whose square is contained in 141752 ;

(4) Thence to find the required number.

Example 3. Find the square root of 3976185249.

Putting a comma before every second figure *from the right* we have

39,76,18,52,49.

Now, by the corollary to the last example the successive stages of the required operation will be as follows :—

(1) The greatest integer whose square is contained in 39 is 6.

(2) Hence let us find the greatest integer whose square is contained in 39,76 :—

$$\begin{array}{rrl}
 & 39,76 & (63 \\
6^2 = & 36 & \\
\hline
123) & 376 & \\
 & 369 & \\
\hline
 & 7 &
\end{array}$$

Thus the greatest integer whose square is contained in 3976 is 63.

(3) Hence let us find the greatest integer whose square is contained in 3976,18 :—

```
                 3976,18(630
      (63)² =    3969
               1260)7 18
                   )   0
                   7 18
```

Thus the greatest integer whose square is contained in 397618 is 630.

(4) Hence let us find the greatest integer whose square is contained in 397618,52 :—

```
              397618,52(6305
   (630)² =   396900
          12605)718 52
               )630 25
                 88 27
```

Thus the greatest integer whose square is contained in 39761852 is 6305.

(5) Hence let us find the square root of 39761852,49 :—

```
             39761852,49(63057
   (6305)² = 39753025
         126107)882749
               )882749
```

Thus the required root is 63057.

NOTE. 1. It may be observed that the remainder in (2), *viz.*, 7, is the same as the remainder left by subtracting $(63)^2$ from 3976 in (3) ; that the remainder in (3), *viz.*, 718, is the same as the remainder left by subtracting $(630)^2$ from 397618 in (4) ; and that the remainder in (4), *viz.*, 8827, is the same as the remainder left by subtracting $(6305)^2$ from 39761852 in (5). Hence the five stages of the process shewn above may be most conveniently combined into one as follows :—

```
           39,76,18,52,49(63057
           36
        123) 3 76
           ) 3 69
         1260)718
             )  0
        12605)71852
             )63025
       126107)882749
             )882749
```

Note. 2. The groups of figures separated by commas are called *periods* thus the given number in the present example has been divided into five periods altogether, namely, 39, 76, 18, 52 and 49. Hence the operation performed above amounts to the carrying out of the following directions :—

(1) Find the greatest integer whose square is contained in the first period, *viz.*, 39. This is found to be 6.

(2) Put down 6 as the first figure of the root and subtract its square from the first period.

(3) To the remainder, 3, bring down the second period, *viz.*, 76, thus getting 376.

(4) On the left of 376 place 12, *i.e.*, *twice* the first figure of the root.

(5) Divide 37, *i. e.*, the number obtained by omitting the last figure of 376, by 12. The quotient is found to be 3.

(6) Annex this 3 to 12, thus getting 123 on the left of 376 and also annex it to 6 as the second figure of the root.

(7) Multiply 123 by 3 and subtract the product from 376.

(8) To the remainder, 7, bring down the next period, *viz.*, 18, thus getting 718.

(9) On the left of 718 place 126, *i.e.*, *twice* the part of the root already found.

(10) Divide 71, *i. e.*, the number obtained by omitting the last figure of 718, by 126. The quotient is found to be zero.

(11) Annex this zero to 126, thus getting 1260 on the left of 718, and also annex it to 63 as the third figure of the root.

(12) Multiply 1260 by zero and subtract the product from 718.

(13) To the remainder, 718, bring down the next period, *viz.*, 52, thus getting 71852.

(14) On the left of 71852 place 1260, *i.e.*, *twice* the part of the root already found.

(15) Divide 7185, *i.e.*, the number obtained by omitting the last figure of 71852, by 1260. The quotient is found to be 5.

(16) Annex this 5 to 1260, thus getting 12605 on the left of 71852, and also annex it to 630 as the fourth figure of the root.

(17) Multiply 12605 by 5 and subtract the product from 71852.

(18) To the remainder, 8827 bring down the next period (which is also the last), *viz.*, 49, thus getting 882749.

(19) On the left of 882749 place 12610, *i.e.*, *twice* the part of the root already found.

(20) Divide 88274, *i.e.*, the number obtained by omitting the last figure of 882749, by 12610. The quotient is found to be 7.

(21) Annex this 7 to 12610, thus getting 126107 on the left of 882749, and also annex it to 6305 as the fifth figure of the root.

(22) Multiply 126107 by 7 and subtract the product from 882749.

(23) As no remainder is left the operation ceases and we obtain the required root = 63057.

Note 3. The first period may consist of one figure only, as in 8, 53, 64. It may also be observed with profit that the number of figures in the root is the same as the number of periods into which the given number is divided.

Example 4. Find the square root of 564·345, to three places of decimals.

We have $564{\cdot}345 = \dfrac{564345000}{10^6}$.

Hence $\sqrt{564{\cdot}345} = \dfrac{\sqrt{564345000}}{10^3}$.

Now let us find the square root of 564345000.

$$\begin{array}{r|l}
 & 5,64,34,50,00\,(23755 \\
 & 4 \\ \hline
43 & 1\ 64 \\
 & 1\ 29 \\ \hline
467 & 35\ 34 \\
 & 32\ 69 \\ \hline
4745 & 2\ 65\ 50 \\
 & 2\ 37\ 25 \\ \hline
47505 & 28\ 25\ 00 \\
 & 23\ 75\ 25 \\ \hline
 & 4\ 49\ 75
\end{array}$$

This shows that the square root can not be exactly found, and all that we have found is the greatest integer whose square is contained in the number.

Hence the required root $= \dfrac{23755}{10^3} = 23{\cdot}755$.

NOTE. 1. If we were asked to find the root to 4 decimal places we would have to multiply and divide the number by 10^8; if to 5 decimal places, by 10^{10}; and so on.

NOTE. 2. From the process shewn above it is clear that we might with much advantage proceed as follows:—

(1) Make the decimal places *six* in number by affixing three zeroes to the right of the number; thus, 564·345000.

(2) On the left of the decimal point put a comma *before* every second figure, and on the right of the point put a comma *after* every second figure, thus, 5,64·34,50,00. The whole number is thus divided into *five* periods altogether, of which two are on the left, and three on the right, of the decimal point.

(3) Now extract the square root of this number as if it were an integer and put the decimal point after the first two figures in the root thus found.

NOTE. 3. It should be observed with care that the number of figures on the left of the decimal point in the root is the same as the number of periods on the left of the decimal point in the proposed number. It should also be observed that if we were asked to find the root to *four* or *five* decimal places, we would have to make the decimal places respectively *eight* or *ten* in number by affixing zeroes.

Example 5. Find the square root of 3 to *five* places of decimals.

3 00,00,00,00,00(1·73205
1
27)200
189
343)1100
1029
3462)7100
6924
346405)1760000
1732025
27975

Thus the required root = 1·73205.

Exercise (3).

Find the square root of :—

1. 37249. **2.** 824464. **3.** 2819041

4. 22071204. **5.** 97535376. **6.** 550183936.

7. 28·8369. **8.** ·000729. **9.** 236·144689.

10. ·00139876.

Find to four places of decimals the square root of :—

11. 16·245. **12.** 2. **13.** ·00064. **14.** 5. **15.** 20.

4. If the square root of a number consists of $2n+1$ figures, the first $n+1$ of them being found by the ordinary method the remaining n may be obtained by division only.

[Let us take a particular case. Suppose we have to find the square root of 4556235397156. This number when divided into periods stands thus :—4,55,62,35,39,71,56. Hence evidently the square root will consist of 7 figures, and hence in this case $n = 3$.

If we proceed by the ordinary method the square root will be found to be 2134534.

We have to shew then that after the first four figures, 2134. have been obtained by the ordinary method, the remaining three, 534, may be obtained by a simple operation of division.

The given number being 4,55,62,35,39,71,56, by example 3 we know that 2134 is the greatest integer whose square is contained in 4,55,62,35. Hence the remainder at this stage of the process

$$= \left.\begin{matrix} 4556235 \\ -(2134)^2 \end{matrix}\right\} = \left.\begin{matrix} 4556235 \\ -4553956 \end{matrix}\right\} = 2279$$

to which the next period, *viz.*, 39, would be annexed if the ordinary method were continued.

The divisor at this stage = 2 × 2134 = 4268. We have to shew now that if instead of annexing to 2279 the next period only, *viz.*, 39, we annex to it all the remaining three periods and divide the number thus formed by the number formed by annexing *three* zeroes to 4268, *i.e.*, divide 2279397156 by 4268000, the quotient will be equal to 534 (the remaining three figures of the root).

By actual division we find this to be the case and we are inclined to accept the proposition as true.

It may be observed that

$$2279397156 = \left.\begin{matrix} 4556235397156 \\ -4553956000000 \end{matrix}\right\}$$
$$= 4556235397156 - (2134).^2 10^6.$$

Hence, it is clear that the remaining three figures of the root are given by the quotient of

$$4556235397156 - \{(2134).10^3\}^2 \text{ by } 2 \times (2134).10^3.$$

Now we shall give an algebraical proof of the proposition.]

Let N represent the given number, a the number formed by the first $n+1$ figures of the root (found by the ordinary method), and x the number formed by the remaining n figures of the root.

Then we must have

$$\sqrt{N} = a.10^n + x,$$
$$\therefore N = a^2.10^{2n} + 2a.10^n.x + x^2.$$
$$\therefore \frac{N - a^2.10^{2n}}{2a.10^n} = x + \frac{x^2}{2a.10^n}.$$

Now $N - a^2.10^{2n}$ is the number formed by annexing all the remaining periods of N to the remainder after the group of root-figures represented by a has been found.

Thus we find that if the above number be divided by $2a.10^n$, the quotient is equal to x together with $\dfrac{x^2}{2a.10^n}$.

Now since x consists of n digits. x^2 consists of $2n$ digits *at most*; also since $a.10^n$ consists of $2n+1$ digits, $2a.10^n$ consists of $2n+1$ digits, *at least*. Hence, $\dfrac{x^2}{2a.10^n}$ is a proper fraction and therefore it forms no part of the arithmetical quotient.

Thus it is clear that when $N - a^2\,10^{2n}$ is divided by $2a.10^n$, the arithmetical quotient is the remaining part of the root; which proves the proposition.

NOTE. The proposition evidently applies to integers which are perfect squares. It may also be shewn to be true (with some exceptions) for integers which are *not* perfect squares.

Suppose N is an integer which is not a perfect square and that it is divided into $2n+1$ periods. Suppose also that a represents the number formed by the first $n+1$ figures of its square root and that x represents the number formed by the next n figures. If now $N - a^2.10^{2n}$ be divided by $2a.10^n$ will x be the quotient?

Let N′ be the perfect square next below N. Then by the present article we must have

$$\frac{N' - a^2.10^{2n}}{2a.10^n} = x + \frac{x^2}{2a.10^n},$$

and ∴ we must have

$$\begin{aligned}\frac{N - a^2.10^{2n}}{2a.10^n} &= x + \frac{x^2}{2a.10^n} + \frac{N - N'}{2a.10^n}\\ &= x + \frac{x^3 + (N - N')}{2a.10^n}.\end{aligned}$$

Hence it is clear that our answer to the above query can be affirmative *only when* $x^2 + (N - N')$ is less than $2a.10^n$.

Example. Find the square root of 7 to six places of decimals.

The answer will evidently be found by extracting the square root of

7,00,00,00,00,00,00

and putting a decimal point after the first figure of the root.

Now the square root of the above number will evidently consist of 7 digits, and hence if four of them be found by the ordinary method the remaining three can be obtained by division.

```
 7,00,00,00,00,00,00(2645
 4
  ---------
46)300
   276
  ---------
524)2400
    2096
  ---------
5285)30400
     26425
  ---------
5290)3975
```

Now that we have found the first four figures of the root, the remaining three will be obtained by dividing 3975,00,00,00 by 5290,000, *i e*, by dividing 3975000 by 5290.

```
5290)3975000(751
     37030
     -----
      27200
      26450
      -----
        7500
        5290
        ----
        2210
```

Thus the square root of the above number = 2645751 and ∴ the required root = 2·645751

5. The ordinary method of finding the cube root of a compound algebraical expression.

Evidently we have $(ax^2+bx+c)^3$

$$= (ax^2+bx)^3 + 3(ax^2+bx)^2c + 3(ax^2+bx)c^2 + c^3$$
$$= a^3x^6 + 3(a^2x^4)(bx) + 3(ax^2)(bx)^2 + (bx)^3$$
$$+ 3(ax^2+bx)^2c + 3(ax^2+bx)c^2 + c^3.$$

Hence if we are asked to find the cube root of the above expression we see that we have the following means of discovering successively the several terms of the root :—

The first term of the root, *viz.*, ax^2 is evidently the cube root of the first term of the given expression, which is a^3x^6.

If we subtract a^3x^6 from the given expression the term containing the highest power of x in the remainder is $3(a^2x^4)(bx)$, *i.e.*, equal to three times the square of the first term of the root *into* the second term; the second term is therefore discovered.

If from the above remainder we now subtract $\{3(a^2x^4) + 3(ax^2)(bx) + (bx)^2\}(bx)$, the second remainder is $3(ax^2 + bx)^2c + 3(ax^2 + bx)c^2 + c^3$; the term containing the highest power of x in this remainder is $3a^2x^4c$, *i.e.*, equal to three times the square of the first term of the root *into* the third.

Hence the third term is discovered.

If from the second remainder we now subtract $\{3(ax^2 + bx)^2 + 3(ax^2 + bx)c + c^2\}c$, nothing is left and we obtain the required root $= ax^2 + bx + c$.

Let us illustrate the process by an example.

Example. Find the cube root of

$$x^6 - 6x^5y + 24x^4y^2 - 56x^3y^3 + 96x^2y^4 - 96xy^5 + 64y^6.$$

The given expression stands arranged according to descending powers of x; we need not therefore change the order of the terms.

The second term of the root, *viz.*, $-2xy$, as shewn on the next page, is obtained by dividing $-6x^5y$, by $3x^4$ (*i.e.*, three times the square of the first term).

Then the divisor, $3x^4 - 6x^3y + 4x^2y^2$, is formed as shewn on the next page.

The product of this divisor by $(-2xy)$, *viz.*, $-6x^5y + 12x^4y^2 - 8x^3y^3$, is now subtracted from the expression which stands above it, and the remainder is put down below the line.

Now take three times the square of the part of the root already obtained and put down the result, $3x^4 - 12x^3y + 12x^2y^2$, as part of a divisor.

The third term of the root, *viz*, $4y^2$, is obtained by dividing $12x^4y^2$, the first term of the remainder, by $3x^4$, the first term of the divisor.

The complete divisor is then formed as shewn on the next page, and the product of this divisor by the third term of the root is subtracted from the expression which stands above it.

As no remainder is now left we find the required root $^2 - 2xy + 4y^2$.

$$x^6 - 6x^5y + 24x^4y^2 - 56x^3y^3 + 96x^2y^4 - 96xy^5 + 64y^6 \,\big(\, x^2 - 2xy + 4y^2$$

$$x^6$$

$$3 \times (x^2)^2 = 3x^4 \qquad\qquad - 6x^5y + 24x^4y^2 - 56x^3y^3 + 96x^2y^4 - 96xy^5 + 64y^6$$

$$3 \times x^2 \times (-2xy) = \quad - 6x^3y$$

$$(-2xy)^2 = \quad + 4x^2y^2$$

$$3x^4 - 6x^3y + 4x^2y^2 \qquad - 6x^5y + 12x^4y^2 - 8x^3y^3$$

$$3 \times (x^2 - 2xy)^2 = 3x^4 - 12x^3y + 12x^2y^2 \qquad 12x^4y^2 - 48x^3y^3 + 96x^2y^4 - 96xy^5 + 64y^6$$

$$3 \times (x^2 - 2xy) \times (4y^2) = \quad + 12x^2y^2 - 24xy^3$$

$$(4y^2)^2 = \quad + 16y^4$$

$$3x^4 - 12x^3y + 24x^2y^2 - 24xy^3 + 16y^4 \qquad 12x^4y^2 - 48x^3y^3 + 96x^2y^4 - 96xy^5 + 64y^6$$

Exercise (4).

Find the cube root of :—

1. $x^3+27x^2+243x+729$.
2. $27x^3-216x^2+576x-512$.
3. $64a^3-144a^2b+108ab^2-27b^3$.
4. $33x^4-36x+x^6-63x^3+8-9x^5+66x^2$.
5. $8x^6+12x^5-30x^4-35x^3+45x^2+27x-27$.
6. $1-9x^2+33x^4-63x^6+66x^8-36x^{10}+8x^{12}$.
7. $c^6-63c^3x^3+8x^6-9c^5x+66c^2x^4-36cx^5+33c^4x^2$.

6. Cube Roots of Numbers.

Example 1. Given that the greatest number whose cube is contained in 9718142 is 213, find the cube root of 9718142,104.

$$\begin{aligned} \text{Since } 9718142 &> (213)^3 \\ \text{and } &< (214)^3 \\ \therefore\ 9718142,104 &> (2130)^3 \\ \text{and } &< (2140)^3. \end{aligned}$$

Evidently therefore the required root must lie between 2130 and 2140 ; let it therefore be represented by $2130+x$, where x is some integer less than 10.

Hence, we must have

$$\begin{aligned} 9718142,104 &= (2130+x)^3 \\ &= (2130)^3+3.(2130)^2x+3.(2130)x^2+x^3 \\ &= 9663597000+3\times 4536900x \\ &\qquad +3\times 2130x^2+x^3, \\ \text{or,}\quad 54545104 &= (3\times 4536900+3\times 2130x+x^2)x. \end{aligned}$$

Dividing 54545104 by 3×4536900 we get 4 as the quotient, and we also find that $x=4$ satisfies the above relation.

Hence we have the required root = 2134. The process may be shewn as follows :—

$$\begin{array}{rcr|l} & & & 9718142,104\,(213+4 \\ & & (2130)^3 = & 9663597\,000 \\ \hline 3\times(2130)^2 & = & 13610700 & 54545\,104 \\ 3\times 2130\times 4 & = & 25560 & \\ 4^2 & = & 16 & \\ \cline{3-3} & & 13636276 & \\ & & 4 & \\ \cline{3-3} & & 54545104 & 54545\,104 \\ \hline \end{array}$$

Or, more briefly thus :—

$$
\begin{array}{rcr|l}
 & & 9718142,104 & \!\!\big(\,2134 \\
(213)^3 & = & 9663597 & \\
\hline
3\times(2130)^2 & = & 13610700 & 54545104 \\
3\times2130\times4 & = & 25560 & \\
4^2 & = & 16 & \\
\cline{3-3}
 & & 13636276 & \\
 & & 4 & \\
\cline{3-3}
 & & 54545104 & 54545104 \\
\end{array}
$$

NOTE. 1. It should be carefully remembered that the figure 4 of the root is found by dividing 54545104 by 13610700. Sometimes the figure so found may prove too large and then we have to try the next smaller figure.

NOTE. 2. By a method similar to that of example 2, article 3, we can easily shew that when the greatest integer whose cube is contained in a given number is known, we can determine the greatest integer whose cube is contained in a number formed by annexing three more digits to the given number ; *i e.*, when the greatest integer whose cube is contained in a number like 34567 is known, we can find the greatest integer whose cube is contained in 34567,432.

Hence we have the following means of determining the cube root of any number, say 3278975416.

(1) Divide the number into a number of periods by placing a comma before every *third* figure from the right, thus :—3,278,975,416.

(2) Find the greatest number whose cube is contained in the first period, namely 3.

(3) Hence find the greatest integer whose cube is contained in 3,278.

(4) Hence find the greatest integer whose cube is contained in 3,278,975.

(5) Hence find the greatest integer whose cube is contained in 3,278,975,416. If the number be a perfect cube this integer will be its cube root and no remainder will be left after the operation.

Example 2. Find the cube root of 68417929.

Putting a comma before every third figure *from the right* we have 68,417,929.

Then, by Note 2, last example, the successive stages of the required operation will be as follows :—

(1) The greatest number whose cube is contained in the first period, namely 68, is 4.

(2) Hence let us find the greatest number whose cube is contained in 68,417 :—

$$\begin{array}{rrl|l}
 & & 68{,}417 & (40 \\
 & 4^3 = & 64 & \\
\hline
3\times(40)^2 & = & 4800 & 4417 \\
3\times 40\times 0 & = & 0 & \\
0^2 & = & 0 & \\
\hline
 & & 4800 & \\
 & & 0 & \\
\hline
 & & 0 & 0 \\
\hline
 & & & 4417
\end{array}$$

Since 4417 is not divisible by 4800, the second figure of the root must be zero.

Thus the greatest integer whose cube is contained in 68,417 is 40.

(3) Hence let us find the cube root of 68,417,929.

$$\begin{array}{rrl|l}
 & & 68{,}417{,}929 & (409 \\
 & (40)^3 = & 64\,000 & \\
\hline
3\times(400)^2 & = & 480000 & 4417929 \\
3\times 400\times 9 & = & 10800 & \\
9^2 & = & 81 & \\
\hline
 & & 490881 & \\
 & & 9 & \\
\hline
 & & 4417929 & 4417929 \\
\hline
\end{array}$$

Dividing 4417929 by 480000 we have 9 for the quotient, and so 9 is the next figure of the root.

Thus the required root $= 409$.

NOTE. 1. Since the remainder in (2), namely 4417, is the same as the remainder left when $(40)^3$ is subtracted from 68,417 in (3), it is clear that the three different stages might very well be combined into one as in example 3, article 3.

NOTE. 2. The number of figures in the cube root is the same as the number of periods into which the given number is divided.

Example 3. Find the cube root of 5687432·75 to two places of decimals.

$$\text{We have } 5687432{\cdot}75 = \frac{5687432750000}{(100)^3}.$$

$$\text{Hence } \sqrt[3]{5687432{\cdot}75} = \frac{\sqrt[3]{5687432750000}}{100}.$$

Now let us find the cube root of 5687432750000 or rather the greatest integer whose cube is contained in it.

$$
\begin{array}{rr|l}
 & & 5,687,432,750,000\,(17850 \\
1^3 = & 1 & \\
\hline
 & & 4687 \\
3\times(10)^2 = & 300 & \\
3\times 10\times 7 = & 210 & \\
7^2 = & 49 & \\
\cline{2-2}
 & 559 & \\
 & 7 & \\
\cline{2-2}
 & 3913 & 3913 \\
\hline
3\times(170)^2 = & 86700 & 774432 \\
3\times 170\times 8 = & 4080 & \\
8^2 = & 64 & \\
\cline{2-2}
 & 90844 & \\
 & 8 & \\
\cline{2-2}
 & 726752 & 726752 \\
\hline
3\times(1780)^2 = & 9505200 & 47680750 \\
3\times 1780\times 5 = & 26700 & \\
5^2 = & 25 & \\
\cline{2-2}
 & 9531925 & \\
 & 5 & \\
\cline{2-2}
 & 47659625 & 47659625 \\
\cline{3-3}
3\times(17850)^2 = & 955867500 & 21125000
\end{array}
$$

Dividing 4687 by 300 we get 15, which of course we reject since the next figure of the root cannot be greater than 9. Trying successively 9 and 8 we ultimately arrive at numbers too large to be subtracted from 4687; so we next try 7 and succeed.

As 21125000 is not divisible by 955867500 we must take zero for the next figure of the root and stop here.

Thus the greatest number whose cube is contained in 5,687,432,750,000 is 17850; and therefore the required root = 178·50.

NOTE. 1. If we were asked to find the cube root to three places of decimals we would have to multiply and divide the given number by $(1000)^3$, and if to four places of decimals, by $(10000)^3$ or $(10^4)^3$, and so on.

NOTE. 2. From the process shown above it is clear that we might with much advantage proceed as follows:—

(1) Since the root is to be found to *two* places of decimals, make the decimal places *six* in number by affixing four zeroes to the right of the proposed number, thus :—5687432·750000.

(2) On the left of the decimal point put a comma *before* every third figure, and on the right of the point put a comma *after* every third figure, thus :—5,687,432·750,000. The whole number is thus divided into five periods altogether of which three are on the left, and two on the right, of the decimal point.

(3) Now extract the cube root of this number as if it were an integer. and put the decimal point after the first three figures in the root thus found.

NOTE. 3. It should be observed with care that the number of figures on the left of the decimal point in the root is the same as the number of periods on the left of the decimal point in the proposed number. It should also be observed that if we were asked to find the root to *three* or *four* places of decimals we would have to make the decimal places in the proposed number respectively *nine* or *twelve* in number by affixing zeroes.

Exercise (5).

Find the cube root of :—

1. 15625.
2. 110592.
3. 941192.
4. 8365427.
5. 28934443.
6. 95443993.
7. 194104539.
8. 223648543.
9. 843908625.
10. 673373097125.
11. 32·461759.
12. 27054·036008.

Find to three places of decimals the cube root of :—

13. 44.6.
14. 3·00415.
15. 576.

7. If the cube root of a number consists of $2n+2$ figures, the first $n+2$ of them being found by the ordinary method the remaining n may be obtained by division only.

Let N represent the given number, a the number formed by the first $n+2$ figures of the root (found by the ordinary method), and x the number formed by the remaining n figures of the root

Then we must have

$$\sqrt[3]{N} = a.10^n + x,$$

$$\therefore\quad N = a^3.10^{3n} + 3a^2.10^{2n}.x + 3a.10^n.x^2 + x^3\,;$$

$$\therefore\quad \frac{N - a^3.10^{3n}}{3a^2.10^{2n}} = x + \frac{x^2}{a.10^n} + \frac{x^3}{3a^2.10^{2n}}.$$

Now, $N - a^3.10^{3n}$ is the number formed by annexing all the remaining periods of N to the remainder after the group of root-figures represented by a has been found.

Thus we find that if the above number be divided by $3a^2.10^{2n}$, the quotient is equal to x together with $\dfrac{x^2}{a.10^n} + \dfrac{x^3}{3a^2.10^{2n}}$.

Now, since x consists of n digits it is less than 10^n and $\therefore x^2 < 10^{2n}$; also since a consists of $n+2$ figures it is greater than 10^{n+1} and $\therefore a.10^n > 10^{2n+1}$; hence

$$\frac{x^2}{a.10^n} < \frac{10^{2n}}{10^{2n+1}}, \quad i.e., < \frac{1}{10},$$

and also $$\frac{x^3}{3a^2.10^{2n}} < \frac{10^{3n}}{3 \times 10^{4n+2}}, \quad i.e., < \frac{1}{3 \times 10^{n+2}}.$$

Hence, $\dfrac{x^2}{a.10^n} + \dfrac{x^3}{3a^2.10^{2n}}$ is clearly a proper fraction and

$\therefore$ it forms no part of the arithmetical quotient.

Thus when $N - a^3.10^{3n}$ is divided by $3a^2.10^{2n}$, the arithmetical quotient is x, the remaining part of the root; which proves the proposition.

NOTE. Remarks similar to those made in the note to article 4 apply to the present article.

Miscellaneous examples.

Example 1. If $x^6 + 3dx^5 + ex^4 + fx^3 + gx^2 + hx + k^3$ be a perfect cube, find its cube root and determine the co-efficients e, f, g, h in terms of d and k.

(Bombay University P. E. Paper, 1889.)

The given expression stands arranged according to descending powers of x.

Hence the first and last terms of the root must be x^2 and k, the cube roots of the first and last terms of the given expression.

Also as the second term of the root is to be found by dividing the second term of the given expression by three times the square of the first term of the root, $\therefore$ the second term of the root must be $= 3dx^5 \div 3x^4 = dx$.

Thus we have

$$\left.\begin{array}{ll}\text{the 1st term of the root} & = x^2\\ \quad\text{the 2nd term} & = dx\\ \quad\text{and the \textit{last} term} & = k,\end{array}\right\}\begin{array}{l}\text{and evidently there cannot be any}\\ \text{more term of the root between } dx\\ \text{and } k.\end{array}$$

Hence, the required root $= x^2+dx+k$.

Therefore we must have

$$\begin{aligned} & x^6+3dx^5+ex^4+fx^3+gx^2+hx+k^3 \\ \textit{identically} &= (x^2+dx+k)^3 \\ &= x^6+3x^4(dx+k)+3x^2(dx+k)^2+(dx+k)^3 \\ &= x^6+3dx^5+3(k+d^2)x^4+d(6k+d^2)x^3 \\ &\qquad +3k(k+d^2)x^2+3dk^2x+k^3. \end{aligned}$$

Hence, equating the co-efficients of like powers of x on both sides, we have

$$e = 3(k+d^2), \qquad f = d(6k+d^2),$$
$$g = 3k(k+d^2), \qquad h = 3dk^2.$$

Example 2. If a be the greatest integer contained in $N^{\frac{1}{3}}$, and the difference be so small that its cube may be neglected, prove that a nearer approximate value of $N^{\frac{1}{3}}$ will be

$$\frac{1}{2}\left\{a+\left(\frac{4N-a^3}{3a}\right)^{\frac{1}{2}}\right\}$$

(Bombay University P. E. Paper, 1885.)

Let $N^{\frac{1}{3}} = a+x$, where x is, by hypothesis, so small that its cube may be neglected.

Then we have

$$N = a^3+3a^2x+3ax^2+x^3.$$

Hence, neglecting x^3, we have the following equation from which the value of x can be determined approximately :—

$$3ax^2+3a^2x+a^3 = N.$$

$$\text{Hence,} \qquad x^2+ax = \frac{N-a^3}{3a};$$

$$\text{or,} \qquad x^2+ax+\frac{a^2}{4} = \frac{N-a^3}{3a}+\frac{a^2}{4};$$

$$\text{or,} \qquad \left(x+\frac{a}{2}\right)^2 = \frac{4N-a^3}{4.3a};$$

$$\therefore \quad x+\frac{a}{2} = \frac{1}{2}\left(\frac{4N-a^3}{3a}\right)^{\frac{1}{2}};$$

[The negative sign is rejected since x is, by hypothesis, positive.]

$$\therefore \quad x+a = \frac{a}{2}+\frac{1}{2}.\left(\frac{4N-a^3}{3a}\right)^{\frac{1}{2}}.$$

Thus the approximate value of $a+x$

$$= \frac{1}{2}\left\{a + \left(\frac{4N - a^3}{3a}\right)^{\frac{1}{2}}\right\};$$

and this is therefore the required value of $N^{\frac{1}{3}}$.

CHAPTER II.

INDICES.

1. Definition. The product of m factors each equal to a is represented by a^m. [See Art. 11, page 9.]

Thus the meaning of a^m is clear when m is a *positive integer.*

2. The Index Law and the truths necessarily following from it.

To prove that $a^m \times a^n = a^{m+n}$, where m and n are any two positive integers.

$$\begin{aligned}
\text{Since}\quad a^m &= a.\ a.\ a.\ \ldots\ \ldots\ \ldots\ \ldots\ \text{to } m \text{ factors}\\
\text{and}\quad a^n &= a.\ a.\ a.\ a.\ \ldots\ \ldots\ \ldots\ \text{to } n \text{ factors,}\\
\therefore\quad a^m \times a^n &= (a.\ a.\ a.\ \ldots\ \ldots\ \ldots\ \ldots\ \text{to } m \text{ factors})\\
&\quad \times (a.\ a.\ a.\ a.\ \ldots\ \ldots\ \ldots\ \text{to } n \text{ factors})\\
&= a.\ a.\ a.\ a.\ a.\ a.\ a.\ \ldots\ \ldots\ \text{to } (m+n) \text{ factors}\\
&= a^{m+n}.
\end{aligned}$$

This result is called the *Index Law.*

Cor. 1. $a^m \times a^n \times a^p = a^{m+n+p}$, when m, n and p are positive integers.

For $a^m \times a^n = a^{m+n}$, $\therefore$ $a^m \times a^n \times a^p = a^{m+n} \times a^p = a^{(m+n)+p}$ $= a^{m+n+p}$.

Hence, $a^m \times a^n \times a^p \times a^q \cdots\cdots = a^{m+n+p+q+\cdots\cdots}$.

Thus, *the product of any number of powers of a given quantity is that power of the quantity whose index is equal to the sum of the indices of the factors.*

Cor. 2. $(a^m)^n = a^{mn}$, when m and n are any two positive integers.

For $(a^m)^n = a^m \times a^m \times a^m \times \ldots$ to n factors

$= a^{m+m+m+\ldots\ldots}$ to n terms [by Cor. 1.]

and $\therefore \quad = a^{mn}$.

Cor. 3. $a^m \div a^n = a^{m-n}$, when m and n are positive integers and m is greater than n.

For $a^{m-n} \times a^n = a^{(m-n)+n}$ [because $m-n$ is a positive integer.]

$= a^m$,

$\therefore \quad a^m \div a^n = a^{m-n}$.

3. Assuming the formula $a^m \times a^n = a^{m+n}$ to be true for *all* values of m and n, to find meanings for quantities with fractional or negative indices.

(i) To find the meaning of $a^{\frac{p}{q}}$, when p and q are any two positive integers.

Since $a^m \times a^n = a^{m+n}$ for *all* values of m and n, putting $\frac{p}{q}$ for each of them, we have

$$a^{\frac{p}{q}} \times a^{\frac{p}{q}} = a^{\frac{p}{q}+\frac{p}{q}} = a^{\frac{2p}{q}}.$$

Similarly, $a^{\frac{p}{q}} \times a^{\frac{p}{q}} \times a^{\frac{p}{q}} = a^{\frac{2p}{q}} \times a^{\frac{p}{q}} = a^{\frac{3p}{q}}$, and so on.

Hence, $a^{\frac{p}{q}} \times a^{\frac{p}{q}} \times a^{\frac{p}{q}} \ldots\ldots\ldots$ to q factors

$$= a^{\frac{qp}{q}} = a^p.$$

Thus $a^{\frac{p}{q}}$ is equal to the q^{th} root of a^p, and is therefore equivalent to $\sqrt[q]{a^p}$.

Cor. Hence $a^{\frac{1}{2}} = \sqrt{a}$, $a^{\frac{1}{3}} = \sqrt[3]{a}$, $a^{\frac{1}{4}} = \sqrt[4]{a}$, and so on.

Generally, $a^{\frac{1}{n}} = \sqrt[n]{a}$.

NOTE. From the Index Law it is also easy to see that $a^{\frac{1}{q}} \times a^{\frac{1}{q}} \times a^{\frac{1}{q}} \times \ldots$ to p factors $= a^{\frac{p}{q}}$. Thus $a^{\frac{p}{q}}$ may as well be regarded as the p^{th} power of $a^{\frac{1}{q}}$, *i.e.*, equivalent to $(\sqrt[q]{a})^p$. Thus $a^{\frac{p}{q}}$ may be interpreted either as the q^{th} root of the p^{th} power of a, or as the p^{th} power of the q^{th} root of a.

(ii) To find the meaning of a^0.

Since $a^m \times a^n = a^{m+n}$ is true for *all* values of m and n, putting $m = 0$ we have

$$a^0 \times a^n = a^{0+n} = a^n;$$

$$\therefore \quad a^0 = a^n \div a^n = 1.$$

Thus any quantity raised to the power zero is equivalent to 1.

(iii) To find the meaning of a^{-n}, where n is any positive integer.

Since $a^m \times a^n = a^{m+n}$ is true for *all* values of m and n, putting $m = -n$, we have

$$a^{-n} \times a^n = a^{-n+n} = a^0 = 1;$$

$$\therefore \quad a^{-n} = \frac{1}{a^n}, \text{ and } a^n = \frac{1}{a^{-n}}.$$

Cor. Hence $a^m \div a^n = a^{m-n}$ for *all* values of m and n.

For $a^m \div a^n = \frac{a^m}{a^n} = a^m \times a^{-n} = a^{m-n}$.

Example 1. Find the value of $8^{\frac{5}{3}}$.

$$8^{\frac{5}{3}} = \left(\sqrt[3]{8}\right)^5 = 2^5 = 32.$$

Example 2. Find the value of $4^{-\frac{5}{2}}$.

$$4^{-\frac{5}{2}} = \frac{1}{4^{\frac{5}{2}}} = \frac{1}{(\sqrt{4})^5} = \frac{1}{2^5} = \frac{1}{32}.$$

Example 3. Multiply together $\sqrt{a^5}$, $a^{\frac{3}{4}}$, $\sqrt[4]{a^{-5}}$ and $\frac{1}{a^{-3}}$.

$$\begin{aligned} \text{The required product} &= a^{\frac{5}{2}} \times a^{\frac{3}{4}} \times a^{-\frac{5}{4}} \times a^3 \\ &= a^{\frac{5}{2}+\frac{3}{4}-\frac{5}{4}+3} \\ &= a^{\frac{5}{2}-\frac{1}{2}+3} = a^{2+3} = a^5. \end{aligned}$$

Exercise (6)

Express the following avoiding fractional or negative indices :—

1. $a^{\frac{5}{7}}$.
2. $x^{-\frac{3}{2}}$.
3. $\dfrac{3}{x^{-\frac{4}{5}}}$.
4. $x^{-\frac{2}{5}} \times 3a^{-\frac{1}{2}}$.
5. $8m^{-2} \times m^{-\frac{2}{3}}$.
6. $x^{-\frac{4}{5}} \div 3a^{-\frac{5}{4}}$.
7. $x^{-\frac{2}{3}} \div 2x^{-\frac{1}{2}}$.
8. $\sqrt[5]{x^2} \div \sqrt[5]{x^{-a}}$.
9. $\sqrt[2m]{a^{-5}} \times \sqrt[m]{a^8}$.
10. $\sqrt[4a]{x^6} \div \sqrt[2a]{x^{-5}}$.

Express the following avoiding radical signs and negative indices :—

11. $\left(\sqrt[3]{x}\right)^7$.
12. $\left(\sqrt[4]{a}\right)^{-6}$.
13. $\dfrac{1}{\sqrt[3]{x^{-2}}}$.
14. $\dfrac{1}{(\sqrt[5]{a})^{-2}}$.
15. $\sqrt[3]{x^4} \div \left(\sqrt[6]{x}\right)^{-1}$.
16. $\sqrt[4]{a^{-3}} \div \left(\sqrt[8]{a}\right)^{-12}$.

Find the value of :—

17. $4^{-\frac{3}{2}}$.
18. $8^{\frac{2}{3}}$.
19. $9^{\frac{3}{2}}$.
20. $16^{\frac{5}{4}}$.
21. $81^{-\frac{3}{4}}$.
22. $\dfrac{1}{6^{-2}}$.
23. $(125)^{-\frac{2}{3}}$.
24. $\left(\dfrac{1}{27}\right)^{-\frac{4}{3}}$.
25. $\left(\dfrac{1}{216}\right)^{-\frac{2}{3}}$.
26. Simplify $\dfrac{x^{m+2n}x^{3m-8n}}{x^{5m-6n}}$.

4. To prove that $(a^m)^n = a^{mn}$ is true for *all* values of m and n.

(i) Let n be a *positive integer*. Then, whatever may be the value of m, we have

$$(a^m)^n = a^m \times a^m \times a^m \times \ldots\ldots \text{ to } n \text{ factors}$$
$$= a^{m+m+m+\ldots\ldots} \text{ to } n \text{ terms}$$
$$= a^{mn}.$$

(ii) Let n be a *positive* fraction equal to $\frac{p}{q}$, where p and q are positive integers. Then we have

$$\left(a^m\right)^n = \left(a^m\right)^{\frac{p}{q}} = \sqrt[q]{\left(a^m\right)^p} \qquad \text{[Art. 3, (i)]}$$

$$= \sqrt[q]{a^{mp}} \qquad \text{[by (i)]}$$

$$= a^{\frac{mp}{q}} \qquad \text{[Art. 3, (i)]}$$

$$= a^{mn}.$$

(iii) Let n be *any negative* quantity, equal to $-p$, where p is *positive*. Then we have

$$(a^m)^n = (a^m)^{-p} = \frac{1}{(a^m)^p} \qquad \text{[Art. 3, (iii)]}$$

$$= \frac{1}{a^{mp}} \qquad \text{[by (i) and (ii)]}$$

$$= a^{-mp} \qquad \text{[Art. 3, (iii)]}$$

$$= a^{m(-p)} = a^{mn}.$$

Thus the proposition is established.

5. To prove that $a^n b^n = (ab)^n$ for *all* values of n.

(i) Let n be a positive integer. Then we have

$$\begin{aligned} a^n b^n &= (a.\ a.\ a.\ \ldots\ \ldots \text{ to } n \text{ factors}) \\ &\quad \times (b.\ b.\ b.\ \ldots \text{ to } n \text{ factors}) \\ &= (ab.\ ab.\ ab.\ \ldots \text{ to } n \text{ factors}) \\ &= (ab)^n. \end{aligned}$$

(ii) Let n be a *positive fraction* equal to $\frac{p}{q}$, where p and q are positive integers. Then putting x for $a^n b^n$ we have

$$x = a^{\frac{p}{q}} b^{\frac{p}{q}},$$

$$\therefore\ x^q = \left(a^{\frac{p}{q}} b^{\frac{p}{q}}\right)^q$$

$$= \left(a^{\frac{p}{q}}\right)^q \times \left(b^{\frac{p}{q}}\right)^q \qquad \text{[by (i)]}$$

$$= a^p \times b^p \qquad [\text{Art. } 4]$$
$$= (ab)^p ; \qquad [\text{by (i)}]$$

$$\therefore \quad x = (ab)^{\frac{p}{q}} ; \ i.e., \ a^n b^n = (ab)^n.$$

(iii) Let n be *any negative* quantity, equal to $-p$, where p is positive. Then we have

$$a^n b^n = a^{-p} b^{-p}$$
$$= \frac{1}{a^p b^p} \qquad [\text{Art. 3, (iii)}]$$
$$= \frac{1}{(ab)^p} \qquad [\text{by (i) and (iii)}]$$
$$= (ab)^{-p} \qquad [\text{Art. 3, (iii)}]$$
$$= (ab)^n.$$

Thus the proposition is established.

Cor. 1. $\frac{a^n}{b^n} = a^n b^{-n} = a^n . \left(b^{-1}\right)^n = \left(ab^{-1}\right)^n = \left(\frac{a}{b}\right)^n.$

Cor. 2. $a^n b^n c^n = (ab)^n c^n = (abc)^n ;$
generally, $a^n b^n c^n d^n \ldots . = (abcd \ldots .)^n.$

6. Applications of the results proved in the last two articles.

Example 1. Simplify $(a^8 b^{\frac{5}{3}})^{-\frac{3}{4}}$.

$$(a^8 b^{\frac{5}{3}})^{-\frac{3}{4}} = (a^8)^{-\frac{3}{4}} \times (b^{\frac{5}{3}})^{-\frac{3}{4}}$$
$$= a^{8.(-\frac{3}{4})} \times b^{\frac{5}{3}.(-\frac{3}{4})}$$
$$= a^{-6} b^{-\frac{5}{4}}.$$

Example 2. Simplify $\sqrt{a^{-2}b} \times \sqrt[3]{ab^{-3}}$.

$$\sqrt{a^{-2}b} = (a^{-2}b)^{\frac{1}{2}} = (a^{-2})^{\frac{1}{2}} \times b^{\frac{1}{2}} = a^{-1}b^{\frac{1}{2}} ;$$

$$\sqrt[3]{ab^{-3}} = (ab^{-3})^{\frac{1}{3}} = a^{\frac{1}{3}} \times (b^{-3})^{\frac{1}{3}} = a^{\frac{1}{3}}b^{-1}.$$

Hence, the given expression

$$= a^{-1}b^{\frac{1}{2}} \times a^{\frac{1}{3}}b^{-1}$$

$$= a^{-1+\frac{1}{3}} \times b^{\frac{1}{2}-1} = a^{-\frac{2}{3}}b^{-\frac{1}{2}}.$$

Example 3. Simplify $\sqrt{a^3b^{-\frac{2}{3}}c^{-\frac{7}{6}}} \div \sqrt[3]{a^4b^{-1}c^{\frac{5}{4}}}$

$$\sqrt{a^3b^{-\frac{2}{3}}c^{-\frac{7}{6}}} = \left(a^3b^{-\frac{2}{3}}c^{-\frac{7}{6}}\right)^{\frac{1}{2}}$$

$$= \left(a^3\right)^{\frac{1}{2}}\left(b^{-\frac{2}{3}}\right)^{\frac{1}{2}}\left(c^{-\frac{7}{6}}\right)^{\frac{1}{2}}$$

$$= a^{\frac{3}{2}}b^{-\frac{1}{3}}c^{-\frac{7}{12}};$$

and $\sqrt[3]{a^4b^{-1}c^{\frac{5}{4}}} = \left(a^4b^{-1}c^{\frac{5}{4}}\right)^{\frac{1}{3}}$

$$= \left(a^4\right)^{\frac{1}{3}}\left(b^{-1}\right)^{\frac{1}{3}}\left(c^{\frac{5}{4}}\right)^{\frac{1}{3}}$$

$$= a^{\frac{4}{3}}b^{-\frac{1}{3}}c^{\frac{5}{12}}.$$

Hence, the given expression

$$= a^{\frac{3}{2}}b^{-\frac{1}{3}}c^{-\frac{7}{12}} \div a^{\frac{4}{3}}b^{-\frac{1}{3}}c^{\frac{5}{12}}$$

$$= a^{\frac{3}{2}}b^{-\frac{1}{3}}c^{-\frac{7}{12}} \times a^{-\frac{4}{3}}b^{\frac{1}{3}}c^{-\frac{5}{12}}$$

$$= a^{\frac{3}{2}-\frac{4}{3}}b^{-\frac{1}{3}+\frac{1}{3}}c^{-\frac{7}{12}-\frac{5}{12}}$$

$$= a^{\frac{1}{6}}b^0c^{-1} = a^{\frac{1}{6}}c^{-1}.$$

Exercise (7).

Simplify—

1. $\left(a^{-\frac{3}{4}}\right)^8$. 2. $\left(a^{-\frac{2}{3}}b^{\frac{5}{6}}\right)^{\frac{3}{4}}$. 3. $\left(a^{-\frac{1}{2}}b^{-3}\right)^{-2}$.

4. $\left(a^6b^{\frac{5}{4}}\right)^{-\frac{4}{3}}$. 5. $\left(\sqrt[3]{a^4b^3}\right)^6$. 6. $\left(\sqrt[6]{x^9y^{-8}}\right)^{-3}$.

7. $\sqrt[8]{x^2 . \sqrt[4]{x^{-3}}}$.

8. $\sqrt{a^{-3}b^4} \times \sqrt[4]{a^2 b^{-8}}$.

9. $\sqrt[4]{x^{-2}\sqrt{y^5}} \times \sqrt{x\sqrt[4]{y^3}}$.

10. $(8x^3 \div 27a^{-3})^{\frac{2}{3}}$.

11. $(64x^3 \div 27a^{-3})^{-\frac{2}{3}}$.

12. $\sqrt[3]{a^6 b^{-2} c^{-4}} \times \sqrt[4]{a^{-6} b^4 c^8}$.

13. $\sqrt{a^{-\frac{2}{3}} b^4 c^{-\frac{1}{3}}} \div \sqrt[3]{a^2 b^4 c^{-1}}$.

14. $\sqrt{ab^{-2}c^3} \div \left(\sqrt[3]{a^3 b^2 c^{-3}}\right)^{-1}$.

15. $\left(\frac{a^{-1}b^2}{a^2 b^{-4}}\right)^7 \div \left(\frac{a^3 b^{-5}}{a^{-2} b^3}\right)^{-5}$.

7. Miscellaneous Examples.

Example 1. Divide $a+b+c+3a^{\frac{1}{3}}b^{\frac{2}{3}}+3a^{\frac{2}{3}}b^{\frac{1}{3}}$ by $a^{\frac{1}{3}}+b^{\frac{1}{3}}+c^{\frac{1}{3}}$.

Let us proceed by arranging the dividend and the divisor according to descending powers of a :—

$$\begin{array}{l|l|l}
a^{\frac{1}{3}}+(b^{\frac{1}{3}}+c^{\frac{1}{3}}) & a+3a^{\frac{2}{3}}b^{\frac{1}{3}}+3a^{\frac{1}{3}}b^{\frac{2}{3}}+(b+c) & a^{\frac{2}{3}}+a^{\frac{1}{3}}(2b^{\frac{1}{3}}-c^{\frac{1}{3}}) \\
 & a+\ a^{\frac{2}{3}}(b^{\frac{1}{3}}+c^{\frac{1}{3}}) & \quad +(b^{\frac{2}{3}}-b^{\frac{1}{3}}c^{\frac{1}{3}}+c^{\frac{2}{3}}) \\
\hline
 & \quad a^{\frac{2}{3}}(2b^{\frac{1}{3}}-c^{\frac{1}{3}})+3a^{\frac{1}{3}}b^{\frac{2}{3}}+(b+c) & \\
 & \quad a^{\frac{9}{3}}(2b^{\frac{1}{3}}-c^{\frac{1}{3}})+\ a^{\frac{1}{3}}(2b^{\frac{2}{3}}+b^{\frac{1}{3}}c^{\frac{1}{3}}-c^{\frac{2}{3}}) & \\
\hline
 & \qquad\qquad a^{\frac{1}{3}}(b^{\frac{2}{3}}-b^{\frac{1}{3}}c^{\frac{1}{3}}+c^{\frac{2}{3}})+(b+c) & \\
 & \qquad\qquad a^{\frac{1}{3}}(b^{\frac{2}{3}}-b^{\frac{1}{3}}c^{\frac{1}{3}}+c^{\frac{2}{3}})+(b+c) & \\
\hline
\end{array}$$

Thus the required quotient

$$= a^{\frac{2}{3}}+2a^{\frac{1}{3}}b^{\frac{1}{3}}-a^{\frac{1}{3}}c^{\frac{1}{3}}+b^{\frac{2}{3}}-b^{\frac{1}{3}}c^{\frac{1}{3}}+c^{\frac{2}{3}}.$$

NOTE. In multiplication as well as in division the arrangement of the expressions concerned according to ascending or descending powers of some common letter should *never* be overlooked. Such arrangements invariably give neatness to the required operations, if not always indispensable.

Example 2. Divide $x^{2^n}+a^{2^{n-1}}x^{2^{n-1}}+a^{2^n}$ by

$$x^{2^{n-1}}-a^{2^{n-2}}x^{2^{n-2}}+a^{2^{n-1}}.$$

Let $m=x^{2^{n-2}}$ and $n=a^{2^{n-2}}$.

Then $m^2=\left(x^{2^{n-2}}\right)^2=x^{2\times 2^{n-2}}=x^{2^{n-1}}$,

and $m^4=\left(m^2\right)^2=\left(x^{2^{n-1}}\right)^2=x^{2\times 2^{n-1}}=x^{2^n}$.

Similarly, $n^2=a^{2^{n-1}}$ and $n^4=a^{2^n}$.

Hence, $\dfrac{x^{2^n}+a^{2^{n-1}}x^{2^{n-1}}+a^{2^n}}{x^{2^{n-1}}-a^{2^{n-2}}x^{2^{n-2}}+a^{2^{n-1}}}$

$$=\frac{m^4+m^2n^2+n^4}{m^2-mn+n^2}=\frac{(m^2+n^2)^2-m^2n^2}{m^2-mn+n^2}$$

$$=\frac{(m^2+n^2+mn)(m^2+n^2-mn)}{m^2-mn+n^2}$$

$$=m^2+mn+n^2$$

$$=x^{2^{n-1}}+x^{2^{n-2}}a^{2^{n-2}}+a^{2^{n-1}}.$$

Example 3. Shew that

$$\frac{1}{1+x^{m-n}+x^{m-p}}+\frac{1}{1+x^{n-m}+x^{n-p}}+\frac{1}{1+x^{p-m}+x^{p-n}}=1.$$

The 1st term $=\dfrac{x^{-m}}{x^{-m}(1+x^{m-n}+x^{m-p})}$

$$=\frac{x^{-m}}{x^{-m}+x^{-n}+x^{-p}};$$

the 2nd term $=\dfrac{x}{x^{-n}(1+x^{n-m}+x^{n-p})}$

$$=\frac{x^{-n}}{x^{-n}+x^{-m}+x^{-p}};$$

and the 3rd term $= \dfrac{x^{-p}}{x^{-p}(1 + x^{p-m} + x^{p-n})}$

$$= \frac{}{x^{-p} + x^{-m} + x^{-n}}$$

Hence, the given expression

$$= \frac{x^{-m}}{x^{-m} + x^{-n} + x^{-p}} + \frac{x^{-}}{x^{-n} + x^{-m} + x^{-p}} + \frac{}{x^{-p} + x^{-m} + x^{-n}}$$

$$= \frac{x^{-m} + x^{-n} + x^{-p}}{x^{-m} + x^{-n} + x^{-p}} = 1.$$

Example 4. Solve the equation

$$a^{-x}.(a^x + b^{-x}) = \frac{a^2b^2 + 1}{a^2b^2}.$$

We have $a^{-x}.a^x + a^{-x}.b^{-x} = 1 + \dfrac{1}{a^2b^2}$,

or, $1 + (ab)^{-x} = 1 + a^{-2}b^{-2}$

$= 1 + (ab)^{-2}$.

Hence $(ab)^{-x} = (ab)^{-2}$,

$\therefore \quad x = 2$.

Example 5. If $a^b = b^a$, shew that $\left(\dfrac{a}{b}\right)^{\frac{a}{b}} = a^{\frac{a}{b}-1}$; and if $a = 2b$, shew that $b = 2$.

Since $a^b = b^a$,

$\therefore \quad a = b^{\frac{a}{b}}$ [extracting the bth root of both sides]

Hence, $\left(\dfrac{a}{b}\right)^{\frac{a}{b}} = \dfrac{a^{\frac{a}{b}}}{b^{\frac{a}{b}}} = \dfrac{a^{\frac{a}{b}}}{a} = a^{\frac{a}{b}-1}$.

If $a = 2b$, from the given relation we have

$$(2b)^b = (b)^{2b} = (b^2)^b,$$

$\therefore \quad 2b = b^2, \quad \therefore \quad b = 2$.

Example 6. If $x = \left(a + \sqrt{a^2+b^3}\right)^{\frac{1}{3}} + \left(a - \sqrt{a^2+b^3}\right)^{\frac{1}{3}}$,

shew that $x^3 + 3bx - 2a = 0$.

Putting m for $a + \sqrt{a^2+b^3}$,

and n for $a - \sqrt{a^2+b^3}$, we have

$$\begin{aligned} x^3 &= (m^{\frac{1}{3}} + n^{\frac{1}{3}})^3 \\ &= (m^{\frac{1}{3}})^3 + (n^{\frac{1}{3}})^3 + 3m^{\frac{1}{3}}.n^{\frac{1}{3}}(m^{\frac{1}{3}} + n^{\frac{1}{3}}) \\ &= m + n + 3(mn)^{\frac{1}{3}}.(m^{\frac{1}{3}} + n^{\frac{1}{3}}) \\ &= m + n + 3(mn)^{\frac{1}{3}}.x. \end{aligned}$$

But $m + n = 2a$,

and $(mn)^{\frac{1}{3}} = \{a^2 - (a^2 + b^3)\}^{\frac{1}{3}}$

$= (-b^3)^{\frac{1}{3}} = -b$;

$\therefore\quad x^3 = 2a - 3bx$,

$\therefore\quad x^3 + 3bx - 2a = 0$.

Exercise (8).

Multiply :—

1. $x^{\frac{2}{3}} + 2x^{\frac{1}{2}} + 3x^{\frac{1}{3}} + 2x^{\frac{1}{6}} + 1$ by $x^{\frac{1}{3}} - 2x^{\frac{1}{6}} + 1$.

2. $a^{\frac{2}{3}} + 3a^{\frac{1}{3}}b^{\frac{1}{3}} + 9b^{\frac{2}{3}}$ by $a^{\frac{1}{3}} - 3b^{\frac{1}{3}}$.

3. $1 + ab^{-1} + a^2b^{-2}$ by $1 - ab^{-1} + a^2b^{-2}$.

4. $x + 2y^{\frac{1}{2}} + 3z^{\frac{1}{3}}$ by $x - 2y^{\frac{1}{2}} + 3z^{\frac{1}{3}}$.

5. $x^{-1} + x^{-\frac{1}{2}}y^{-\frac{1}{2}} + y^{-1}$ by $x^{-1} - x^{-\frac{1}{2}}y^{-\frac{1}{2}} + y^{-1}$.

6. $a^{\frac{2}{3}} - a^{\frac{1}{3}} + 1 - a^{-\frac{1}{3}} + a^{-\frac{2}{3}}$ by $a^{\frac{1}{3}} + 1 + a^{-\frac{1}{3}}$.

7. $x^{\frac{2}{3}} + y^{\frac{2}{3}} + z^{\frac{2}{3}} - y^{\frac{1}{3}}z^{\frac{1}{3}} - z^{\frac{1}{3}}x^{\frac{1}{3}} - x^{\frac{1}{3}}y^{\frac{1}{3}}$ by $x^{\frac{1}{3}} + y^{\frac{1}{3}} + z^{\frac{1}{3}}$.

8. $a^m + 3b^n - 2c^p$ by $a^m - 3b^n + 2c^p$.

9. $a^{\frac{5}{2}} + 8ab + 4a^{\frac{3}{2}}b^{\frac{2}{3}} + 2a^2b^{\frac{1}{3}} + 32b^{\frac{5}{3}} + 16a^{\frac{1}{2}}b^{\frac{4}{3}}$ by $a^{\frac{1}{2}} - 2b^{\frac{1}{3}}$.

10. $a^{\frac{5}{8}} + a^{\frac{1}{4}}x^{-\frac{3}{8}} + x^{-\frac{5}{8}} + a^{\frac{3}{8}}x^{-\frac{1}{4}} + a^{\frac{1}{8}}x^{-\frac{1}{2}} + a^{\frac{1}{2}}x^{-\frac{1}{8}}$ by

$a^{\frac{3}{8}} + a^{\frac{1}{8}}x^{-\frac{1}{4}} - x^{-\frac{3}{8}} - a^{\frac{1}{4}}x^{-\frac{1}{8}}$.

Divide :—

11. $x^{\frac{5}{2}} - 4x^{\frac{3}{2}} - 2x^{\frac{1}{2}} + 6x - x^2$ by $x^{\frac{3}{2}} + 2 - 4x^{\frac{1}{2}}$.

12. $8 + 12x^{-1} + 2x^{-2} + 2x^{-4}$ by $x^{-2} - 2x^{-1} + 4$.

13. $xy^{-1} + 2x^{\frac{1}{2}}y^{-\frac{1}{2}} + 3 + 2x^{-\frac{1}{2}}y^{\frac{1}{2}} + x^{-1}y$ by $x^{-1} + x^{-\frac{1}{2}}y^{-\frac{1}{2}} + y^{-1}$.

14. $a^{\frac{5}{2}} - a^{\frac{3}{2}}b + ab^{\frac{3}{2}} - 2a^{\frac{1}{2}}b^2 + b^{\frac{5}{2}}$ by $a^{\frac{3}{2}} - ab^{\frac{1}{2}} + a^{\frac{1}{2}}b - b^{\frac{3}{2}}$.

15. $8x^{-n} - 8x^n + 5x^{3n} - 3x^{-3n}$ by $5x^n - 3x^{-n}$.

16. $8x^{\frac{3}{2}} + y^{-\frac{3}{2}} - z + 6x^{\frac{1}{2}}y^{-\frac{1}{2}}z^{\frac{1}{3}}$ by $2x^{\frac{1}{2}} + y^{-\frac{1}{2}} - z^{\frac{1}{3}}$.

17. Show that $x^3 + a^3 + x^{\frac{3}{2}}a^{\frac{3}{2}}$ is divisible by $x^{\frac{3}{4}} + a^{\frac{3}{4}} + x^{\frac{3}{8}}a^{\frac{3}{8}}$.

18. Multiply $x^{2^{n-1}} + a^{2^{n-1}}$ by $x^{2^{n-1}} - a^{2^{n-1}}$.

19. Divide $x^{2^n} - y^{2^n}$ by $x^{2^{n-1}} + y^{2^{n-1}}$.

20. Simplify $\left\{\left(a^m\right)^{m-\frac{1}{m}}\right\}^{\frac{1}{m+1}}$.

21. Divide $2x^{-\frac{1}{4}} + 3x^{\frac{3}{4}} - 7x^{\frac{1}{4}} + x - 2x^{\frac{1}{2}}$ by $x^{\frac{1}{4}} - 2x^{-\frac{1}{4}}$.

22. Find the square of $x^{\frac{3}{4}} - x^{\frac{1}{2}}y^{-\frac{1}{4}} + y^{\frac{1}{2}}$.

23. Divide $x^{\frac{3n}{2}} - a^{\frac{3n}{2}}$ by $x^{\frac{n}{2}} - a^{\frac{n}{2}}$.

24. Find the square of $x^{\frac{1}{3}} - 2x^{\frac{1}{2}} + x^{\frac{5}{6}}$.

25. Divide $ax^{-1} + a^{-1}x + 2$ by $a^{\frac{1}{3}}x^{-\frac{1}{3}} + a^{-\frac{1}{3}}x^{\frac{1}{3}} - 1$.

26. Simplify $\left(\dfrac{a-b}{a^{\frac{1}{2}} - b^{\frac{1}{2}}} - \dfrac{a^{\frac{3}{2}} - b^{\frac{3}{2}}}{a-b}\right)^{-1}$

27. Simplify $\dfrac{x^{\frac{1}{3}}+3y^{\frac{1}{3}}}{x^{\frac{1}{3}}-3y^{\frac{1}{3}}}+\dfrac{x^{\frac{2}{3}}-3x^{\frac{1}{3}}y^{\frac{1}{3}}+9y^{\frac{2}{3}}}{x^{\frac{2}{3}}+3x^{\frac{1}{3}}y^{\frac{1}{3}}+9y^{\frac{2}{3}}}$.

28. Simplify $\dfrac{a^{\frac{3}{2}}-ax^{\frac{1}{2}}+a^{\frac{1}{2}}x-x^{\frac{3}{2}}}{a^{\frac{5}{2}}-a^2x^{\frac{1}{2}}+3a^{\frac{3}{2}}x-3ax^{\frac{3}{2}}+a^{\frac{1}{2}}x^2-x^{\frac{5}{2}}}$.

29. Simplify $\dfrac{a^2+b^2-a^{-2}-b^{-2}}{a^2b^2-a^{-2}b^{-2}}+\dfrac{(a-a^{-1})(b-b^{-1})}{ab+a^{-1}b^{-1}}$.

30. Simplify $\dfrac{x-y}{x^{\frac{3}{4}}+x^{\frac{1}{2}}y^{\frac{1}{4}}}\div\dfrac{x^{\frac{1}{2}}+y^{\frac{1}{2}}}{x^{\frac{1}{3}}y^{\frac{1}{4}}+x^{\frac{1}{4}}y^{\frac{1}{2}}}$.

31. Simplify

$(a+b+c)(a^{-1}+b^{-1}+c^{-1})-a^{-1}b^{-1}c^{-1}(b+c)(c+a)(a+b)$.

Solve :—

32. $2^{x+7}=4^{x+2}$.

33. $(\sqrt{3})^{x+5}=(\sqrt[3]{3})^{2x+5}$.

34. $(\sqrt[5]{4})^{4x+7}=(\sqrt[11]{64})^{2x+7}$.

35. $(\sqrt[3]{25})^{2x+1}=(\sqrt[5]{125})^{x+6}$.

36. $\left.\begin{aligned}2^{3x-1}&=4^{y-1}\\ 3x-y&=1\end{aligned}\right\}$.

37. $\left.\begin{aligned}9^{2x-3}&=(\sqrt{3})^{2y-x}\\ 2^{3x}&=4^{y}\end{aligned}\right\}$.

38. $\left.\begin{aligned}4^{3y-1}&=16^{x+y}\\ 3^{x+3y}&=9^{2x+3}\end{aligned}\right\}$.

39. $\left.\begin{aligned}2^{x+y+z}&=8^{x+z-y}\\ 5^{3y+2}&=25^{x+z}\\ 3^{2x+2z+y}&=9^{3x+y}\end{aligned}\right\}$

40. $\left.\begin{aligned}(\sqrt{a})^{x+y}&=(\sqrt[3]{a})^{y+z-1}\\ (\sqrt[3]{b})^{x+z-2}&=(\sqrt[5]{b})^{y+z}\\ (\sqrt[4]{c})^{y}&=(\sqrt[7]{c})^{x+y+1}\end{aligned}\right\}$.

CHAPTER III.

SURDS.

1. Definition. Any root of any arithmetical number which cannot be exactly found is called a *surd* or an *irrational quantity*. Thus $\sqrt{2}$, $\sqrt{6}$, $\sqrt[3]{4}$, and $\sqrt[4]{5}$ are all surds.

NOTE 1. Quantities which are not surds are called *rational quantities*. Hence every root of an arithmetical number is either *rational* or *irrational*. Thus $\sqrt[3]{8}$, $\sqrt{25}$ and $\sqrt[4]{16}$ are *rational* quantities, whilst $\sqrt{2}$, $\sqrt[3]{5}$ and $\sqrt[4]{9}$ are all *irrational* quantities.

NOTE 2. An algebraical expression also, such as $\sqrt{x}$, is called a surd although the value of x may be such that $\sqrt{x}$ is not in reality a surd. For instance, if $x = 4$, $\sqrt{x} = \sqrt{4} = 2$, and is therefore not really a surd.

2. To express in the form of a surd the product of a rational quantity and a surd.

Example 1.

$$5\sqrt{3} = \left(5^2\right)^{\frac{1}{2}} \times 3^{\frac{1}{2}}$$

$$= \left(5^2 \times 3\right)^{\frac{1}{2}} \qquad \text{[Art. 5, Chap. II.]}$$

$$= \sqrt{5^2 \times 3} = \sqrt{75}.$$

Example 2.

$$2\sqrt[3]{9} = \left(2^3\right)^{\frac{1}{3}} \times 9^{\frac{1}{3}}$$

$$= \left(2^3 \times 9\right)^{\frac{1}{3}} \qquad \text{[Art. 5, Chap. II.]}$$

$$= \sqrt[3]{2^3 \times 9} = \sqrt[3]{72}.$$

Exercise (9).

Express as a complete surd :—

1. $3\sqrt{5}$. **2.** $2\sqrt[3]{3}$. **3.** $2\sqrt[4]{6}$. **4.** $4\sqrt[4]{5}$.

5. $a\sqrt[x]{b}$. **6.** $x^3\sqrt[4]{y}$. **7.** $a^4\sqrt[5]{b^2}$.

3. A surd may sometimes be expressed as the product of a rational quantity and a surd.

Example 1.
$$\sqrt{32} = \sqrt{16 \times 2}$$
$$= (4^2 \times 2)^{\frac{1}{2}}$$
$$= (4^2)^{\frac{1}{2}} \times 2^{\frac{1}{2}} \qquad \text{[Art. 5, Chap. II.]}$$
$$= 4 \times 2^{\frac{1}{2}} = 4\sqrt{2}$$

Example 2.
$$\sqrt[3]{40} = \sqrt[3]{8 \times 5}$$
$$= (2^3 \times 5)^{\frac{1}{3}}$$
$$= (2^3)^{\frac{1}{3}} \times 5^{\frac{1}{3}} \qquad \text{[Art. 5, Chap. II.]}$$
$$= 2 \times 5^{\frac{1}{3}}$$
$$= 2\sqrt[3]{5}.$$

Exercise (10.)

Simplify :—

1. $\sqrt{18}$.
2. $\sqrt{80}$.
3. $\sqrt[3]{250}$.
4. $\sqrt[5]{128}$.
5. $\sqrt[4]{405}$.
6. $\sqrt[3]{1372}$.
7. $\sqrt[4]{1875}$.
8. $\sqrt[3]{a^6 b}$.
9. $\sqrt[n]{x^{4n} a}$.
10. $\sqrt[3]{-2560}$.
11. $\sqrt[3]{-192a^3b^4}$.
12. $\sqrt[3]{500a^7x^4}$.

4. Similar surds. Two or more surds are said to be *similar* or **like** when they can be so reduced as to have the same irrational factor. Thus $\sqrt{45}$ and $\sqrt{80}$ are similar surds for they are respectively equivalent to $3\sqrt{5}$ and $4\sqrt{5}$. The sum of any number of similar surds may be found as follows :—

Example 1.
$$\sqrt{147} + \sqrt{27}$$
$$= \sqrt{49 \times 3} + \sqrt{9 \times 3}$$
$$= 7\sqrt{3} + 3\sqrt{3} = 10\sqrt{3}.$$

Example 2.
$$\sqrt[3]{625} - \sqrt[3]{135} + \sqrt[3]{40}$$
$$= \sqrt[3]{125 \times 5} - \sqrt[3]{27 \times 5} + \sqrt[3]{8 \times 5}$$

$$= \sqrt[3]{5^3 \times 5} - \sqrt[3]{3^3 \times 5} + \sqrt[3]{2^3 \times 5}$$
$$= 5\sqrt[3]{5} - 3\sqrt[3]{5} + 2\sqrt[3]{5}$$
$$= 4\sqrt[3]{5}.$$

Exercise (11).

Simplify :—

1. $\sqrt{12} + \sqrt{75}$. 2. $\sqrt{18} + \sqrt{32}$. 3. $\sqrt{20} + \sqrt{180}$.
4. $\sqrt{98} - \sqrt{50}$. 5. $\sqrt[3]{128} - \sqrt[3]{54}$. 6. $\sqrt[4]{80} + \sqrt[4]{405}$.
7. $\sqrt[4]{768} - \sqrt[4]{243}$. 8. $2\sqrt{27} - \sqrt{75} + \sqrt{12}$.
9. $2\sqrt{405} - 3\sqrt{125} + \sqrt{45}$. 10. $4\sqrt[3]{192} - 4\sqrt[3]{375} + 2\sqrt[3]{24}$.
11. $3\sqrt[3]{40} + 2\sqrt[3]{625} - 4\sqrt[3]{320}$.
12. $5\sqrt[3]{-54} - 2\sqrt[3]{-16} + 4\sqrt[3]{686}$.
13. $\sqrt{45x^3} + \sqrt{80x^3} + \sqrt{5xy^2}$.
14. $x\sqrt[3]{x^3a} + y\sqrt[3]{-8y^3a} - z\sqrt[3]{-27z^3a}$.
15. $2\sqrt[4]{32a^4x} + 3\sqrt[4]{512a^4x} - 4a\sqrt[4]{162x}$.

5. Surds of the same order. Surds are said to be *of the same order* or **equiradical** when they have all got the same root symbol. Thus $\sqrt{5}$, $\sqrt{a^3}$ and $(a+x)^{\frac{5}{2}}$ are all surds of the same (the *second*) order.

A surd of the second order is often called a **quadratic surd**; whilst one of the third order, as $\sqrt[3]{4}$ or $\sqrt[3]{a^2}$, is called a **cubic surd.**

Surds of different orders may be reduced to equivalent surds of the same order.

Example 1. Reduce $\sqrt{5}$ and $\sqrt[3]{4}$ to surds of the same order.

The given surds are respectively of the 2nd and 3rd orders; and the L. C. M. of 2 and 3 is 6. Hence we can at once reduce them to surds of the 6th order, thus :—

$$\sqrt{5} = 5^{\frac{1}{2}} = 5^{\frac{3}{6}} = \sqrt[6]{5^3} = \sqrt[6]{125}$$
$$\sqrt[3]{4} = 4^{\frac{1}{3}} = 4^{\frac{2}{6}} = \sqrt[6]{4^2} = \sqrt[6]{16}.$$

Thus the required surds are $\sqrt[6]{125}$ and $\sqrt[6]{16}$.

Example 2. Reduce $\sqrt[6]{3}$ and $\sqrt[8]{2}$ to surds of the same order.

The L. C. M. of 6 and 8 is 24.

Thus we have

$$\sqrt[6]{3} = 3^{\frac{1}{6}} = 3^{\frac{4}{24}} = \sqrt[24]{3^4} = \sqrt[24]{81},$$

and $\sqrt[8]{2} = 2^{\frac{1}{8}} = 2^{\frac{3}{24}} = \sqrt[24]{2^3} = \sqrt[24]{8}$.

Thus the required surds are $\sqrt[24]{81}$ and $\sqrt[24]{3}$.

Example 3. Which is the greater $\sqrt[3]{9}$ or $\sqrt[4]{20}$?

We have $\sqrt[3]{9} = 9^{\frac{1}{3}} = 9^{\frac{4}{12}} = \sqrt[12]{9^4} = \sqrt[12]{6561}$,

and $\sqrt[4]{20} = 20^{\frac{1}{4}} = 20^{\frac{3}{12}} = \sqrt[12]{20^3} = \sqrt[12]{8000}$.

Thus the given surds are respectively equivalent to $\sqrt[12]{6561}$ and $\sqrt[12]{8000}$, and as the latter is greater than the former, therefore $\sqrt[4]{20} > \sqrt[3]{9}$.

Exercise (12).

Reduce to surds of the same order :—

1. $\sqrt{3}$ and $\sqrt[3]{2}$. **2.** $\sqrt[3]{4}$ and $\sqrt[4]{5}$. **3.** $\sqrt[5]{2}$ and $\sqrt[3]{3}$.

4. $\sqrt[4]{3}$ and $\sqrt[6]{5}$. **5.** $\sqrt[6]{4}$ and $\sqrt[8]{6}$.

Which is the greater :—

6. $\sqrt{2}$ or $\sqrt[3]{3}$? **7.** $\sqrt[3]{3}$ or $\sqrt[4]{4}$? **8.** $\sqrt[3]{6}$ or $\sqrt[4]{10}$?

Arrange according to descending order of magnitude :—

9. $\sqrt[4]{6}$, $\sqrt{2}$ and $\sqrt[3]{4}$. **10.** $\sqrt[4]{3}$, $\sqrt[8]{10}$ and $\sqrt[12]{25}$.

6. Multiplication and division of surds.

Example 1. $\sqrt[3]{6} \times \sqrt[3]{10} = 6^{\frac{1}{3}} \times 10^{\frac{1}{3}}$

$$= (6 \times 10)^{\frac{1}{3}}$$

$$= \sqrt[3]{60}.$$

NOTE. In this example the given surds are of the *same* order.

Example 2.

$$\begin{aligned}\sqrt[4]{5}\times\sqrt[6]{8} &= 5^{\frac{1}{4}}\times 8^{\frac{1}{6}}\\ &= 5^{\frac{3}{12}}\times 8^{\frac{2}{12}}\\ &= \left(5^3\right)^{\frac{1}{12}}\times\left(8^2\right)^{\frac{1}{12}} && \text{[Art. 4, Chap. II.]}\\ &= \left(5^3\times 8^2\right)^{\frac{1}{12}} && \text{[Art. 5, Chap. II.]}\\ &= \sqrt[12]{125\times 64}\\ &= \sqrt[12]{8000}.\end{aligned}$$

NOTE. In this example the given surds are of *different* orders.

Example 3.

$$\begin{aligned}\sqrt[3]{2}\times\sqrt[5]{2} &= 2^{\frac{1}{3}}\times 2^{\frac{1}{5}}\\ &= 2^{\frac{1}{3}+\frac{1}{5}}\\ &= 2^{\frac{8}{15}} = \sqrt[15]{2^8}\\ &\qquad\;\; = \sqrt[15]{256}.\end{aligned}$$

NOTE. In this example the given surds have got the same quantity under the radical sign. They may as well be regarded as surds of different orders and treated like those in the last example.

Example 4. $4\sqrt{18}\times\sqrt{75}$

$$\begin{aligned}&= 4.3\sqrt{2}\times 5\sqrt{3}\\ &= 60\sqrt{2}.\sqrt{3} = 60\sqrt{6}.\end{aligned}$$

NOTE. In this example the given surds have been reduced to simpler forms before multiplication.

Example 5.

$$\begin{aligned}\sqrt[6]{4}\div\sqrt[4]{6} &= 4^{\frac{1}{6}}\div 6^{\frac{1}{4}}\\ &= 4^{\frac{2}{12}}\div 6^{\frac{3}{12}}\\ &= \frac{\left(4^2\right)^{\frac{1}{12}}}{\left(6^3\right)^{\frac{1}{12}}} && \text{[Art. 4, Chap. II.]}\\ &= \left(\frac{4^2}{6^3}\right)^{\frac{1}{12}} && \text{[Cor. 1, Art. 5, Chap. II.]}\\ &= \sqrt[12]{\frac{2}{27}}.\end{aligned}$$

Example 6. Express $\sqrt{5} \div 3\sqrt{3}$ as a fraction with a rational denominator.

We have $$\sqrt{5} \div 3\sqrt{3} = \frac{\sqrt{5}}{3\sqrt{3}} = \frac{\sqrt{5} \times \sqrt{3}}{3\sqrt{3} \times \sqrt{3}}$$
$$= \frac{\sqrt{15}}{3 \times 3} = \frac{\sqrt{15}}{9}.$$

NOTE. For arithmetical calculations it is always most convenient to reduce the quotient of one surd by another to the form of a fraction with a rational denominator. Hence even when the numerical value of a surd fraction is not required it is usual to express it in the above form.

Exercise (13).

Simplify :—

1. $\sqrt{5} \times \sqrt{10}$. **2.** $\sqrt{8} \times \sqrt{6}$. **3.** $\sqrt{27} \times \sqrt{3}$.
4. $\sqrt{15} \times \sqrt{6}$. **5.** $\sqrt{20} \times \sqrt{45}$. **6.** $\sqrt[3]{5} \times \sqrt[3]{25}$.
7. $\sqrt[3]{6ax} \times \sqrt[3]{27a^4x^3}$. **8.** $\sqrt[3]{2} \times \sqrt[2]{6}$. **9.** $\sqrt[2]{2} \times \sqrt[3]{6}$.
10. $\sqrt[3]{4} \times \sqrt[2]{8}$. **11.** $\sqrt[3]{9} \times \sqrt[2]{27}$. **12.** $\sqrt[6]{2} \times \sqrt[9]{3}$.
13. $\sqrt[4]{3} \times \sqrt[8]{3}$. **14.** $\sqrt[6]{2} \times \sqrt[9]{2}$. **15.** $\sqrt[4]{4} \times \sqrt[6]{4}$.
16. $5\sqrt{8} \times 2\sqrt{6}$. **17.** $8\sqrt{12} \times 3\sqrt{24}$.
18. $4\sqrt[3]{72} \times 5\sqrt[3]{576}$. **19.** $7\sqrt[3]{8a^3x^2} \times 5\sqrt[3]{27b^3x^2}$.
20. $8\sqrt{10} \div 4\sqrt{15}$. **21.** $3\sqrt{12} \div 6\sqrt{27}$.
22. $\sqrt[3]{36} \div \sqrt[3]{48}$. **23.** $\sqrt[9]{8} \div \sqrt[6]{6}$.

Given $\sqrt{2} = 1·414$, $\sqrt{3} = 1·732$, $\sqrt{5} = 2\,236$, find to 3 places of decimals the numerical value of :—

24. $\sqrt{2} \div \sqrt{6}$. **25.** $\sqrt{72} \div \sqrt{40}$. **26.** $\sqrt{275} \div \sqrt{22}$.
27. $10\sqrt{108} \div \sqrt{15}$.

7. Compound Surds. An expression consisting of two or more simple surds connected by the sign + or − is called a *compound* surd. Thus $5\sqrt{2}$ and $4\sqrt{3}$ are simple surds, but $5\sqrt{2} + 4\sqrt{3}$ and $5\sqrt{2} - 4\sqrt{3}$ are compound surds.

Two or more compound surds are multiplied together in the same way as two or more compound algebraical expressions.

Example 1. Multiply $3\sqrt{x}+2\sqrt{3}$ by $\sqrt{x}-\sqrt{3}$.

$$\begin{aligned}(3\sqrt{x}+2\sqrt{3})(\sqrt{x}-\sqrt{3}) &= 3\sqrt{x}.\sqrt{x}+2\sqrt{3}.\sqrt{x} \\ &\quad -3\sqrt{x}.\sqrt{3}-2\sqrt{3}.\sqrt{3} \\ &= 3x+2\sqrt{3x}-3\sqrt{3x}-6 \\ &= 3x-\sqrt{3x}-6.\end{aligned}$$

Example 2. Multiply $7\sqrt{2}+\sqrt{3}$ by $7\sqrt{2}-\sqrt{3}$.

$$\begin{aligned}(7\sqrt{2}+\sqrt{3})(7\sqrt{2}-\sqrt{3}) &= (7\sqrt{2})^2-(\sqrt{3})^2 \\ &= 49.2-3 \\ &= 98-3 = 95.\end{aligned}$$

Example 3. Find the square of $\sqrt{3a+x}+\sqrt{3a-x}$.

$$\begin{aligned}(\sqrt{3a+x}+\sqrt{3a-x})^2 &= (\sqrt{3a+x})^2+(\sqrt{3a-x})^2 \\ &\quad +2\sqrt{3a+x}.\sqrt{3a-x} \\ &= (3a+x)+(3a-x)+2\sqrt{9a^2-x^2} \\ &= 6a+2\sqrt{9a^2-x^2}.\end{aligned}$$

Exercise (14).

Multiply :—

1. $\sqrt{a}+\sqrt{b}$ by $\sqrt{ab}$.
2. $\sqrt{a}+\sqrt{b}$ by $\sqrt{a}-\sqrt{b}$.
3. $3\sqrt{a}-5$ by $2\sqrt{a}$.
4. $4\sqrt{x}+3\sqrt{y}$ by $4\sqrt{x}-3\sqrt{y}$.
5. $2\sqrt{x-5}+4$ by $3\sqrt{x-5}-6$.
6. $3\sqrt{5}-4\sqrt{2}$ by $2\sqrt{5}+3\sqrt{2}$.
7. $\sqrt{2}+2\sqrt{3}+\sqrt{7}$ by $\sqrt{2}+2\sqrt{3}-\sqrt{7}$.
8. $3-\sqrt{5}+\sqrt{8}$ by $3-\sqrt{5}-\sqrt{8}$.
9. $\sqrt{11}+\sqrt{6}-\sqrt{3}$ by $\sqrt{11}-\sqrt{6}+\sqrt{3}$.
10. $\sqrt[3]{4}+\sqrt[3]{9}+\sqrt[3]{48}$ by $\sqrt[3]{2}+\sqrt[3]{3}$.

Find the square of :—

11. $\sqrt{x+a}-\sqrt{x-a}$.
12. $2\sqrt{8}+5\sqrt{6}$.
13. $2\sqrt{5}+3\sqrt{7}$.
14. $\sqrt{a^2+2b^2}-\sqrt{a^2-2b^2}$.
15. $2\sqrt{x^2+y^2}+5\sqrt{x^2-y^2}$.

8. Rationalisation. If two surds be such that their product is rational, each of them is said to be rationalised when multiplied by the other. Thus $2\sqrt{5}$ and $\sqrt{3}+\sqrt{2}$ are rationalised when respectively multiplied by $\sqrt{5}$ and $\sqrt{3}-\sqrt{2}$;

for, $$2\sqrt{5}\times\sqrt{5} = 10,$$

and $$(\sqrt{3}+\sqrt{2})(\sqrt{3}-\sqrt{2}) = 3-2 = 1.$$

Two binomial quadratic surds which differ only in the sign which connects their terms are said to be **conjugate** or **complementary** to each other. Thus $\sqrt{3}+\sqrt{2}$ and $2\sqrt{5}-\sqrt{7}$ are respectively *conjugate* or *complementary* to $\sqrt{3}-\sqrt{2}$ and $2\sqrt{5}+\sqrt{7}$.

Evidently therefore every binomial quadratic surd is rationalised when multiplied by the complementary surd.

Hence a fraction with a binomial quadratic surd for its denominator can be easily reduced to an equivalent fraction with a rational denominator.

Example 1. Given $\sqrt{2} = 1\cdot414$, find to three places of decimals the value of $\dfrac{1+\sqrt{2}}{3-2\sqrt{2}}$.

$$\frac{1+\sqrt{2}}{3-2\sqrt{2}} = \frac{(1+\sqrt{2})(3+2\sqrt{2})}{(3-2\sqrt{2})(3+2\sqrt{2})}$$
$$= \frac{3+3\sqrt{2}+2\sqrt{2}+4}{9-8}$$
$$= 7+5\sqrt{2}$$
$$= 7+5\times1\cdot414$$
$$= 7+7\cdot070 = 14\cdot070.$$

Example 2. Rationalise the denominator of
$$\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}.$$

The given expression
$$= \frac{(\sqrt{1+x^2}-\sqrt{1-x^2})^2}{(\sqrt{1+x^2}+\sqrt{1-x^2})(\sqrt{1+x^2}-\sqrt{1-x^2})}$$
$$= \frac{(1+x^2)+(1-x^2)-2\sqrt{1-x^4}}{(1+x^2)-(1-x^2)}$$
$$= \frac{2-2\sqrt{1-x^4}}{2x^2} = \frac{1-\sqrt{1-x^4}}{x^2}$$

Example 3. Simplify $\dfrac{3+\sqrt{6}}{5\sqrt{3}-2\sqrt{12}-\sqrt{32}+\sqrt{50}}$.

The denominator $= 5\sqrt{3}-2\times 2\sqrt{3}-4\sqrt{2}+5\sqrt{2}$

$= \sqrt{3}+\sqrt{2}$.

Hence, the given fraction

$$= \frac{3+\sqrt{6}}{\sqrt{3}+\sqrt{2}}$$

$$= \frac{(3+\sqrt{6})(\sqrt{3}-\sqrt{2})}{(\sqrt{3}+\sqrt{2})(\sqrt{3}-\sqrt{2})}$$

$$= \frac{3\sqrt{3}-3\sqrt{2}+3\sqrt{2}-2\sqrt{3}}{3-2}$$

$$= \sqrt{3}.$$

Example 4. Simplify $\dfrac{3\sqrt{2}}{\sqrt{3}+\sqrt{6}}-\dfrac{4\sqrt{3}}{\sqrt{6}+\sqrt{2}}+\dfrac{\sqrt{6}}{\sqrt{2}+\sqrt{3}}$.

The 1st term

$$= \frac{3\sqrt{2}}{\sqrt{3}(1+\sqrt{2})} = \frac{\sqrt{6}}{\sqrt{2}+1} = \frac{\sqrt{6}(\sqrt{2}-1)}{(\sqrt{2}+1)(\sqrt{2}-1)}$$

$$= 2\sqrt{3}-\sqrt{6}$$

The 2nd term

$$= \frac{4\sqrt{3}}{\sqrt{2}(\sqrt{3}+1)} = \frac{2\sqrt{6}}{\sqrt{3}+1} = \frac{2\sqrt{6}(\sqrt{3}-1)}{(\sqrt{3}+1)(\sqrt{3}-1)}$$

$$= \frac{2(3\sqrt{2}-\sqrt{6})}{2} = 3\sqrt{2}-\sqrt{6}.$$

The 3rd term

$$\frac{\sqrt{6}}{\sqrt{3}+\sqrt{2}} = \frac{\sqrt{6}(\sqrt{3}-\sqrt{2})}{(\sqrt{3}+\sqrt{2})(\sqrt{3}-\sqrt{2})} = 3\sqrt{2}-2\sqrt{3}.$$

Hence the given expression

$= (2\sqrt{3}-\sqrt{6})-(3\sqrt{2}-\sqrt{6})+(3\sqrt{2}-2\sqrt{3})$

$= 0$

Example 5. Bring $\dfrac{7}{2^{\frac{1}{2}}+2^{\frac{1}{4}}+1}$ to a form with a rational denominator.

(Bombay University P. E. Paper, 1890.)

The given expression

$$= \frac{7}{(2^{\frac{1}{2}}+1)+2^{\frac{1}{4}}} = \frac{7\{(2^{\frac{1}{2}}+1)-2^{\frac{1}{4}}\}}{\{(2^{\frac{1}{2}}+1)+2^{\frac{1}{4}}\}\{(2^{\frac{1}{2}}+1)-2^{\frac{1}{4}}\}}$$

of which the denominator

$$= (2^{\frac{1}{2}}+1)^2 - 2^{\frac{1}{2}}$$

$$= (2+2.2^{\frac{1}{2}}+1) - 2^{\frac{1}{2}} = 3+2^{\frac{1}{2}}.$$

Hence, the given expression

$$= \frac{7\{(2^{\frac{1}{2}}+1)-2^{\frac{1}{4}}\}}{3+2^{\frac{1}{2}}}$$

$$= \frac{7(1-2^{\frac{1}{4}}+2^{\frac{1}{2}})(3-2^{\frac{1}{2}})}{(3+2^{\frac{1}{2}})(3-2^{\frac{1}{2}})} = \frac{7(3-3.2^{\frac{1}{4}}+3.2^{\frac{1}{2}}-2^{\frac{1}{2}}+2^{\frac{3}{4}}-2)}{9-2}$$

$$= 1-3.2^{\frac{1}{4}}+2.2^{\frac{1}{2}}+2^{\frac{3}{4}}.$$

NOTE. It may be observed in this connection that any fraction of the form $\frac{x}{\sqrt{a}+\sqrt{b}+\sqrt{c}}$ can be reduced to an equivalent fraction with a rational denominator as follows :—Multiply the numerator and denominator by $\sqrt{a}+\sqrt{b}-\sqrt{c}$; the denominator thus becomes $(\sqrt{a}+\sqrt{b})^2-(\sqrt{c})^2$ or $a+b-c+2\sqrt{ab}$. Now multiply both numerator and denominator by $(a+b-c)-2\sqrt{ab}$, on which the denominator becomes $(a+b-c)^2-4ab$, which is rational.

Exercise (15).

Reduce to an equivalent fraction with a rational denominator :—

1. $\frac{5\sqrt{3}+\sqrt{7}}{4\sqrt{3}+2\sqrt{7}}$.

2 $\frac{3\sqrt{2}+2\sqrt{3}}{3\sqrt{2}-2\sqrt{3}}$.

3. $\frac{4+3\sqrt{2}}{3-2\sqrt{2}}$.

4. $\frac{3\sqrt{5}+\sqrt{3}}{\sqrt{5}-\sqrt{3}}$.

5. $\frac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}}$.

6. $\frac{\sqrt{x^2+1}-\sqrt{x^2-1}}{\sqrt{x^2+1}+\sqrt{x^2-1}}$.

7. $\frac{1}{1+\sqrt{2}+\sqrt{3}}$.

8. $\frac{12}{3+\sqrt{5}-2\sqrt{2}}$.

9. $\dfrac{(\sqrt{3}+\sqrt{5})(\sqrt{5}+\sqrt{2})}{\sqrt{2}+\sqrt{3}+\sqrt{5}}$. 10. $\dfrac{1}{\sqrt{10}+\sqrt{14}+\sqrt{15}+\sqrt{21}}$.

Given $\sqrt{2} = 1{\cdot}414$, $\sqrt{3} = 1{\cdot}732$, $\sqrt{5} = 2{\cdot}236$, find to three places of decimals the value of :—

11. $\dfrac{\sqrt{2}+1}{\sqrt{2}-1}$. 12. $\dfrac{\sqrt{3}}{2-\sqrt{3}}$. 13. $\dfrac{8-5\sqrt{2}}{3-2\sqrt{2}}$.

14. $\dfrac{3}{\sqrt{5}-\sqrt{2}}$. 15. $\dfrac{3+\sqrt{5}}{3-\sqrt{5}}$. 16. $\dfrac{\sqrt{5}+\sqrt{3}}{4+\sqrt{15}}$.

Simplify :

17. $\dfrac{1}{x+\sqrt{x^2-1}}+\dfrac{1}{x-\sqrt{x^2-1}}$.

18. $\dfrac{15}{\sqrt{10}+\sqrt{20}+\sqrt{40}-\sqrt{5}-\sqrt{80}}$.

19. $\dfrac{\sqrt{2}(\sqrt{3}+1)(2-\sqrt{3})}{(\sqrt{2}-1)(3\sqrt{3}-5)(2+\sqrt{2})}$.

20. $(3+2\sqrt{2})^{-3}+(3-2\sqrt{2})^{-3}$.

21. $\dfrac{x+\sqrt{x^2-1}}{x-\sqrt{x^2-1}}-\dfrac{x-\sqrt{x^2-1}}{x+\sqrt{x^2-1}}$.

22. $\dfrac{\sqrt{x^2+1}+\sqrt{x^2-1}}{\sqrt{x^2+1}-\sqrt{x^2-1}}+\dfrac{\sqrt{x^2+1}-\sqrt{x^2-1}}{\sqrt{x^2+1}+\sqrt{x^2-1}}$.

9. To find a factor which will rationalise any given binomial surd.

Every binomial surd must be of the form $\sqrt[p]{a}+\sqrt[q]{b}$ or $\sqrt[p]{a}-\sqrt[q]{b}$. For instance, $3^{\frac{3}{5}}+2^{\frac{4}{7}} = \sqrt[5]{3^3}+\sqrt[7]{2^4} = \sqrt[5]{27}+\sqrt[7]{16}$; and $4^{\frac{2}{5}}-3^{\frac{4}{11}} = \sqrt[5]{4^2}-\sqrt[11]{3^4} = \sqrt[5]{16}-\sqrt[11]{81}$.

I. Let it be of the form $\sqrt[p]{a}+\sqrt[q]{b}$.

Let $\sqrt[p]{a} = x$, and $\sqrt[q]{b} = y$; and let n be the L. C. M. of p and q.

Then x^n and y^n are clearly rational quantities.

Hence, if a factor P be found such that $(x+y).\mathrm{P} = x^n+y^n$ or, x^n-y^n, P will be the very factor we seek.

Now, we know that if n be an even integer $x+y$ divides $x^n - y^n$ *only*, and we have $x^n - y^n = (x+y)(x^{n-1} - x^{n-2}y + \dots + xy^{n-2} - y^{n-1})$; and if n be an odd integer $x+y$ divides $x^n + y^n$ *only*, and we have

$$x^n + y^n = (x+y)(x^{n-1} - x^{n-2}y + \dots - xy^{n-2} + y^{n-1}).$$

Hence, if n be even, the rationalising factor and the rational product are respectively $x^{n-1} - x^{n-2}y + \dots + xy^{n-2} - y^{n-1}$, and $x^n - y^n$; whilst if n be odd, these quantities are respectively $x^{n-1} - x^{n-2}y + \dots - xy^{n-2} + y^{n-1}$, and $x^n + y^n$.

II. If the binomial surd be of the form $\sqrt[p]{a} - \sqrt[q]{b}$, taking x, y and n to mean the same as before, we see that a factor Q will rationalise $x - y$ if Q be such that $(x-y).Q = x^n - y^n$ or $x^n + y^n$.

Now, we know that $x - y$ divides $x^n - y^n$ *only* for all values of n, and that $x^n - y^n = (x-y)(x^{n-1} + x^{n-2}y + \dots + xy^{n-2} + y^{n-1})$.

Hence, in this case, whether n be odd or even, the rationalising factor is $x^{n-1} + x^{n-2}y + \dots + xy^{n-2} + y^{n-1}$, and the rational product is $x^n - y^n$.

Example 1. Find the factor which will rationalise $\sqrt[3]{3} + \sqrt{2}$.

The given surd $= 3^{\frac{1}{3}} + 2^{\frac{1}{2}}$.

Putting $x = 3^{\frac{1}{3}}$, $y = 2^{\frac{1}{2}}$, we see that x^6 and y^6 are both rational, and we also know that $x^6 - y^6 = (x+y)(x^5 - x^4y + x^3y^2 - x^2y^3 + xy^4 - y^5)$.

Hence, the required factor

$$= x^5 - x^4y + x^3y^2 - x^2y^3 + xy^4 - y^5$$

$$= (3^{\frac{1}{3}})^5 - (3^{\frac{1}{3}})^4.2^{\frac{1}{2}} + (3^{\frac{1}{3}})^3.(2^{\frac{1}{2}})^2 - (3^{\frac{1}{3}})^2.(2^{\frac{1}{2}})^3 + 3^{\frac{1}{3}}.(2^{\frac{1}{2}})^4 - (2^{\frac{1}{2}})^5$$

$$= 3^{\frac{5}{3}} - 3^{\frac{4}{3}}\,2^{\frac{1}{2}} + 3.2 - 3^{\frac{2}{3}}.2^{\frac{3}{2}} + 3^{\frac{1}{3}}.2^2 - 2^{\frac{5}{2}};$$

and the rational product

$$= (3^{\frac{1}{3}})^6 - (2^{\frac{1}{2}})^6 = 3^2 - 2^3 = 1.$$

Example 2. Reduce $\dfrac{3\sqrt{3} - \sqrt[3]{5}}{\sqrt{3} - \sqrt[3]{5}}$ to an equivalent fraction with a rational denominator.

The denominator of the given expression

$$= 3^{\frac{1}{2}} - 5^{\frac{1}{3}}.$$

Putting $x = 3^{\frac{1}{2}}$, $y = 5^{\frac{1}{3}}$, we see that x^6 and y^6 are both rational, and we also know that

$$x^6 - y^6 = (x - y)(x^5 + x^4y + x^3y^2 + x^2y^3 + xy^4 + y^5).$$

Hence, the rationalising factor of the denominator

$$= x^5 + x^4y + x^3y^2 + x^2y^3 + xy^4 + y^5$$

$$= (3^{\frac{1}{2}})^5 + (3^{\frac{1}{2}})^4 \cdot 5^{\frac{1}{3}} + (3^{\frac{1}{2}})^3 . (5^{\frac{1}{3}})^2 + (3^{\frac{1}{2}})^2 . (5^{\frac{1}{3}})^3 + 3^{\frac{1}{2}} . (5^{\frac{1}{3}})^4 + (5^{\frac{1}{3}})^5$$

$$= 3^{\frac{5}{2}} + 3^2 . 5^{\frac{1}{3}} + 3^{\frac{3}{2}} . 5^{\frac{2}{3}} + 3.5 + 3^{\frac{1}{2}} . 5^{\frac{4}{3}} + 5^{\frac{5}{3}};$$

and the rationalised denominator

$$= (3^{\frac{1}{2}})^6 - (5^{\frac{1}{3}})^6 = 3^3 - 5^2 = 2.$$

Hence, the given expression

$$= \frac{(3^{\frac{3}{2}} - 5^{\frac{1}{3}})(3^{\frac{5}{2}} + 3^2 . 5^{\frac{1}{3}} + 3^{\frac{3}{2}} . 5^{\frac{2}{3}} + 3.5 + 3^{\frac{1}{2}} . 5^{\frac{4}{3}} + 5^{\frac{5}{3}})}{2}$$

Exercise (16).

Find the factor which will rationalise :—

1. $\sqrt{3} - \sqrt[3]{4}$.
2. $\sqrt{6} + \sqrt[4]{3}$.
3. $3 + \sqrt[5]{4}$.
4. $2^{\frac{5}{4}} - 3^{\frac{3}{8}}$.
5. $3^{\frac{4}{3}} + 4^{\frac{5}{6}}$.
6. $4 - \sqrt[7]{5}$.

Reduce to an equivalent fraction with a rational denominator :—

7. $\dfrac{2 - \sqrt[3]{4}}{2 + \sqrt[3]{4}}$.
8. $\dfrac{\sqrt{3} + \sqrt[3]{2}}{\sqrt{3} - \sqrt[3]{2}}$.
9. $\dfrac{\sqrt{8} - \sqrt[4]{32}}{\sqrt{8} + \sqrt[4]{32}}$.
10. $\dfrac{3 - \sqrt[5]{9}}{3 + \sqrt[5]{9}}$.

10. The square root of a rational quantity cannot be partly rational and partly a quadratic surd.

If possible let $\sqrt{n} = a + \sqrt{m}$.

Then, squaring both sides, we have

$$n = a^2 + m + 2a\sqrt{m},$$

whence, $$\sqrt{m} = \frac{n - a^2 - m}{2a}.$$

Thus a surd is equal to a rational quantity, which is impossible.

11. If $a + \sqrt{b} = x + \sqrt{y}$, where a and x are rational, and $\sqrt{b}$ and $\sqrt{y}$ are irrational, then will $a = x$, and $b = y$.

For if a be not equal to x, let $a = x + m$.

Then we have $x + m + \sqrt{b} = x + \sqrt{y}$;

$$\therefore\ m + \sqrt{b} = \sqrt{y}.$$

Thus $\sqrt{y}$ is partly rational and partly a quadratic surd, which is impossible by the last article.

Therefore $a = x$, and consequently $\sqrt{b} = \sqrt{y}$, or $b = y$.

NOTE. It should be distinctly borne in mind that the results proved above are true *only when* $\sqrt{b}$ and $\sqrt{y}$ are *really* irrational. For instance, from the relation $5 + \sqrt{9} = 3 + \sqrt{25}$, we cannot conclude that $5 = 3$ and $9 = 25$.

12. To find the square root of $a + \sqrt{b}$, where $\sqrt{b}$ is a surd.

Let $\sqrt{a + \sqrt{b}} = \sqrt{x} + \sqrt{y}$.

Then, squaring both sides, we have

$$a + \sqrt{b} = x + y + 2\sqrt{xy}.$$

Hence, by the last article,

$$\left.\begin{aligned} a &= x + y \\ \text{and}\quad \sqrt{b} &= 2\sqrt{xy} \end{aligned}\right\} \quad \ldots\ldots \quad (1)$$

$$\begin{aligned} \text{Hence,}\quad a^2 - b &= (x + y)^2 - 4xy \\ &= (x - y)^2, \end{aligned}$$

$$\therefore\quad \sqrt{a^2 - b} = x - y.$$

$$\left.\begin{aligned} \text{Thus we have}\quad x + y &= a \\ \text{and}\quad x - y &= \sqrt{a^2 - b} \end{aligned}\right\}$$

Hence, by addition and subtraction,

$$2x = a + \sqrt{a^2 - b}, \text{ and } 2y = a - \sqrt{a^2 - b};$$

$$\therefore\ x = \tfrac{1}{2}(a + \sqrt{a^2 - b}), \text{ and } y = \tfrac{1}{2}(a - \sqrt{a^2 - b}).$$

Thus $\sqrt{a + \sqrt{b}} = \sqrt{\tfrac{1}{2}(a + \sqrt{a^2 - b})} + \sqrt{\tfrac{1}{2}(a - \sqrt{a^2 - b})}.$

NOTE. From the values of x and y found above it is clear that unless $\sqrt{a^2 - b}$ is rational the square root obtained is by far more complicated than the original expression. Thus the process given above is of no great practical value except when $a^2 - b$ is a perfect square.

Cor. From (1) we have $a - \sqrt{b} = x + y - 2\sqrt{xy} = (\sqrt{x} - \sqrt{y})^2$; $\therefore\ \sqrt{a - \sqrt{b}} = \sqrt{x} - \sqrt{y}$. Thus if $\sqrt{a + \sqrt{b}} = \sqrt{x} + \sqrt{y}$ then will $\sqrt{a - \sqrt{b}} = \sqrt{x} - \sqrt{y}$.

Example 1. Find the square root of $7 + 2\sqrt{10}$.

Let $\sqrt{7 + 2\sqrt{10}} = \sqrt{x} + \sqrt{y}$.

Then, squaring both sides,

$$7 + 2\sqrt{10} = x + y + 2\sqrt{xy}.$$

Hence, $\left.\begin{aligned} x + y &= 7 \\ \text{and } xy &= 10 \end{aligned}\right\}$

The relations are evidently satisfied by the numbers 5 and 2.

Hence, the required root $= \sqrt{5} + \sqrt{2}$.

Example 2. Find the square root of $16 - 5\sqrt{7}$.

Let $\sqrt{16 - 5\sqrt{7}} = \sqrt{x} - \sqrt{y}$.

Then $16 - 5\sqrt{7} = x + y - 2\sqrt{xy}$.

Therefore $\left.\begin{aligned} x + y &= 16 \\ 2\sqrt{xy} &= 5\sqrt{7} \end{aligned}\right\}$.

$$\begin{aligned} \text{Hence, } (x - y)^2 &= (x + y)^2 - 4xy \\ &= 16^2 - (5\sqrt{7})^2 \\ &= 256 - 175 = 81; \\ \therefore\ x - y &= 9. \end{aligned}$$

Thus we have $\left.\begin{aligned} x + y &= 16 \\ x - y &= 9 \end{aligned}\right\}$, whence, $x = \frac{25}{2}$, and $y = \frac{7}{2}$.

Thus the required root $= \sqrt{\frac{25}{2}} - \sqrt{\frac{7}{2}}$.

Example 3. Find the square root of $\sqrt{27} + \sqrt{15}$.

$$\sqrt{27} + \sqrt{15} = 3\sqrt{3} + \sqrt{3}\sqrt{5} = \sqrt{3}(3 + \sqrt{5}).$$

Hence, $\sqrt{\sqrt{27} + \sqrt{15}} = \sqrt[4]{3}\sqrt{3 + \sqrt{5}}$.

Now, proceeding as in the last example, we find that

$$\sqrt{3 + \sqrt{5}} = \sqrt{\frac{1}{2}} + \sqrt{\frac{5}{2}}.$$

Therefore $\sqrt{\sqrt{27} + \sqrt{15}} = \sqrt[4]{3}\left(\sqrt{\frac{1}{2}} + \sqrt{\frac{5}{2}}\right)$.

Exercise (17).

Find the square root of :—

1. $8 + 2\sqrt{15}$.
2. $14 - 6\sqrt{5}$.
3. $17 + 12\sqrt{2}$.
4. $37 - 20\sqrt{3}$.
5. $31 + 4\sqrt{21}$.
6. $73 - 12\sqrt{35}$.
7. $47 + 4\sqrt{33}$.
8. $6 - \sqrt{35}$.
9. $\sqrt{18} - \sqrt{16}$.
10. $\sqrt{32} - \sqrt{24}$.
11. $\sqrt{27} + \sqrt{24}$.
12. $5\sqrt{5} + \sqrt{120}$.
13. $a^2 + 2x\sqrt{a^2 - x^2}$.
14. $2a + 2\sqrt{a^2 - b^2}$.
15. $a + x + \sqrt{2ax + x^2}$.
16. $2x - 1 + 2\sqrt{x^2 - x - 6}$.
17. $x + y + z + 2\sqrt{xz + yz}$.

13. If $\sqrt[3]{a + \sqrt{b}} = x + \sqrt{y}$, then will $\sqrt[3]{a - \sqrt{b}} = x - \sqrt{y}$.

Since $\sqrt[3]{a + \sqrt{b}} = x + \sqrt{y}$,

cubing both sides, we have

$$a + \sqrt{b} = x^3 + 3x^2\sqrt{y} + 3xy + y\sqrt{y}.$$

Hence, equating rational and irrational parts, we have

$$\left.\begin{aligned} a &= x^3 + 3xy \\ \text{and} \quad \sqrt{b} &= 3x^2\sqrt{y} + y\sqrt{y} \end{aligned}\right\}$$

Hence, $a - \sqrt{b} = x^3 - 3x^2\sqrt{y} + 3xy - y\sqrt{y}$.

$$\therefore \quad \sqrt[3]{a - \sqrt{b}} = \sqrt{x} - \sqrt{y}.$$

Cor. Conversely if $\sqrt[3]{a - \sqrt{b}} = x - \sqrt{y}$, then $\sqrt[3]{a + \sqrt{b}} = x + \sqrt{y}$.

Example 1. Find the cube root of $38 + 17\sqrt{5}$.

Let $\sqrt[3]{38 + 17\sqrt{5}} = x + \sqrt{y}$,

then $\sqrt[3]{38 - 17\sqrt{5}} = x - \sqrt{y}$.

Therefore, by multiplication,

$$x^2 - y = \sqrt[3]{1444 - 1445} = \sqrt[3]{-1} = -1.$$

Again, since $\sqrt[3]{38 + 17\sqrt{5}} = x + \sqrt{y}$,

$$\therefore \quad 38 + 17\sqrt{5} = x^3 + 3x^2\sqrt{y} + 3xy + y\sqrt{y},$$

whence $38 = x^3 + 3xy$.

Thus we have
$$\left.\begin{aligned} x^3 + 3xy &= 38 \\ \text{and} \quad x^2 - y &= -1 \end{aligned}\right\}$$

Hence, $x^3 + 3x(x^2 + 1) = 38$

or, $4x^3 + 3x = 38$.

By trial, we find $x = 2$, and hence $y = x^2 + 1 = 5$.

Thus the required root $= 2 + \sqrt{5}$.

NOTE. The method shewn above is practically of no use unless the value of $x^2 - y$ as found above be rational.

Example 2. Find the cube root of $21\sqrt{6} - 23\sqrt{5}$.

$$\begin{aligned} 21\sqrt{6} - 23\sqrt{5} &= 6\sqrt{6}\left(\frac{21}{6} - \frac{23\sqrt{5}}{6\sqrt{6}}\right) \\ &= (\sqrt{6})^3 \cdot \left(\frac{7}{2} - \frac{23}{6}\sqrt{\frac{5}{6}}\right); \end{aligned}$$

$$\therefore \quad \sqrt[3]{21\sqrt{6} - 23\sqrt{5}} = \sqrt{6}\sqrt[3]{\frac{7}{2} - \frac{23}{6}\sqrt{\frac{5}{6}}}.$$

Let $\sqrt[3]{\frac{7}{2} - \frac{23}{6}\sqrt{\frac{5}{6}}} = x - \sqrt{y}$,

then $\sqrt[3]{\frac{7}{2} + \frac{23}{6}\sqrt{\frac{5}{6}}} = x + \sqrt{y}$.

Hence $x^2 - y = \sqrt[3]{\frac{49}{4} - \frac{2645}{216}} = \sqrt[3]{\frac{1}{216}} = \frac{1}{6} \cdots (1)$

also, $\frac{7}{2} - \frac{23}{6}\sqrt{\frac{5}{6}} = x^3 - 3x^2\sqrt{y} + 3xy - y\sqrt{y}$;

$$\therefore \quad x^3 + 3xy = \frac{7}{2} \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot (2)$$

Hence, from (1) and (2), we have

$$x^3 + 3x\left(x^2 - \frac{1}{6}\right) = \frac{7}{2};$$

or, $\quad 8x^3 - x = 7.$

By trial, we find $x = 1$, and $\therefore y = \frac{5}{6}$.

Thus $\sqrt[3]{\frac{7}{2} - \frac{23}{6}\sqrt{\frac{5}{6}}} = 1 - \sqrt{\frac{5}{6}}$;

and $\therefore$ the required root $= \sqrt{6}\left(1 - \sqrt{\frac{5}{6}}\right) = \sqrt{6} - \sqrt{5}$.

Exercise (18).

Find the cube root of :—

1. $19 + 9\sqrt{6}$.
2. $26 - 15\sqrt{3}$.
3. $11\sqrt{5} + 17\sqrt{2}$.
4. $99\sqrt{2} - 59\sqrt{5}$.
5. $264\sqrt{3} + 150\sqrt{6}$.

14. Miscellaneous Examples.

Example 1. Find the square root of $6 + \sqrt{12} - \sqrt{24} - \sqrt{8}$.

Assume $\sqrt{6 + \sqrt{12} - \sqrt{24} - \sqrt{8}} = \sqrt{x} + \sqrt{y} - \sqrt{z}$.

Then we must have

$$6 + \sqrt{12} - \sqrt{24} - \sqrt{8} = x + y + z + 2\sqrt{xy} - 2\sqrt{yz} - 2\sqrt{zx}.$$

If now, x, y, z be such that

$$\left.\begin{aligned} 2\sqrt{xy} &= \sqrt{12} \\ 2\sqrt{yz} &= \sqrt{24} \\ 2\sqrt{zx} &= \sqrt{8} \end{aligned}\right\}$$

and also $x + y + z = 6$, then the required root will be found.

From the first three equations we have

$$\left.\begin{aligned}\sqrt{xy} &= \sqrt{3} \quad \ldots (1)\\ \sqrt{yz} &= \sqrt{2}.\sqrt{3} \ldots (2)\\ \sqrt{zx} &= \sqrt{2} \quad \ldots (3)\end{aligned}\right\} \quad \begin{aligned}&\therefore \text{ by multiplication,}\\ &xyz = 2.3 = 6;\\ &\therefore \sqrt{xyz} = \sqrt{6} \ldots (4)\end{aligned}$$

Dividing (4) by (2), (3) and (1) respectively, we have

$$\sqrt{x} = \sqrt{1}, \quad \sqrt{y} = \sqrt{3}, \quad \sqrt{z} = \sqrt{2};$$

and these values of x, y, z also satisfy the equation $x + y + z = 6$.

Hence the required root $= 1 + \sqrt{3} - \sqrt{2}$.

Example 2. Simplify $\dfrac{2+\sqrt{3}}{\sqrt{2}+\sqrt{2+\sqrt{3}}} + \dfrac{2-\sqrt{3}}{\sqrt{2}-\sqrt{2-\sqrt{3}}}$.

(Bombay University P. E. Paper, 1888).

The 1st term of the given expression

$$\begin{aligned}
&= \frac{\sqrt{2}(2+\sqrt{3})}{2+\sqrt{4+2\sqrt{3}}}\\
&= \frac{\sqrt{2}(2+\sqrt{3})}{2+(1+\sqrt{3})}\\
&= \frac{\sqrt{2}(2+\sqrt{3})}{\sqrt{3}(\sqrt{3}+1)}\\
&= \frac{4+2\sqrt{3}}{\sqrt{2}.\sqrt{3}(\sqrt{3}+1)}\\
&= \frac{(\sqrt{3}+1)^2}{\sqrt{6}(\sqrt{3}+1)}\\
&= \frac{\sqrt{3}+1}{\sqrt{6}};
\end{aligned}$$

and the 2nd term

$$\begin{aligned}
&= \frac{\sqrt{2}(2-\sqrt{3})}{2-\sqrt{4-2\sqrt{3}}}\\
&= \frac{\sqrt{2}(2-\sqrt{3})}{2-(\sqrt{3}-1)}\\
&= \frac{\sqrt{2}(2-\sqrt{3})}{\sqrt{3}(\sqrt{3}-1)}\\
&= \frac{4-2\sqrt{3}}{\sqrt{2}.\sqrt{3}(\sqrt{3}-1)}
\end{aligned}$$

$$= \frac{(\sqrt{3}-1)^2}{\sqrt{6}(\sqrt{3}-1)}$$

$$= \frac{\sqrt{3}-1}{\sqrt{6}}.$$

Hence, the given expression

$$= \frac{\sqrt{3}+1}{\sqrt{6}} + \frac{\sqrt{3}-1}{\sqrt{6}} = \frac{2\sqrt{3}}{\sqrt{6}} = \sqrt{2}.$$

[illegible] $= \dfrac{\sqrt{2}+1}{\sqrt{2}-1}$ and $y = \dfrac{\sqrt{2}-1}{\sqrt{2}+1}$, find the value of $x^2 + xy + y^2$.

We have

$$\left.\begin{aligned} x &= \frac{\sqrt{2}+1}{\sqrt{2}-1} = (\sqrt{2}+1)^2 \\ \text{and } y &= \frac{\sqrt{2}-1}{\sqrt{2}+1} = (\sqrt{2}-1)^2 \end{aligned}\right\}$$

Therefore

$$\left.\begin{aligned} x+y &= 6 \\ \text{and } xy &= 1 \end{aligned}\right\}$$

Hence

$$x^2 + xy + y^2 = (x+y)^2 - xy$$

$$= 6^2 - 1 = 35.$$

Example 4. If $X = \sqrt[3]{r + \sqrt{r^2 + q^3}} + \sqrt[3]{r - \sqrt{r^2 + q^3}}$, find the value of $X^3 + 3qX - 2r$.

(Madras University F. A. Paper, 1885).

Putting a for $\sqrt[3]{r + \sqrt{r^2 + q^3}}$ and b for $\sqrt[3]{r - \sqrt{r^2 + q^3}}$,

we have $X = a + b$;

$$\begin{aligned} \therefore \quad X^3 &= a^3 + b^3 + 3ab(a+b) \\ &= a^3 + b^3 + 3ab.X \\ &= (r + \sqrt{r^2 + q^3}) + (r - \sqrt{r^2 + q^3}) \\ &\quad + 3\sqrt[3]{r^2 - (r^2 + q^3)}.X \\ &= 2r + 3\sqrt[3]{-q^3}.X \\ &= 2r - 3qX. \end{aligned}$$

$$\therefore \quad X^3 + 3qX - 2r = 0.$$

Example 5. If $(x+\sqrt{x^2-bc})(y+\sqrt{y^2-ca})(z+\sqrt{z^2-ab})$ $=(x-\sqrt{x^2-bc})(y-\sqrt{y^2-ca})(z-\sqrt{z^2-ab})$, shew that each of these expressions $=\pm abc$. (Bombay University P E. Paper, 1889.)

Let each of the given expressions = K, then we have

$$\text{K} = (x+\sqrt{x^2-bc})(y+\sqrt{y^2-ca})(z+\sqrt{z^2-ab})$$

and also $$\text{K} = (x-\sqrt{x^2-bc})(y-\sqrt{y^2-ca})(z-\sqrt{z^2-ab}).$$

$$\therefore \quad \text{K}^2 = \{x^2-(x^2-bc)\}\{y^2-(y^2-ca)\}\{z^2-(z^2-ab)\}$$
$$= bc.\,ca.\,ab = a^2b^2c^2\,;$$
$$\therefore \quad \text{K} = \pm abc,$$

i.e., each of the given expressions $=\pm abc$.

Example 6. If $ax=\dfrac{2pq}{1+q^2}$, find the value of

$$\frac{\sqrt{\dfrac{p}{a}+x}+\sqrt{\dfrac{p}{a}-x}}{\sqrt{\dfrac{p}{a}+x}-\sqrt{\dfrac{p}{a}-x}}.$$

The given expression

$$=\frac{\left(\sqrt{\dfrac{p}{a}+x}+\sqrt{\dfrac{p}{a}-x}\right)^2}{\left(\dfrac{p}{a}+x\right)-\left(\dfrac{p}{a}-x\right)}$$

$$=\frac{\dfrac{2p}{a}+2\sqrt{\dfrac{p^2}{a^2}-x^2}}{2x}=\frac{p}{ax}+\sqrt{\frac{p^2}{a^2x^2}-1}.$$

Now since $ax=\dfrac{2pq}{1+q^2}$, $\therefore \dfrac{p}{ax}=\dfrac{1+q^2}{2q}$.

Hence, the given expression

$$=\frac{1+q^2}{2q}+\sqrt{\frac{(1+q^2)^2}{4q^2}-1}=\frac{1+q^2}{2q}+\sqrt{\frac{1+q^4-2q^2}{4q^2}}$$

$$=\frac{1+q^2}{2q}+\frac{1-q^2}{2q}=\frac{2}{2q}=\frac{1}{q}$$

Example 7. If $$\sqrt{(x-\sqrt{a^2-b^2})^2+y^2}+\sqrt{(x+\sqrt{a^2-b^2})^2+y^2} = 2a,$$

shew that $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1.$ (Madras University F. A. Paper, 1886)

By transposition, we have

$$\sqrt{(x-\sqrt{a^2-b^2})^2+y^2} = 2a-\sqrt{(x+\sqrt{a^2-b^2})^2+y^2}.$$

Squaring both sides, we have

$$(x-\sqrt{a^2-b^2})^2+y^2 = 4a^2+(x+\sqrt{a^2-b^2})^2+y^2 - 4a\sqrt{(x+\sqrt{a^2-b^2})^2+y^2},$$

or, $$-2x\sqrt{a^2-b^2} = 4a^2+2x\sqrt{a^2-b^2} - 4a\sqrt{(x+\sqrt{a^2-b^2})^2+y^2};$$

therefore, by transposition,

$$4a\sqrt{(x+\sqrt{a^2-b^2})^2+y^2} = 4a^2+4x\sqrt{a^2-b^2}.$$

Removing the common factor 4 from both sides and squaring, we have

$$a^2(x+\sqrt{a^2-b^2})^2+a^2y^2 = a^4+x^2(a^2-b^2)+2a^2x\sqrt{a^2-b^2};$$

or, $$a^2(a^2-b^2)+a^2y^2 = a^4-b^2x^2,$$

or, $$b^2x^2+a^2y^2 = a^2b^2.$$

Hence, dividing both sides by a^2b^2, we have

$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1.$$

Otherwise :—

We have

$$\sqrt{(x-\sqrt{a^2-b^2})^2+y^2}+\sqrt{(x+\sqrt{a^2-b^2})^2+y^2} = 2a \quad \ldots (1)$$

and also *identically*

$$\{(x-\sqrt{a^2-b^2})^2-y^2\}-\{(x+\sqrt{a^2-b^2})^2+y^2\} = -4x.\sqrt{a^2-b^2} \quad .. (2)$$

Dividing (2) by (1), we have

$$\sqrt{(x-\sqrt{a^2-b^2})^2+y^2}-\sqrt{(x+\sqrt{a^2-b^2})^2+y^2}$$
$$=-\frac{2x}{a}\sqrt{a^2-b^2} \quad \ldots \quad (3)$$

Adding (1) and (3), we have

$$2\sqrt{(x-\sqrt{a^2-b^2})^2+y^2} = 2a-\frac{2x}{a}\sqrt{a^2-b^2},$$

or, $$a\sqrt{(x-\sqrt{a^2-b^2})^2+y^2} = a^2-x\sqrt{a^2-b^2}.$$

Squaring both sides, we have

$$a^2(x-\sqrt{a^2-b^2})^2+a^2y^2 = a^4+x^2(a^2-b^2)$$
$$-2a^2x\sqrt{a^2-b^2},$$

or, $$a^2(a^2-b^2)+a^2y^2 = a^4-b^2x^2,$$

or, $$b^2x^2+a^2y^2 = a^2b^2.$$

Hence, dividing both sides by a^2b^2, we have

$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1.$$

Exercise (10).

Find the square root of :—

1. $8+2\sqrt2+2\sqrt5+2\sqrt{10}$. **2.** $10+2\sqrt6+2\sqrt{10}+2\sqrt{15}$.

3. $11+6\sqrt2+4\sqrt3+2\sqrt6$. **4.** $21-4\sqrt5+8\sqrt3-4\sqrt{15}$.

Simplify :—

5. $\dfrac{\sqrt3+\sqrt2}{\sqrt2+\sqrt{2+\sqrt3}}-\dfrac{\sqrt3-\sqrt2}{\sqrt2-\sqrt{2+\sqrt3}}$.

6. Given $\sqrt5=2{\cdot}23607$, find the value of $\dfrac{\sqrt3-\sqrt5}{\sqrt2+\sqrt{7-3\sqrt5}}$.

7. $\dfrac{2\sqrt8\sqrt{3+\sqrt5}}{4+\sqrt{10}-\sqrt2}$. (Bombay University P. E. Paper, 1887.)

8. Find the value of

$$\frac{1+x}{1+\sqrt{1+x}}+\frac{1-x}{1-\sqrt{1-x}}, \text{ when } x=\frac{\sqrt3}{2}.$$

9. Find the value of $\dfrac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}$, when $x = \dfrac{\sqrt{3}}{2}$.

(Bombay University P. E. Paper, 1883.)

10. Find the value of $\dfrac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}}$, when $x = \dfrac{2ab}{b^2+1}$.

11. Find the value of

$$\frac{2a\sqrt{1+x^2}}{x+\sqrt{1+x^2}}, \text{ when } x = \frac{1}{2}\left(\sqrt{\frac{a}{b}}-\sqrt{\frac{b}{a}}\right).$$

12. Reduce $\dfrac{4}{\sqrt[3]{9}-\sqrt[3]{3}+1}$, to an equivalent fraction with a rational denominator.

13. Simplify $(72-32\sqrt{5})^{-\frac{2}{3}}-(72+32\sqrt{5})^{-\frac{2}{3}}$.

14. If $x = \dfrac{1}{3-\sqrt{8}}$, $y = \dfrac{1}{3+\sqrt{8}}$, find the value of $3x^2+23xy+3y^2$.

15. If $x = \dfrac{\sqrt{7}-\sqrt{5}}{\sqrt{7}+\sqrt{5}}$, $y = \dfrac{\sqrt{7}+\sqrt{5}}{\sqrt{7}-\sqrt{5}}$, find the value of x^3+y^3.

CHAPTER IV.

RATIO AND PROPORTION.

1. Definitions. The ratio of one quantity to another of the same kind is defined to be the *abstract number* (integral or fractional) which expresses what multiple, part or parts, the former is of the latter. Thus,

since 2 hours is a portion of time which is three times as large as 40 minutes, the ratio of 2 hours to 40 minutes = 3;

since a length of 9 inches is a fourth part of 3 feet, the ratio of 9 inches to 3 feet = $\frac{1}{4}$;

since the sum of £1. 4*s*. is obtained by dividing 18*s*. into 3 equal parts and taking 4 of those parts, the ratio of £1. 4*s*. to 18*s*. = $\frac{4}{3}$;

and so on.

Hence it is clear that the ratio of one *concrete* quantity to another (of the same kind) is a fraction, of which the numerator and denominator are respectively the *measures* of those quantities (*referred to one and the same unit*) ; and the ratio of one *abstract* quantity to another is a fraction of which the numerator and denominator are respectively the quantities themselves.

The ratio of any number a to any other number b is usually expressed by the notation $a : b$; thus $a : b$ is the same as $\frac{a}{b}$. The quantities a and b are respectively called the **antecedent** and the **consequent** (or the *first term* and the *second term*) of the ratio $a : b$.

A ratio is called a ratio of *greater inequality*, of *less inequality* , or of *equality*, according as it is *greater* than, *less* than, or *equal to*, 1.

NOTE. Since a ratio is only a fraction there is no difficutly in seeing that the value of a ratio remains unaltered if its terms be multiplied or divided by the same number. Thus the ratios $3 : 4$, $6 : 8$, $15 : 20$, and $3n : 4n$ are all equal to one another. Hence also two or more ratios can be easily compared with one another ; for instance, the ratios $2 : 3$, $4 : 5$, and $7 : 10$ being respectively equivalent to $20 : 30$, $24 : 30$ and $21 : 30$, we see at once that the second of them is the greatest and the first the least.

2. A ratio of less inequality is increased, and a ratio of greater inequality is diminished, by adding the same number to both its terms.

Let $\frac{a}{b}$ be any given ratio, and let $\frac{a+x}{b+x}$ be the new ratio formed by adding x to both its terms.

Then, $$\frac{a+x}{b+x} - \frac{a}{b} = \frac{x(b-a)}{b(b+x)},$$

and therefore it is positive or negative according as a is less or greater than b.

$$\left.\begin{array}{ll} \text{Hence, if } a<b, & \dfrac{a+x}{b+x} > \dfrac{a}{b} \\ \text{and} \quad \text{if } a>b, & \dfrac{a+x}{b+x} < \dfrac{a}{b} \end{array}\right\} \text{which proves the proposition.}$$

NOTE. Similarly it can be proved that *a ratio of less inequality is diminished, and a ratio of greater inequality is increased by subtracting from both its terms any number which is less than each of those terms.* This is left as an exercise for the student.

3. Composition of ratios. The ratio of the product of the antecedents of any number of ratios to the product of their consequents is called the ratio *compounded* of the given ratios. Thus, the ratio compounded of the three ratios—

$$3 : 4, \qquad 8 : 9, \qquad 2x : 3y,$$

is

$$3 \times 8 \times 2x : 4 \times 9 \times 3y,$$

$$\text{or, } 4x : 9y.$$

When the ratio $a : b$ is compounded with itself the resulting ratio $a^2 : b^2$ is called the *duplicate* ratio of $a : b$. Similarly $a^3 : b^3$ is called the *triplicate* ratio of $a : b$; $a^{\frac{1}{2}} : b^{\frac{1}{2}}$ is called the *subduplicate* ratio of $a : b$; and $a^{\frac{1}{3}} : b^{\frac{1}{3}}$ is called the *subtriplicate* ratio of $a : b$.

4. Approximate values of ratios. If x is *very small* compared with a, to show that the ratio $(a+x)^2 : a^2$ is approximately the same as $a + 2x : a$.

We have $\dfrac{(a+x)^2}{a^2} = \dfrac{a^2 + 2ax + x^2}{a^2} = 1 + \dfrac{2x}{a} + \dfrac{x^2}{a^2}$,

and $\therefore$ *approximately* $= 1 + \dfrac{2x}{a}$,

since $\dfrac{x^2}{a^2}\left(\text{which} = \dfrac{x}{a} \times \dfrac{x}{a}\right)$ is very small compared with $\dfrac{2x}{a}$, and smaller still than 1.

Thus approximately we have

$$\frac{(a+x)^2}{a^2} = 1 + \frac{2x}{a} = \frac{a+2x}{a} \quad \ldots (1)$$

Cor. From (1) we have $\sqrt{\dfrac{a+2x}{a}} = \dfrac{a+x}{a}$. Hence, if x is very small compared with a, we have

$$\sqrt{a+x} : \sqrt{a} = a + \tfrac{1}{2}x : a.$$

NOTE. By a similar mode of reasoning it can be shown that when x is very small compared with a, $(a+x)^3 : a^3 = a+3x : a$; $(a+x)^4 : a^4 = a+4x : a$; $(a+x)^{\frac{1}{3}} : a^{\frac{1}{3}} = a + \frac{1}{3}x : a$, and so on.

5. Incommensurable quantities. If two quantities be such that their ratio cannot be exactly expressed by the ratio of two integers, they are said to be *incommensurable quantities*. Thus $\sqrt{3}$ and 2 are incommensurable quantities, since no two integers can be found whose ratio is *exactly* equal to $\sqrt{3} : 2$.

Although the ratio of two incommensurable quantities cannot be *exactly* expressed by the ratio of two integers, we can always find two integers however, whose ratio differs from such a ratio by as small a quantity as we please.

For instance, $\frac{\sqrt{3}}{2} = \frac{1{\cdot}73205\ \ldots\ldots}{2} = {\cdot}86602\ldots\ldots\ldots\ldots$

and therefore $\frac{\sqrt{3}}{2} > \frac{86602}{100000}$ and $< \frac{86603}{100000}$;

thus $\sqrt{3} : 2$ differs from either $86602 : 100000$ or $86603 : 100000$ by even less than a hundred-thousandth part of unity. A further approximation might evidently be arrived at by calculating the value of $\sqrt{3}$ to more places of decimals.

NOTE. Any number which cannot be *exactly* expressed as the ratio of two whole numbers is also sometimes called incommensurable. From this point of view every surd is an incommensurable quantity.

Examples.

Example 1. Two numbers are in the ratio of 2 to 3, and if 9 be added to each they are in the ratio of 3 to 4. Find the numbers.

Since the numbers are in the ratio of 2 to 3, evidently we can represent them by $2x$ and $3x$ respectively.

Hence, by the second condition, we have

$$\frac{2x+9}{3x+9} = \frac{3}{4}.$$

Hence $8x+36 = 9x+27$, whence $x = 9$.

Therefore the numbers are 18 and 27.

Example 2. What is the ratio of x to y, if

$$10x+3y : 5x+2y = 9 : 5\,?$$

We have $\frac{9}{5} = \frac{10x+3y}{5x+2y} = \frac{10.\frac{x}{y}+3}{5.\frac{x}{y}+2}.$

Hence, $45.\frac{x}{y}+18 = 50\,\frac{x}{y}+15,$

$$\therefore\ 5.\frac{x}{y} = 3,\quad \therefore\ \frac{x}{y} = \frac{3}{5}$$

Example 3. Which is the greater

$$x^3+y^3 : x^2+y^2, \quad \text{or} \quad x^2+y^2 : x+y\,?$$

We have

$$\frac{x^3+y^3}{x^2+y^2}-\frac{x^2+y^2}{x+y}=\frac{xy^3+x^3y-2x^2y^2}{(x^2+y^2)(x+y)}$$
$$=\frac{xy(x-y)^2}{(x^2+y^2)(x+y)},$$

which evidently is a positive quantity, since $(x-y)^2$ is positive whether x is greater or less than y. Hence,

$$x^3+y^3 : x^2+y^2 > x^2+y^2 : x+y.$$

Example 4. Two armies number 11000 and 7000 men respectively; before they fight each is reinforced by 1000 men; in favour of which army is the increase?

The new strength of the 1st army : its original strength

$$= 12000 : 11000 = 12 : 11,$$

whilst, the new strength of the 2nd army : its original strength

$$= 8000 : 7000 = 8 : 7.$$

Now, since $12 : 11 = 84 : 77$,

and $8 : 7 = 88 : 77$,

it is clear that $8 : 7 > 12 : 11$.

Thus, *compared* with the original strength the new strength of the second army is greater than that of the first.

Hence the increase is in favour of the 2nd army.

Exercise (20).

Which is the greater :—

1. $4 : 5$ or $7 : 8$? **2.** $7 : 10$ or $11 : 14$

3. $9 : 5$ or $13 : 8$? **4.** $22 : 27$ or $32 : 45$?

5. $28 : 39$ or $49 : 65$?

Find the ratio compounded of :—

6. $a : b$, $b : c$ and $c : d$.

7. $3 : 5$, $7 : 9$ and $15 : 28$

8. $a+x : a-x$, $a^2+x^2 : (a+x)^2$, and $(a^2-x^2)^2 : a^4-x^4$.

9. $16 : 5$, the triplicate ratio of $5 : 4$, and the subduplicate ratio of $9 : 4$.

10. 25 : 18, the subduplicate ratio of 81 : 49, the triplicate ratio of 2 : 3, and the duplicate ratio of 7 : 5.

11. If $2x+5y : 3x+5y=9 : 10$, find $x : y$.

12. If $x : y = 3 : 4$, find the value of $5x+9y : 16x+5y$.

13. Two numbers are in the ratio of 7 : 8, and their sum is 135. Find the numbers.

14. Find the two numbers which are in the ratio of 5 : 3 and whose difference is 34.

15. Two numbers are in the ratio of 4 : 5, and if 7 be added to each the sums are in the ratio of 5 : 6 Find the numbers.

16. Two numbers are in the ratio of 7 : 9, and if 10 be subtracted from each the remainders are in the ratio of 8 : 11. Find the numbers.

17. For what value of x will the ratio $23+x : 19+x$ be equal to 2 ?

18. What number must be added to each term of the ratio 27 : 35 that it may become equal to 5 : 6 ?

19. What number must be taken from each term of the ratio 29 : 38 that it may become equal to 4 : 7 ?

20. What quantity must be added to each of the terms of the ratio $a : b$, that it may become equal to $c : d$?

21. Show that if $a>x$, the ratio $a^2-x^2 : a^2+x^2$ is greater than the ratio $a-x : a+x$.

22. Show that the ratio $a^2+b^2 : a+b$ is less than the ratio $a^2-b^2 : a-b$.

Find approximately the value of :—

23. $(226)^3 : (225)^3$.

24. $\sqrt{3546} : \sqrt{3542}$.

25. A, B, C are three school boys, getting monthly allowances of Rs 15, Rs. 20, and Rs. 25 respectively ; out of these amounts they respectively spend Rs. $8\frac{3}{4}$, Rs. $11\frac{1}{4}$, and Rs. $15\frac{5}{8}$ per month. Which of them is the most frugal ?

PROPORTION.

6. Definitions. Four quantities are said to be *proportionals* when the ratio of the first to the second is equal to the ratio of the third to the fourth. Thus a, b, c, d are proportionals, if $a : b = c : d$. This is often expressed as $a : b :: c : d$, and is read "a is to b as c is to d."

The terms a and b are called the *extremes*, and the terms b and c, the *means*. The term d is also called the *fourth proportional* to a, b, c.

Three or more quantities are said to be in continued proportion when the first is to the second, as the second is to the third, as the third is to the fourth, and so on. Thus a, b, c, d are in continued proportion when $a : b = b : c = \; : c \; d$

If three quantities a, b, c are in continued proportion, ($a : b :: b : c$), then b is called the *mean proportional* between a and c, and c is called the *third proportional to* a and b.

7. If $a : b :: c : d$, then will $ad = bc$.

Since $$\frac{a}{b} = \frac{c}{d},$$

multiplying both sides by bd, we have $ad = bc$.

Thus, *if four quantities are proportionals, the product of the extremes is equal to the product of the means.*

[Conversely, if $ad = bc$, then $a : b :: c : d$. This is obvious by dividing both sides of the equation by bd.]

Cor. If $a : b :: b : c$, then $ac = b^2$; *i.e., if three quantities are in continued proportion, the product of the extremes is equal to the square of the mean.*

NOTE. From the results above established we can at once find a third proportional to, or a mean proportional between, two given quantities, as well as a fourth proportional to three given quantities.

Exercise (21).

Find a third proportional to :—

1. 9, 6. **2.** 8, 12. **3.** 6, 15. **4.** 16, 24.

Find the fourth proportional to :—

5. 6, 8, 15. **6.** 14, 24, 35. **7.** ·0014, 1·4, ·02.

Find a mean proportional between :—

8. 4, 9. **9.** 7, 28. **10.** 6, 54.

8. If $a : b :: b : c$, then $a : c :: a^2 : b^2$.

For $$\frac{a}{b} = \frac{b}{c};$$

$$\therefore \quad \frac{a}{b} \times \frac{b}{c} = \frac{a}{b} \times \frac{a}{b}$$

$$\text{or,} \quad \frac{a}{c} = \frac{a^2}{b^2}.$$

Thus, *if three quantities are in continued proportion, the first is to the third in the duplicate ratio of the first to the second.*

NOTE. Similarly, if $a : b = b : c = c : d$, it can be easily proved that $a : d = a^3 : b^3$, which is left as an exercise for the student.

9. If $a : b :: c : d$, then $b : a :: d : c$.

For $$\frac{a}{b} = \frac{c}{d};$$

$$\therefore \quad 1 \div \frac{a}{b} = 1 \div \frac{c}{d}, \text{ whence } \frac{b}{a} = \frac{d}{c}.$$

Thus, *if four quantities be proportionals, they are also proportionals when taken inversely.*

This operation is called **Invertendo.**

10. If $a : b :: c : d$, then $a : c :: b : d$.

For $$\frac{a}{b} = \frac{c}{d};$$

$$\therefore \quad \frac{a}{b} \times \frac{b}{c} = \frac{c}{d} \times \frac{b}{c}, \text{ or, } \frac{a}{c} = \frac{b}{d}.$$

Thus, *if four quantities be proportionals, they are proportionals when taken alternately.*

This operation is called **Alternando.**

11. If $a : b :: c : d$, then $a+b : b :: c+d : d$.

For $$\frac{a}{b} = \frac{c}{d};$$

$$\therefore \quad \frac{a}{b} + 1 = \frac{c}{d} + 1, \text{ or, } \frac{a+b}{b} = \frac{c+d}{d}.$$

Thus *when four quantities are proportionals, the first together with the second is to the second as the third together with the fourth is to the fourth.*

This operation is called **Componendo.**

12. If $a : b :: c : d$, then $a - b : b :: c - d : d$.

For
$$\frac{a}{b} = \frac{c}{d};$$

$$\therefore \quad \frac{a}{b} - 1 = \frac{c}{d} - 1, \text{ or, } \frac{a-b}{b} = \frac{c-d}{d}.$$

Thus, *when four quantities are proportionals, the excess of the first over the second is to the second as the excess of the third over the fourth is to the fourth.*

This operation is called **Dividendo.**

***Cor.* If $a : b :: c : d$, then $a : a - b :: c : c - d$.**

For
$$\frac{a-b}{b} = \frac{c-d}{d}; \therefore \text{ inversely, } \frac{b}{a-b} = \frac{d}{c-d}.$$

Hence,
$$\frac{b}{a-b} \times \frac{a}{b} = \frac{d}{c-d} \times \frac{c}{d},$$

or,
$$\frac{a}{a-b} = \frac{c}{c-d}.$$

Thus, *when four quantities are proportionals, the first is to the excess of the first above the second as the third is to the excess of the third above the fourth.*

This operation is called **Convertendo.**

13. If $a : b :: c : d$, then $a + b : a - b :: c + d : c - d$.

From Art 12, $\frac{a+b}{b} = \frac{c+d}{d}$...... (1)

From Art 13, $\frac{a-b}{b} = \frac{c-d}{d}$............(2)

Hence, dividing (1) by (2), $\frac{a+b}{a-b} = \frac{c+d}{c-d}$.

Otherwise :—

Since $\frac{a}{b} = \frac{c}{d}$, putting each $= k$, we hrve

$$\left.\begin{aligned} a &= bk \\ \text{and } c &= dk \end{aligned}\right\}.$$

Hence, $$\frac{a+b}{a-b} = \frac{b(k+1)}{b(k-1)} = \frac{k+1}{k-1};$$

also, $$\frac{c+d}{c-d} = \frac{d(k+1)}{d(k-1)} = \frac{k+1}{k-1}.$$

Therefore $$\frac{a+b}{a-b} = \frac{c+d}{c-d}.$$

Thus, *when four quantities are proportionals, the sum of the first and second is to their difference as the sum of the third and fourth is to their difference.*

This result is often spoken of as **Componendo and Dividendo.**

NOTE. The result proved in this article is of great use in solving a certain class of equations. This will be illustrated in some of the following examples.

Example 1. Solve $$\frac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}} = b.$$

By componendo and dividendo, we have

$$\frac{2\sqrt{a+x}}{2\sqrt{a-x}} = \frac{b+1}{b-1}.$$

Hence, $$\frac{a+x}{a-x} = \left(\frac{b+1}{b-1}\right)^2 = \frac{b^2+2b+1}{b^2-2b+1}$$

Again applying componendo and dividendo,

$$\frac{2a}{2x} = \frac{2(b^2+1)}{4b}$$

or, $$\frac{a}{x} = \frac{b^2+1}{2b};$$

$$\therefore \quad x(b^2+1) = 2ab, \ \therefore \ x = \frac{2ab}{b^2+1}.$$

Example 2. Solve $\dfrac{1-ax}{1+ax}\sqrt{\dfrac{1+bx}{1-bx}} = 1.$

We have $$\sqrt{\frac{1+bx}{1-bx}} = \frac{1+ax}{1-ax};$$

$$\therefore \quad \frac{1+bx}{1-bx} = \frac{1+2ax+a^2x^2}{1-2ax+a^2x^2}.$$

Hence, by componendo and dividendo,

$$\frac{1}{bx} = \frac{1+a^2x^2}{2ax};$$

$$\therefore \quad b(1+a^2x^2) = 2a,$$

$$\text{or, } a^2x^2 = \frac{2a}{b} - 1;$$

$$\therefore \quad x = \frac{1}{a}\sqrt{\frac{2a}{b} - 1}.$$

Example 3. Find the value of $\dfrac{x+2a}{x-2a} + \dfrac{x+2b}{x-2b}$,

when $x = \dfrac{4ab}{a+b}$.

From the given relation, we have

$$\frac{x}{2a} = \frac{2b}{a+b}, \text{ and } \frac{x}{2b} = \frac{2a}{a+b}.$$

Hence, by componendo and dividendo,

$$\frac{x+2a}{x-2a} = \frac{a+3b}{b-a}, \text{ and } \frac{x+2b}{x-2b} = \frac{3a+b}{a-b}.$$

Hence, the given expression

$$= \frac{-(a+3b)}{a-b} + \frac{3a+b}{a-b}$$

$$= \frac{2(a-b)}{a-b} = 2.$$

Example 4. If $(pa+qb+rc+sd)(pa-qb-rc+sd)$ $= (pa-qb+rc-sd)(pa+qb-rc-sd)$, shew that bc, ad, ps, qr are in proportion. (Bombay University P. E. Paper, 1890.)

From the given relation, we have

$$\frac{pa+qb+rc+sd}{pa+qb-rc-sd} = \frac{pa-qb+rc-sd}{pa-qb-rc+sd}.$$

Hence, by componendo and dividendo,

$$\frac{pa+qb}{rc+sd} = \frac{pa-qb}{rc-sd},$$

$$\therefore \quad \frac{pa+qb}{pa-qb} = \frac{rc+sd}{rc-sd}; \quad \text{[alternately]}$$

whence, by a second application of componendo and dividendo,

$$\frac{pa}{qb} = \frac{rc}{sd}.$$

Hence, $qb.rc = pa.sd$,

or, $bc.qr = ad.ps$; $\quad \therefore \quad bc : ad :: ps : qr.$

Example 5. If $x = \dfrac{\sqrt[3]{m+1}+\sqrt[3]{m-1}}{\sqrt[3]{m+1}-\sqrt[3]{m-1}}$, shew that

$$x^3 - 3mx^2 + 3x - m = 0.$$

From the given relation, by componendo and dividendo, we have

$$\frac{x+1}{x-1} = \frac{\sqrt[3]{m+1}}{\sqrt[3]{m-1}};$$

$$\therefore \quad \frac{m+1}{m-1} = \frac{(x+1)^3}{(x-1)^3} = \frac{x^3+3x^2+3x+1}{x^3-3x^2+3x-1}.$$

Hence, by a second application of componendo and dividendo,

$$\frac{m}{1} = \frac{x^3+3x}{3x^2+1};$$

$$\therefore \quad m(3x^2+1) = x^3+3x,$$

whence, $x^3 - 3mx^2 + 3x - m = 0$.

Exercise (22).

Solve the following equations :—

1. $\left.\begin{aligned} \frac{x+y}{x-y} &= 5 \\ 2x+3y &= 36 \end{aligned}\right\}$

2. $\left.\begin{aligned} \frac{3x-5y}{3x+5y} &= \tfrac{1}{4} \\ 4x-9y &= 19 \end{aligned}\right\}$

3. $\left.\begin{aligned} \frac{5x-7y}{5x+7y} &= \tfrac{1}{7} \\ 3x-5y &= 18 \end{aligned}\right\}$

4. $16\left(\dfrac{a-x}{a+x}\right)^3 = \dfrac{a+x}{a-x}.$

5. $\dfrac{2x+\sqrt{4x^2-1}}{2x-\sqrt{4x^2-1}} = 4.$ 6. $\dfrac{1-\sqrt{1-x}}{1+\sqrt{1-x}} = \frac{1}{3}.$

7. $\dfrac{\sqrt{36x+1}+\sqrt{36x}}{\sqrt{36x+1}-\sqrt{36x}} = 9.$ 8. $\dfrac{1+x+x^2}{1-x+x^2} = \dfrac{62}{63}\cdot\dfrac{1+x}{1-x}.$

9. $\dfrac{\sqrt{5}+\sqrt{5-x}}{\sqrt{5}-\sqrt{5-x}} = 5.$ 10. $\dfrac{a+x+\sqrt{a^2-x^2}}{a+x-\sqrt{a^2-x^2}} = \dfrac{b}{x}.$

11. $\dfrac{a^{\frac{1}{2}}-\{a-(a^2-ax)^{\frac{1}{2}}\}^{\frac{1}{2}}}{a^{\frac{1}{2}}+\{a-(a^2-ax)^{\frac{1}{2}}\}^{\frac{1}{2}}} = b.$

Prove that $a : b :: c : d$—

12. If $(a+3b+2c+6d)(a-3b-2c+6d)$
$= (a-3b+2c-6d)(a+3b-2c-6d).$

13. If $(2a+b+4c+2d)(2a-b-4c+2d)$
$= (2a-b+4c-2d)(2a+b-4c-2d).$

14. If $\dfrac{7x-5y}{7x+5y} = \dfrac{11}{31}$, find the value of $\dfrac{5x^2-4y^2}{5x^2+4y^2}$.

15. If $x = \dfrac{\sqrt{2a+3b}+\sqrt{2a-3b}}{\sqrt{2a+3b}-\sqrt{2a-3b}}$, show that

$$3bx^2-4ax+3b = 0.$$

16. If $x = \dfrac{2\sqrt{24}}{\sqrt{2}+\sqrt{3}}$, find the value of $\dfrac{x+\sqrt{8}}{x-\sqrt{8}}+\dfrac{x+\sqrt{12}}{x-\sqrt{12}}$.

14. An Important Theorem. If $\dfrac{a}{b} = \dfrac{c}{d} = \dfrac{e}{f}$, then each of these ratios $= \left(\dfrac{pa^n+qc^n+re^n}{pb^n+qd^n+rf^n}\right)^{\frac{1}{n}}$, where p, q, r, n, are any quantities whatever.

Supposing each of the given ratios $= k$, we have $a = bk$, $c = dk$, $e = fk$.

Hence,

$$\left.\begin{aligned} pa^n &= p(bk)^n = pb^n.k^n \\ qc^n &= q(dk)^n = qd^n.k^n \\ re^n &= r(fk)^n = rf^n.k^n \end{aligned}\right\} \begin{aligned} \therefore\ & pa^n+qc^n+re^n \\ &= (pb^n+qd^n+rf^n)k^n \end{aligned}$$

whence, $k^n = \dfrac{pa^n+qc^n+re^n}{pb^n+qd^n+rf^n}$, and $\therefore \quad k = \left(\dfrac{pa^n + qc^n + re^n}{pb^n + qd^n + rf^n}\right)^{\frac{1}{n}}$, which proves the proposition.

Cor. As a particular case, if p, q, r, n be each equal to 1, we have each of the given ratios $= \dfrac{a + c + e}{b + d + f}$.

Similarly, giving different sets of values to p, q, r, n, several particular cases may be at once deduced.

NOTE. What is proved above for *three* equal ratios is obviously true for *any number* of equal ratios, the same reasoning being applicable to all cases. It is always a very good exercise for the student however to work out independently every fresh example of this class, applying the mode of demonstration illustrated above. Hence an exercise is added below with a recommendation to the student that he should find the result in *each* case *without using the formula* established in this article.

Exercise (23).

If $\dfrac{a}{b} = \dfrac{c}{d} = \dfrac{e}{f}$, prove that each of these ratios is equal to—

1. $\dfrac{a - c + e}{b - d + f}$. 2. $\dfrac{a + 3c - 5e}{b + 3d - 5f}$. 3. $\dfrac{5a - 7c - 13e}{5b - 7d - 13f}$. 4. $\dfrac{ka + lc + me}{kb + ld + mf}$.

5. $\left(\dfrac{a^2 + c^2 + e^2}{b^2 + d^2 + f^2}\right)^{\frac{1}{2}}$. 6. $\left(\dfrac{a^3 - 2c^3 + 3e^3}{b^3 - 2d^3 + 3f^3}\right)^{\frac{1}{3}}$. 7. $\dfrac{\sqrt[3]{a^3 + c^3 + e^3}}{\sqrt[3]{b^3 + d^3 + f^3}}$.

If $\dfrac{a}{b} = \dfrac{c}{d} = \dfrac{e}{f} = \dfrac{g}{h}$, prove that each of these ratios is equal to—

8. $\left(\dfrac{a^{-1} + c^{-1} + e^{-1} + g^{-1}}{b^{-1} + d^{-1} + f^{-1} + h^{-1}}\right)^{-1}$. 9. $\sqrt[4]{\dfrac{a^4 - 2c^4 + 3e^4 - 4g^4}{b^4 - 2d^4 + 3f^4 - 4h^4}}$.

10. $\sqrt{\left(\dfrac{3a^{-2} - 7c^{-2} - 8e^{-2} + 15g^{-2}}{3b^{-2} - 7d^{-2} - 8f^{-2} + 15h^{-2}}\right)^{-1}}$.

15. Miscellaneous Examples.

Example 1. If $x : y :: m^2 : n^2$, and

$m : n :: \sqrt{p^2 + x^2} : \sqrt{p^2 - y^2}$, then $p^2 : xy :: x + y : x - y$.

We have $\dfrac{x}{y} = \dfrac{m^2}{n^2} = \dfrac{p^2 + x^2}{p^2 - y^2}$;

$\therefore \quad x(p^2 - y^2) = y(p^2 + x^2)$, [Art 7.]

or, $p^2(x - y) = xy(x + y)$,

$$\therefore \quad \frac{p^2}{xy} = \frac{x+y}{x-y}, \qquad \text{[Art. 7, converse.]}$$

$$i.e., \quad p^2 : xy :: x+y : x-y.$$

Example 2. If $\dfrac{x}{(b-c)(b+c-2a)} = \dfrac{y}{(c-a)(c+a-2b)}$

$= \dfrac{z}{(a-b)(a+b-2c)}$, find the value of $x+y+z$.

Let each of the given ratios $= k$.

Then,
$$x = k(b-c)(b+c-2a) = k\{(b^2-c^2)-2a(b-c)\},$$
$$y = k(c-a)(c+a-2b) = k\{(c^2-a^2)-2b(c-a)\},$$
$$z = k(a-b)(a+b-2c) = k\{(a^2-b^2)-2c(a-b)\}.$$

Hence $x+y+z$
$$= k[\{(b^2-c^2)+(c^2-a^2)+(a^2-b^2)\} - 2\{a(b-c)+b(c-a)+c(a-b)\}]$$
$$= 0.$$

Example 3. If $\dfrac{ay-bx}{c} = \dfrac{cx-az}{b} = \dfrac{bz-cy}{a}$,

shew that $\dfrac{x}{a} = \dfrac{y}{b} = \dfrac{z}{c}$.

Let each of the given ratios $= k$.

Then, we have
$$(ay-bx)c = kc^2,$$
$$(cx-az)b = kb^2,$$
$$(bz-cy)a = ka^2.$$

Hence, by addition, $k(a^2+b^2+c^2) = 0$; $\therefore k = 0$.

Hence, $ay-bx = 0$, $\therefore ay = bx$, $\therefore \dfrac{x}{a} = \dfrac{y}{b}$ ... (1)

also, $cx-az = 0$, $\therefore cx = az$, $\therefore \dfrac{x}{a} = \dfrac{z}{c}$ ... (2)

Hence, from (1) and (2), $\dfrac{x}{a} = \dfrac{y}{b} = \dfrac{z}{c}$.

Example 4. If $\dfrac{a}{y+z} = \dfrac{b}{z+x} = \dfrac{c}{x+y}$,

prove that $\dfrac{a(b-c)}{y^2-z^2} = \dfrac{b(c-a)}{z^2-x^2} = \dfrac{c(a-b)}{x^2-y^2}$.

(Madras University F. A. Paper, 1887.)

Let each of the given ratios $= k$.

Then we have
$$\left.\begin{aligned} a &= k(y+z) \\ b &= k(z+x) \\ c &= k(x+y) \end{aligned}\right\}$$

Hence, we have

$$\left.\begin{aligned} a(b-c) &= k(y+z).k(z-y) &= -k^2(y^2-z^2) \\ b(c-a) &= k(z+x)\,k(x-z) &= -k^2(z^2-x^2) \\ c(a-b) &= k(x+y).k(y-x) &= -k^2(x^2-y^2) \end{aligned}\right\}$$

whence

$$\frac{a(b-c)}{y^2-z^2} = \frac{b(c-a)}{z^2-x^2} = \frac{c(a-b)}{x^2-y^2},$$

each of them being equal to $-k^2$.

Example 5. If $ab = cd = ef$, shew that

$$\frac{ac+ce+ea}{bdf(b+d+f)} = \frac{a^2+c^2+e^2}{d^2f^2+f^2b^2+b^2d^2}.$$

(Bombay University F. A. Paper, 1889.)

Let $ab = cd = ef = k$.

Then we have $a = \dfrac{k}{b}$, $c = \dfrac{k}{d}$, $e = \dfrac{k}{f}$.

Hence
$$\begin{aligned} \frac{ac+ce+ea}{bdf(b+d+f)} &= \frac{k^2\left(\dfrac{1}{bd}+\dfrac{1}{df}+\dfrac{1}{fb}\right)}{bdf(b+d+f)} \\ &= \frac{k^2(f+b+d)}{b^2d^2f^2(b+d+f)} \\ &= \frac{k^2}{b^2d^2f^2} \qquad \ldots \qquad \ldots \qquad (1) \end{aligned}$$

also,
$$\begin{aligned} \frac{a^2+c^2+e^2}{d^2f^2+f^2b^2+b^2d^2} &= \frac{k^2\left(\dfrac{1}{b^2}+\dfrac{1}{d^2}+\dfrac{1}{f^2}\right)}{d^2f^2+f^2b^2+b^2d^2)} \\ &= \frac{k^2(d^2f^2+f^2b^2+b^2d^2)}{b^2d^2f^2(d^2f^2+f^2b^2+b^2d^2)} \\ &= \frac{k^2}{b^2d^2f^2} \qquad \ldots \qquad \ldots \qquad (2) \end{aligned}$$

Hence, from (1) and (2),

$$\frac{ac+ce+ea}{bdf(b+d+f)} = \frac{a^2+c^2+e^2}{d^2f^2+f^2b^2+b^2d^2}.$$

Example 6. If $x : a = y : b = z : c$, show that

$$\frac{x^3}{a^2}+\frac{y^3}{b^2}+\frac{z^3}{c^2} = \frac{(x+y+z)^3}{(a+b+c)^2}.$$

Let each of the given ratios $= k$. Then we have $x = ak$, $y = bk$, and $z = ck$.

Hence,

$$\begin{aligned}\frac{x^3}{a^2}+\frac{y^3}{b^2}+\frac{z^3}{c^2} &= \frac{a^3k^3}{a^2}+\frac{b^3k^3}{b^2}+\frac{c^3k^3}{c^2}\\ &= (a+b+c)\,k^3\\ &= \frac{(a+b+c)^3.k^3}{(a+b+c)^2}\\ &= \frac{(ak+bk+ck)^3}{(a+b+c)^2}\\ &= \frac{(x+y+z)^3}{(a+b+c)^2}.\end{aligned}$$

Example 7. If, a, b, c, d be in continued proportion prove that

$$\sqrt{ab}-\sqrt{bc}+\sqrt{cd} = \sqrt{(a-b+c)(b-c+d)}.$$

(Madras University F. A. Paper, 1890).

We have $\frac{a}{b} = \frac{b}{c} = \frac{c}{d}$, and supposing each of them $= k$, we have

$$a = bk,\ b = ck,\ c = dk.$$

Hence,

$$\begin{aligned}&\sqrt{ab}-\sqrt{bc}+\sqrt{cd}\\ &= \sqrt{b^2k}-\sqrt{c^2k}+\sqrt{d^2k}\\ &= (b-c+d)\sqrt{k} \qquad \dots \qquad \dots \qquad (1)\end{aligned}$$

also

$$\begin{aligned}&\sqrt{(a-b+c)(b-c+d)}\\ &= \sqrt{(b-c+d)k.(b-c+d)}\\ &= \sqrt{(b-c+d)^2.k}\\ &= (b-c+d)\sqrt{k} \qquad \dots \qquad (2)\end{aligned}$$

Hence, from (1) and (2),

$$\begin{aligned}&\sqrt{ab}-\sqrt{bc}+\sqrt{cd}\\ &= \sqrt{(a-b+c)(b-c+d)}.\end{aligned}$$

Exercise (24).

If a be the greatest of the four quantities a, b, c, d, and if $a : b : : c : d$, shew that—

1. b and c are each $> d$.

2. $a - b > c - d$. 3. $a + d > b + c$.

If $a : b : : c : d$, shew that—

4. $m + nb : b : : mc + nd : d$.

5. $ma + nb : mc + nd : : pa - qb : pc - qd$.

6. $a^2 + c^2 : b^2 + d^2 : : ac : bd$.

7. $(a + c)^3 : (b + d)^3 = a(a - c)^2 : b(b - d)^2$.

8. $a(a + c) : c^2 : : b(b + d) : d^2$.

9. $a + b : c + d = \sqrt{a^2 + b^2} : \sqrt{c^2 + d^2}$.

10. $a + b : c + d : : \sqrt{3a^2 + 5b^2} : \sqrt{3c^2 + 5d^2}$.

11. $a^2 + ab + b^2 : a^2 - ab + b^2 : : c^2 + cd + d^2 : c^2 - cd + d^2$.

12. $a^2 + ac + c^2 : a^2 - ac + c^2 : : b^2 + bd + d^2 : b^2 - bd + d^2$.

13. $\frac{1}{ma} + \frac{1}{nb} + \frac{1}{pc} + \frac{1}{qd} = \frac{1}{bc}\left\{\frac{a}{q} + \frac{b}{p} + \frac{c}{n} + \frac{d}{m}\right\}$.

(Bombay University P. E. Paper, 1881.)

If $a : b = c : d = e : f$, shew that—

14. $\frac{ma + nb}{mc + nd} = \frac{b^2c}{d^2a}$.

15. $ac : bd : : 2a^2 + 3c^2 + 5e^2 : 2b^2 + 3d^2 + 5f^2$.

16. $a^2 + c^2 + e^2 \quad b^2 + d^2 + f^2 : : ce : df$.

17. $pa + qc + re \quad pb + qd + rf : : \sqrt[3]{ace} : \sqrt[3]{bdf}$.

18. $a^3 + c^3 + e^3 \quad b^3 + d^3 + f^3 : : ace : bdf$.

19. $\sqrt{a^3c^3 + c^3e^3 + e^3a^3} : \sqrt{b^3d^3 + d^3f^3 + f^3b^3} : : ace : bdf$.

20. If, a, b, c, d, e be in continued proportion, shew that $a : e : : a^4 : b^4$.

21. If $\frac{}{b + c - a} - \frac{y}{c + a - b} = \frac{z}{a + b - c}$, find the value of $(b - c)x + (c - a)y + (a - b)z$.

22. If $a : b : : c : d$, prove that

$$a^2+c^2 : b^2+d^2 : : \sqrt{a^4+c^4} : \sqrt{b^4+d^4}.$$

23. If $a : b = c : d = e : f$, shew that

$$27(a+b)(c+d)(e+f) = bdf\left(\frac{a+b}{b}+\frac{c+d}{d}+\frac{e+f}{f}\right)^3.$$

24. If $a : b : : c : d$, show that

$$ad+bc : 2bd : : a^2+c^2 : ab+cd.$$

25. If $a : b : : c : d$, show that

$$a^2+b^2 : ab+ad-bc : : c^2+d^2 : cd-ad+bc.$$

If $a : b : : b : c$, show that

26. $a^2+ab+b^2 : b^2+bc+c^2 = a : c.$

27. $a-2b+c = \dfrac{(a-b)^2}{a} = \dfrac{(b-c)^2}{c}.$

28. $a^2b^2c^2\left(\dfrac{1}{a^3}+\dfrac{1}{b^3}+\dfrac{1}{c^3}\right) = a^3+b^3+c^3.$

If $a : b = b : c = c : d$, show that

29. $(b+c)(b+d) = (c+a)(c+d).$

30. $(a+d)(b+c)-(a+c)(b+d) = (b-c)^2.$

31. $\left(\dfrac{a-b}{c}+\dfrac{a-c}{b}\right)^2-\left(\dfrac{d-b}{c}+\dfrac{d-c}{b}\right)^2 = (a-d)^2\left(\dfrac{1}{c^2}-\dfrac{1}{b^2}\right).$

32. $a : b = \dfrac{1}{b}+\dfrac{1}{c}+\dfrac{1}{d} : \dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}.$

33. $a : d : : a^3+b^3+c^3 : b^3+c^3+d^3.$

34. If $a : b : : c : d$, show that

$$a^2+ab : c^2+cd : : b^2-2ab : d^2-2cd.$$

35. If $a : b = c : d = e : f$, show that—

$$(a^2+b^2)(ce+df)^2 = (c^2+d^2)(ae+bf)^2 = (e^2+f^2)(ac+bd)^2.$$

36. If $\sqrt{(x-x')^2+(y-y')^2}$

$$= \sqrt{x^2+y^2}-\sqrt{x'^2+y'^2},$$

then x, y, x', y', are in proportion.

(Madras University F. A. Paper, 1888).

37. If $x : y : : a : b$, show that—

$$\frac{x^3+a^3}{x^2+a^2}+\frac{y^3+b^3}{y^2+b^2} = \frac{(x+y)^3+(a+b)^3}{(x+y)^2+(a+b)^2}.$$

38. If $x : a = y : b = z : c$, show that—

$$\frac{x^4+a^4}{x^3+a^3}+\frac{y^4+b^4}{y^3+b^3}+\frac{z^4+c^4}{z^3+c^3}$$
$$=\frac{(x+y+z)^4+(a+b+c)^4}{(x+y+z)^3+(a+b+c)^3}.$$

39. If $\dfrac{2a+5b-7c}{a+3b+c}=\dfrac{a-2b+c}{2a-b-4c}=\dfrac{3a-11b+10c}{7a-6b-13c}$, show that $a : b : c = 2 : 3 : 1$.

40. Shew that the ratio of $pa_1+qa_2+ra_3+\ldots$ to $pb_1+qb_2+rb_3+\ldots$ is greater than the least and less than the greatest of the ratios $a_1 : b_1$, $a_2 : b_2$, $a_3 : b_3$, &c., the quantities being all positive.

CHAPTER V.

VARIATION.

1. Definition. One quantity is said to *vary directly* as another when the two quantities are so related that if one of them be changed the other is changed *in the same ratio*; or, in other words, if a, a' be *any two* values of a quantity A, and b, b' the *corresponding* values of a second quantity B, then A is said to vary directly as B when $a : a' = b : b'$.

For instance, suppose the measure of the area of a triangle is a, when that of the base is b; now if, the height remaining unchanged, the base be increased to $2b$, then as we know from Euclid, the area will become $2a$; if the base becomes $3b$, the area will be $3a$; and so on. Thus the height remaining the same if the base is doubled, trebled, quadrupled, &c., the area also becomes doubled, trebled, quadrupled, &c, (*i.e.*, the area changes *in the same ratio* as the base) and so we say that if the height of a triangle remains unaltered, the area *varies directly* as the base.

NOTE 1. The word *directly* is often omitted, so that when we say *A varies as B* it is implied that A varies directly as B.

NOTE 2. The symbol $\propto$ is used to express variation; thus $A \propto B$ stands for "*A* varies as *B*."

2. If A varies as B, then the numerical measure of *any* value of A and that of the *corresponding* value of B are in a constant ratio.

Let a_1, a_2, a_3, &c, be the measures of a series of values of A, and let b_1, b_2, b_3, &c, be the measures of the corresponding values of B.

Then, by definition,

$$\frac{a_1}{a_2} = \frac{b_1}{b_2};\ \frac{a_2}{a_3} = \frac{b_2}{b_3};\ \frac{a_3}{a_4} = \frac{b_3}{b_4},\ \text{and so on.}$$

Hence, $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3} = \frac{a_4}{b_4} =$ &c., which proves the proposition.

NOTE. Putting m for each of the above ratios, we have $a_1 = mb_1$, $a_2 = mb_2$, $a_3 = mb_3$, and so on. Thus when A varies as B, the numerical measure of *any* value of A is equal to that of the corresponding value of B multiplied by a constant. This result is briefly expressed as follows :—"If $A \propto B$, then $A = mB$ where m is a constant.

3 Definitions.—(1) One quantity A is said to *vary inversely* as another B, when A varies directly as the reciprocal of B.

Thus if A varies inversely as B, $A = \frac{m}{B}$, where m is constant.

Illustration :—If 20 men do a certain work in 4 hours, 10 men would do it in 8 hours, 40 men in two hours ; and so on. Thus when the number of men *diminishes*, the time *proportionately increases* ; and *vice-versa*. This is expressed by saying that if the amount of work to be done remains constant, the number of men varies inversely as the time.

(2) One quantity is said to *vary jointly* as a number of others, when it varies directly as their product. Thus if A varies jointly as B and C, $A = m.BC$, where m is constant.

Illustration :—The monthly income of a day labourer varies jointly as his daily earning and the number of days he works in a month.

(3) A is said to vary directly as B and inversely as C when A varies jointly as B and the reciprocal of C, that is, when $A = m.\frac{B}{C}$, where m is constant.

Illustration **:—The time of travelling a distance varies directly as the distance and inversely as the speed of travelling.**

4. An Important Theorem.—

If A varies as B when C is constant, and A varies as C when B is constant, then will A vary as BC when both B and C vary.

Suppose a_1 is the value of A when b_1 is that of B, and c_1 that of C. Suppose also that a_2 is the value of A when b_2 is that of B, and c_2 that of C. Then the proposition will be proved if we can show that $a_1 : a_2 = b_1c_1 : b_2c_2$.

Now, the change of A from a_1 to a_2 is due to *two* causes, namely,

(1) the change of B from b_1 to b_2, and (2) the change of C from c_1 to c_2.

Hence, it is clear that if *one* only of these causes be present (*i.e.*, if either B or C *alone* undergoes the supposed change), A will change from a_1 to some value which is *different* from a_2. Let therefore a' be the value of A when b_2 is that of B, and c_1 that of C.

Thus we have the value of A

$= a_1$ when those of B and C are respectively b_1 and c_1 ... (1)
$= a'$ when those of B and C are respectively b_2 and c_1 ... (2)
$= a_2$ when those of B and C are respectively b_2 and c_2 ... (3)

Hence, from (1) and (2), we see that A changes from a_1 to a', when B changes from b_1 to b_2, C *remaining constant* (*i.e.*, retaining the value c_1), and therefore, by hypothesis,

$$\frac{a_1}{a'} = \frac{b_1}{b_2}; \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (\alpha)$$

and from (2) and (3) we see that A changes from a' to a_2 when C changes from c_1 to c_2, B *remaining constant* (*i.e.*, retaining the value b_2), and therefore, by hypothesis,

$$\frac{a'}{a_2} = \frac{c_1}{c_2} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (\beta)$$

Hence, from (α) and (β), $\dfrac{a_1}{a'} \times \dfrac{a'}{a_2} = \dfrac{b_1}{b_2} \times \dfrac{c_1}{c_2}$

$$\text{or,} \quad \frac{a_1}{a_2} = \frac{b_1c_1}{b_2c_2},$$

which proves the proposition.

Illustration :—(1) Suppose that a number of plants have to be watered : the quantity of water bestowed evidently varies directly as the number of men employed *if the time for watering remains unchanged ?* and also it varies directly as the number of hours for which the men can work, *if the number of men engaged remain the same* ; hence if the number of men and the number of hours be both variable, the quantity of water will vary as the product of the number of men and the number of hours.

(2) The area of a triangle varies directly as the base when the height is constant, and it also varies directly as the height when the base is constant ; hence when both the base and the height are variable the area varies as the product of the numbers which express the base and the height

Cor. If there be any number of quantities B, C, D, &c., each of which varies as another A when the rest are constant ; then if they are all variable, A varies as their product.

5. Some Results worth remembering—

(1) If $A \propto B$ and $B \propto C$, then $A \propto C$.

For, let $A = mB$, and $B = nC$, where m and n are constants ; then $A = mnC$; and $\therefore$ as mn is constant, $A \propto C$.

(2) If $A \propto C$, and $B \propto C$, then $A \pm B \propto C$, and $\sqrt{AB} \propto C$.

For, let $A = mC$, and $B = nC$, where m and n are constants ; then $A + B = (m+n)C$, and $A - B = (m-n)C$; $\therefore \ (A \pm B) \propto C$.

Also $\sqrt{AB} = \sqrt{mn.C^2} = C\sqrt{mn}$; $\therefore \ \sqrt{AB} \propto C$.

(3) If $A \propto BC$, then $B \propto \dfrac{A}{C}$, and $C \propto \dfrac{A}{B}$.

For, let $A = mBC$, then $B = \dfrac{1}{m} \cdot \dfrac{A}{C}$; $\therefore \ B \propto \dfrac{A}{C}$

Similarly $C \propto \dfrac{A}{B}$.

(4) $A \propto B$, and $C \propto D$, then $AC \propto BD$.

For, let $A = mB$, and $C = nD$, then $AC = mn\ BD$

$\therefore \ AC \propto BD$.

(5) If $A \propto B$, then $A^n \propto B^n$.

For let $A = mB$, then $A^n = m^n B^n$;

$\therefore \ A^n \propto B^n$.

(6) If $A \propto B$, then $AP \propto BP$, where P is *any* quantity *variable* or *invariable*.

For, let $A = mB$, then $AP = mBP$;

$$\therefore AP \propto BP.$$

6. Examples—Applications of the principles explained in some of the preceding articles will be illustrated by the following examples.

Example 1. If y varies as x, and $y = 5$ when $x = 12$, find the value of y when $x = 18$.

By supposition $y = mx$, where m is constant.

Putting $y = 5$, $x = 12$, we have

$$5 = m.12, \quad \therefore \quad m = \frac{5}{12}.$$

Hence, x and y are connected by the relation $y = \frac{5}{12}x$.

Hence, when $x = 18$, we have $y = \frac{5}{12}.18 = \frac{15}{2} = 7\frac{1}{2}$.

Example 2. If z varies as $px + y$, and if $z = 3$ when $x = 1$ and $y = 2$, and $z = 5$ when $x = 2$ and $y = 3$, find p.

By supposition $z = m(px + y)$ where m is constant.

Putting $z = 3$, $x = 1$, $y = 2$, we have

$$3 = m(p+2) \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

Again putting $z = 5$, $x = 2$, $y = 3$, we have

$$5 = m(2p+3) \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

Hence, from (1) and (2), by division,

$$\frac{3}{5} = \frac{p+2}{2p+3}, \text{ whence } p = 1.$$

Example 3. If y = the sum of 3 quantities, of which the 1st. $\propto x^2$, the 2nd. $\propto x$, and the 3rd. is constant; and when $x = 1, 2, 3$, $y = 6, 11, 18$ respectively, find the *equation* between x and y.

By supposition $y = mx^2 + nx + p$, where m, n, p are constants.

Now since $y = 6$ when $x = 1$, we have

$$\left.\begin{aligned} 6 &= m+n+p \quad \ldots \quad (1) \\ \text{Similarly, } 11 &= 4m+2n+p \ldots \quad \ldots \quad (2) \\ \text{and } 18 &= 9m+3n+p \ldots \quad \ldots \quad (3) \end{aligned}\right\}$$

From (1) and (2) by subtraction,

$$3m+n = 5 \quad \ldots \quad \ldots \quad (4)$$

Similarly from (2) and (3),

$$5m+n = 7 \quad \ldots \quad \ldots \quad (5)$$

Now subtracting (4) from (5), we have

$$2m = 2, \quad \therefore \quad m = 1;$$

hence from (4), $n = 2$, $\therefore$ from (1), $p = 3$.

Hence, the equation between x and y is $y = x^2+2x+3$.

Example 4. If $a+b \propto a-b$, prove that $a^2+b^2 \propto ab$; and if $a \propto b$, prove that $a^2-b^2 \propto ab$.

(i) By supposition, $a+b = m(a-b)$ where m is constant.

Hence, $(a+b)^2 = m^2(a-b)^2$,

or, $a^2+b^2+2ab = m^2(a^2+b^2-2ab)$,

$$\therefore \quad (m^2-1)(a^2+b^2) = 2ab(1+m^2),$$

$$\therefore \quad a^2+b^2 = \frac{2(m^2+1)}{m^2-1}.ab.$$

But $\dfrac{2(m^2+1)}{m^2-1}$ is constant, $\therefore \quad a^2+b^2 \propto ab$.

(ii) Since $a = mb$,

multiplying both sides by a, we have

$$a^2 = m.ab \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

and also multiplying both sides by b, we have

$$b^2 = \frac{ab}{m} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

Subtracting (2) from (1),

$$a^2-b^2 = \left(m-\frac{1}{m}\right).ab,$$

where $\left(m-\dfrac{1}{m}\right)$ is constant, $\therefore \quad a^2-b^2 \propto ab$.

Example 5. The wages of 5 men for 6 weeks being £14. 5*s.*, how many weeks will 4 men work for £19 ?

Let x denote the wages (in pounds), earned by y men in z weeks.

Then evidently $x \propto y$, when z is constant,

and also $x \propto z$, when y is constant ;

$\therefore$ when y and z are both variable,

$$x \propto yz,$$

i.e., $x = m.yz$, where m is constant.

Now, since $x = 14\frac{1}{4}$, when $y = 5$ and $z = 6$,

$$\therefore \quad 14\tfrac{1}{4} = m \times 5 \times 6 \quad \ldots \quad \ldots \quad (1)$$

Also, If z_1 denote the required number of weeks, then, since the corresponding values of x and y are respectively 19 and 4, we have

$$19 = m \times 4 \times z_1 \quad \ldots \quad \ldots \quad (2)$$

Hence, dividing (1) by (2),

$$\frac{3}{4} = \frac{5 \times 6}{4 \times z_1}, \text{ whence } z_1 = 10;$$

i. e, the required time = 10 weeks.

Example 6. Assuming that the quantity of work done varies as the cube root of the number of agents when the time is the same, and varies as the square root of the time when the number of agents is the same ; find how long 3 men would take to do one-fifth of the work which 24 men can do in 25 hours.

Let x denote the quantity of work done by y men in z hours.

Then by supposition,

$$x \propto y^{\frac{1}{3}} \text{ when } z \text{ and } \therefore z^{\frac{1}{2}} \text{ is constant,}$$

and also, $x \propto z^{\frac{1}{2}}$ when y and $\therefore y^{\frac{1}{3}}$ is constant.

Hence, when both y and z and $\therefore y^{\frac{1}{3}}$ and $z^{\frac{1}{2}}$ are variable,

$$x \propto y^{\frac{1}{3}} z^{\frac{1}{2}},$$

i. e., $x = k.y^{\frac{1}{3}} z^{\frac{1}{2}}$, where k is constant.

Now, since by the problem,

$$x = 1, \text{ when } y = 24 \text{ and } z = 25,$$

$$\therefore \quad 1 = k.\ \sqrt[3]{24}.\ \sqrt{25} \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

Also, if z_1 be the required number of hours, since the corresponding values of x and y are respectively $\frac{1}{5}$ and 3, we have

$$\tfrac{1}{5} = k.\ \sqrt[3]{3}.\ \sqrt{z_1} \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

Hence, dividing (1) by (2),

$$5 = \frac{\sqrt[3]{24} \times 5}{\sqrt[3]{3} \times \sqrt{z_1}} = \frac{\sqrt[3]{8} \times 5}{\sqrt{z_1}}$$

$$\therefore \quad \sqrt{z_1} = 2, \text{ and } \therefore z_1 = 4\ ;$$

i.e., the required time = 4 hours.

Example 7. A sphere of metal is known to have a hollow space above its centre in the form of a concentric sphere, and its weight is $\frac{7}{8}$ of the weight of a solid sphere of the same substance and radius ; compare the inner and outer radii, having given that the weights of spheres of the same substance $\propto$ (radii)^3.

Let R be the outer radius, and W the weight of a solid sphere of the given metal of radius R; also let r be the inner radius (*i.e.*, radius of the spherical cavity), and w the weight of a solid sphere of the given metal of radius r.

Then, by hypothesis,

$$W = KR^3,$$

$$\text{and } w = Kr^3, \text{ where } K \text{ is constant.}$$

Now, since $(W - w)$ is the weight of the given sphere, we have, by the question, $W - w = \frac{7}{8}W$; hence we must have

$$K(R^3 - r^3) = \tfrac{7}{8}KR^3,$$

$$\therefore \quad \tfrac{1}{8}R^3 = r^3 \text{ whence } \frac{r}{R} = \tfrac{1}{2}.$$

Example 8. A point moves with a speed which is different in different miles, but invariable in the same mile, and its speed in any mile varies inversely as the number of miles travelled before it commences this mile. If the second mile be described in 2 hours, find the time occupied in describing the n^{th} mile.

Evidently the time of describing any mile varies *inversely* as the speed in that mile; hence if v_n denote the speed in the

n^{th} mile and t_n the number of hours required to describe the n^{th} mile, we must have

$$t_n = \frac{m}{v_n}, \text{ where } m \text{ is constant.}$$

Also, by hypothesis, $v_n = \frac{K}{n-1}$, where K is constant;

hence, $$t_n = \frac{m}{K}(n-1$$

Evidently then t_n is known if $\frac{m}{K}$ is known; and since the time of describing the 2nd mile is two hours (*i.e.*, $t_n = 2$, when $n = 2$) we have

$$2 = \frac{m}{K}.1; \quad \therefore \quad \frac{m}{K} = 2.$$

Hence, $t_n = 2(n-1)$,

i.e., the n^{th} mile is described in $2(n-1)$ hours.

Example 9. A locomotive engine without a train can go 24 miles an hour, and its speed is diminished by a quantity which varies as the square root of the number of waggons attached. With four waggons its speed is 20 miles an hour. Find the greatest number of waggons which the engine can move.

Let x = the number of waggons attached.

Then the number of miles travelled by the train per hour (*i.e.*, its speed) $= 24 - m\sqrt{x}$ where m is a constant.

Now, since the speed is 20 miles per hour when $x = 4$, we must have

$$\begin{aligned} 20 &= 24 - m\sqrt{4} \\ &= 24 - 2m, \quad \therefore \quad m = 2. \end{aligned}$$

Hence, the speed of the engine with x waggons $= 24 - 2\sqrt{x}$; evidently therefore the speed diminishes as x increases.

Now let us see for what value of x the speed is reduced to nothing. If x_1 be this value, we must have

$$0 = 24 - 2\sqrt{x_1},$$

$$\therefore \quad \sqrt{x_1} = 12, \text{ and } x_1 = 144.$$

Thus when 144 waggons are attached, the engine *just* fails to move the train.

Hence, the *greatest number* of waggons which the engine *can move* = 143.

Example 10. If x, y, z be variable quantities such that $y+z-x$ is constant, and that $(x+y-z)(x+z-y)$ varies as yz, prove that $x+y+z$ varies as yz.

By supposition, we have

$$y+z-x = k \qquad \ldots \qquad \ldots \qquad (1)$$

and $$(x+y-z)(x+z-y) = myz \qquad \ldots \qquad \ldots \qquad (2)$$

where k and m are constants.

Now from (2), we have

$$x^2-(y-z)^2 = myz,$$

$$\therefore \quad x^2-(y+z)^2 = (m-4)yz$$

or, $$(x+y+z)(x-y-z) = (m-4)yz.$$

Hence, from (1),

$$(x+y+z)(-k) = (m-4)yz,$$

$$\therefore \quad x+y+z = \left(\frac{4-m}{k}\right)yz,$$

$$i.e., = (\text{a constant}) \times yz.$$

Hence, $x+y+z \propto yz$.

Exercise (25).

1. If $y \propto x$, and $y = 5$ when $x = 15$, find the *equation* between x and y.

2. If $y \propto x$, and $y = 10$ when $x = 25$, find y when $x = 35$.

3. If P varies inversely as Q, and $Q = 10$ when $P = 2$, what will P become when $Q = 8$?

4. If $P \propto QR$, and the three corresponding values of $P.Q.R$, be 6, 9, 10 respectively, find the value of P when $Q = 5$ and $R = 3$.

5. If the square of x vary as the cube of y, and $x = 2$ when $y = 3$, find the *equation* between x and y.

6. Given that y varies as the sum of two quantities, one of which varies as x directly, the other as x inversely, and that $y = 4$ when $x = 1$, and $y = 5$ when $x = 2$, find the *equation* between x and y.

7. If $xy \propto x^2 + y^2$, and $y = 4$ when $x = 3$, find the *equation* between x and y.

8. Given that y is equal to the sum of two quantities, one of which varies as x, and the other varies inversely as x^2, and when $x = 1, 2$, $y = 6, 5$ respectively. Find the *equation* between x and y.

9. If $y =$ the sum of 3 quantities, of which the 1st. is constant, the 2nd. $\propto x$, and the 3rd. $\propto x^2$, also when $x = 3, 5, 7$, $y = 0, -12, -32$ respectively; find the *equation* between x and y.

10. Given that $y^2 \propto a^2 - x^2$ and when $x = \sqrt{a^2 - b^2}$, $y = \frac{b^2}{a^2}$; find the *equation* between x and y.

11. If $y = r + s$, whilst $r \propto x$, and $s \propto \sqrt{x}$; and if, when $x = 4$, $y = 5$, and when $x = 9$, $y = 10$, shew that $6y = 5(x + \sqrt{x})$.

12. Assuming that the time of oscillation of a pendulum varies as the square root of its length; if the length of a pendulum which oscillates once in a second be 39·2 inches, find the length of one which oscillates 56 times in a minute.

13. If 13 men earn £7 in 15 days of 8 hours each, what will be the wages of 52 men for $12\frac{1}{2}$ days of 9 hours each?

14. Given that the volume of sphere varies as the cube of its radius, prove that the volume of a sphere whose radius is 6 inches is equal to the sum of the volumes of three spheres whose radii are 3, 4, 5 inches.

15. The volume of a pyramid varies jointly as its height and the area of its base; and when the area of the base is 60 square feet and the height 14 feet, the volume is 280 cubic feet. What is the area of the base of a pyramid whose volume is 390 cubic feet and whose height is 26 feet?

16. Given that the area of a circle varies as the square of its radius, and that the area of a circle is 154 square feet, when the radius is 7 feet; find the area of a circle whose radius is 10 feet 6 inches.

17. If the volume of a cone whose height is 12 inches and base 30 square inches be 120 cubic inches, find the volume of another whose height is 20 inches and base 1 square foot: the volume of a cone varying as the height and base jointly.

18. The volume of a circular cylinder varies as the square of the radius of the base when the height is the same, and as the height when the base is the same. The volume is 88 cubic feet when the height is 7 feet, and the radius of the base is 2 feet; what will be the height of a cylinder on a base of radius 9 feet, when the volume is 396 cubic feet?

19. Two circular gold plates, each an inch thick, the diameters of which are 6 inches and 8 inches respectively, are melted and formed into a single circular plate one inch thick. Find its diameter, having given that the area of a circle varies as the square of its diameter.

20 Given that the illustration from a source of light *varies inversely* as the *square* of the distance, how much further from a candle must a book, which is now three inches off, be removed, so as to receive just half as much light?

21. A solid spherical mass of glass, 1 inch in diameter, is blown into a shell bounded by two concentric spheres, the diameter of the outer one being 3 inches. Calculate the thickness of the shell. (The volume of a sphere varies directly as the cube of its diameter).

22. When a body falls from rest, its distance from the starting point varies as the square of the time it has been falling: if a body falls through $402\frac{1}{2}$ feet in 5 seconds, how far does it fall in 10 seconds? Also how far does it fall in the 10th second?

23. If 10 men can reap a field of $7\frac{1}{2}$ acres, in 3 days of 12 hours each, how long will it take 8 men to reap 9 acres, working 16 hours a day?

24. The square of the time of a planet's revolution varies as the cube of its distance from the Sun; find the time of Venus' revolution, assuming the distances of the earth and Venus from the Sun to be $91\frac{1}{4}$ and 66 millions of miles respectively.

[If P be the time of revolution measured in days, and D the distance in millions of miles, we have $P^2 = KD^3$, where K is a constant, &c.]

25. The value of a silver coin varies directly as the square of its diameter while its thickness remains the same, and directly as its thickness while its diameter remains the same. Two silver coins have their diameters in the ratio of

4 : 3 ; find the ratio of their thickness if the value of the first be four times the value of the second.

(Bombay University P. E. Paper, 1885).

26. The value of diamonds $\propto$ the square of their weights, and the square of the value of rubies $\propto$ the cube of their weights. A diamond of a carats is worth m times the value of a ruby of b carats, and both together are worth £c. Required the values of a diamond and of a ruby, each weighing n carats.

27. If $a \propto b$, and $b \propto c$, shew that $(a^2+b^2)^{\frac{3}{2}} \propto c^3$.

28. If $x+y \propto x-y$, shew that $x^2+y^2 \propto xy$ and $x^3+y^3 \propto xy(x \pm y)$.

29. Given that $x+y \propto z+\frac{1}{z}$, and that $x-y \propto z-\frac{1}{z}$, find the relation between x and z provided that $z=2$ when $x=3$ and $y=1$.

(Bombay University P. E. Paper, 1888.)

30. If $x \propto \frac{1}{y}$, prove that $x+y$ is least when $x=y$.

[We have xy = a constant.]

31. The consumption of coal by a locomotive varies as the square of the velocity ; when the speed is 16 miles an hour the consumption of coal per hour is 2 tons : if the price of coal be 10*s*. per ton and the other expenses of the engine be 11*s*. 3*d*. an hour, find the least cost of a journey of 100 miles.

[Apply the preceding example.]

32. If $z \propto y$, and $y \propto x$, shew that

$$x+y+z \propto (yz)^{\frac{1}{2}}+(xz)^{\frac{1}{2}}+(xy)^{\frac{1}{2}}.$$

CHAPTER VI.

QUADRATIC EQUATIONS.

1. Definition.—Any equation which contains the square of the unknown quantity, but no higher power, is called a *quadratic equation*, or an *equation of the second degree*.

If an equation contains *only* the second power of the unknown quantity (and *not the first*) it is called a *pure* quadratic; if it contains the second *as well as* the first power it is called an *adfected* quadratic.

Thus $3x^2 = 75$ is a pure quadratic;

and $3x^2 - 7x = 6$ is an adfected quadratic.

2. Solution of a Pure Quadratic.

—In solving a Pure Quadratic we have to find the *square of the unknown quantity* just in the same way as simple equations are solved and then to extract the square root of the value so found.

Example 1. Solve $5(x^2+1)-2 = 3(x^2+7)$.

$$\begin{aligned} \text{We have}\quad 5x^2+3 &= 3x^2+21\,; \\ \text{hence,}\quad 2x^2 &= 18 \\ \therefore\quad x^2 &= 9\,; \end{aligned}$$

now, since the unknown quantity is one of which the square is 9, it must be *either* $+3$ or -3. (Thus there are *two* values of x satisfying the given equation, as the student can easily verify).

NOTE. The student should carefully observe that the last step of the above solution amounts to answering the following question:—"What quantity is that of which the square is 9?"

Example 2. If $\dfrac{35-2x}{9} + \dfrac{5x^2+7}{5x^2-7} = \dfrac{17-\frac{2}{3}x}{3}$, find x.

By transposition, we have

$$\begin{aligned} \frac{5x^2+7}{5x^2-7} &= \frac{51-2x}{9} - \frac{35-2x}{9} \\ &= \frac{16}{9}\,; \\ \therefore\quad \frac{5x^2}{7} &= \frac{16+9}{16-9} = \frac{25}{7}, \qquad \text{(Componendo and dividendo.)} \\ \therefore\quad x^2 &= 5, \\ \therefore\quad x &= \pm\sqrt{5}. \end{aligned}$$

Example 3. Solve $3\left(\dfrac{x^2-9}{x^2+3}\right) + 4\left(\dfrac{22\frac{1}{2}+x^2}{x^2+9}\right) = 7.$

By transposition, we have

$$4\left(\frac{22\frac{1}{2}+x^2}{x^2+9}\right) - 4 = 3 - 3\left(\frac{x^2-9}{x^2+3}\right)$$

$$\text{or,}\quad 4\left\{\frac{22\frac{1}{2}+x^2}{x^2+9}-1\right\} = 3\left\{1-\frac{x^2-9}{x^2+3}\right\}$$

$$\text{or,}\quad 4\times\frac{13\frac{1}{2}}{x^2+9} = 3\times\frac{12}{x^2+3},$$

$$\therefore\quad \frac{3}{x^2+9} = \frac{2}{x^2+3},$$ [Removing the factor 18 from both sides.]

$$\therefore\quad 3x^2+9 = 2x^2+18,$$

$$\therefore\quad x^2 = 9.$$

$$\therefore\quad x = \pm 3.$$

Example 4. If $\dfrac{1+\sqrt{x^2-1}}{1+2a\sqrt{x^2-1}} = \dfrac{\sqrt{x^2-1}-1}{x^2-2}$, find x.

Put y for $\sqrt{x^2-1}$ and $\therefore$ y^2-1 for x^2-2.

Thus we have $\dfrac{1+y}{1+2ay} = \dfrac{y-1}{y^2-1} = \dfrac{1}{y+1}$.

$$\text{Therefore}\quad (1+y)^2 = 1+2ay$$

$$\text{or,}\quad 1+2y+y^2 = 1+2ay,$$

$$\therefore\quad y+2 = 2a, \quad\text{or } y = 2(a-1)$$

$$i.e.,\quad \sqrt{x^2-1} = 2(a-1),$$

$$\therefore\quad x^2-1 = 4(a-1)^2$$

$$\therefore\quad x = \pm\sqrt{1+4(a-1)^2}.$$

Exercise (26).

Find the values of x in each of the following equations :—

✓1. $8x+\dfrac{7}{x} = \dfrac{65}{7}x.$ 2. $\dfrac{2x^2+10}{15} = 7-\dfrac{50+x^2}{25}$

3. $\dfrac{14x^2+16}{21}-\dfrac{2x^2+8}{8x^2-11} = \dfrac{2x^2}{3}.$

✓4. $\dfrac{x+7}{x(x-7)}-\dfrac{x-7}{x(x+7)} = \dfrac{7}{x^2-73}.$

✓5. $\dfrac{x^3-1}{(x-1)^2}-\dfrac{x^3+1}{(x+1)^2} = 6.$

✓6. $\dfrac{1}{\sqrt{1-x}+1}+\dfrac{1}{\sqrt{1+x}-1}=\dfrac{1}{x}$.

[Rationalise both the terms of the left-hand side and then proceed.]

7. $(1+x+x^2)^{\frac{1}{2}}=a-(1-x+x^2)^{\frac{1}{2}}$.

✓8. $\dfrac{(x-a)(x-b)}{(x-ma)(x-mb)}=\dfrac{(x+a)(x+b)}{(x+ma)(x+mb)}$.

9. $\dfrac{ax+1+(a^2x^2-1)^{\frac{1}{2}}}{ax+1-(a^2x^2-1)^{\frac{1}{2}}}=\dfrac{b^2x}{2}$.

10. $(a+x)^{\frac{2}{3}}+(a-x)^{\frac{2}{3}}=3(a^2-x^2)^{\frac{1}{3}}$.

✓11. $\dfrac{5x^2+17}{x^2-11}+\dfrac{14x^2-117}{2x^2-9}=12$.

✓12. $\dfrac{x^2-1}{x^2-4}-\dfrac{x^2-5}{x^2-8}=\dfrac{x^2-2}{x^2-5}-\dfrac{x^2-6}{x^2-9}$.

3. **The ordinary method of solving an Adfected Quadratic.**—Bring the terms containing the unknown quantity to the left-hand side of the equation, and the known quantities to the right-hand side; if the co-efficient of x^2 be negative, change the sign of every term of the equation, and *then* divide every term by the co-efficient of x^2: thus the equation is reduced to the form $x^2+px=q$. Now add $\dfrac{p^2}{4}$ (*i.e*, square of half the co-efficient of x) to both sides, on which the left-hand side becomes a complete square and we get $\left(x+\dfrac{p}{2}\right)^2=q+\dfrac{p^2}{4}$, whence $x+\dfrac{p}{2}=\pm\sqrt{q+\dfrac{p^2}{4}}$, and therefore

$$x=-\frac{p}{2}\pm\sqrt{q+\frac{p^2}{4}}.$$

Exercise (27).

Solve the following equations :—

✓1. $70x-63=7x^2$.

By transposition, we have

$$-7x^2+70x=63.$$

Since the co-efficient of x^2 is negative, changing the sign of every term we get

$$7x^2 - 70x = -63\ ;$$

dividing both sides by 7,

$$x^2 - 10x = -9.$$

Now adding $\left(\frac{10}{2}\right)^2$ or 25 to both sides, we have

$$x^2 - 10x + 25 = 25 - 9 = 16,$$

$$\text{or,}\quad (x-5)^2 = 16.$$

Hence, $x - 5 = \pm 4$ (because $x - 5$ is a quantity of which the square is 16) ;

$$\therefore\quad x = 5 + 4, \text{ or } 5 - 4,$$

$$i\,e.,\quad x = 9, \text{ or } 1.$$

✓ **2.** $2x^2 - 11x + 5 = 0.$

By transposition,

$$2x^2 - 11x = -5\ ;$$

dividing both sides by 2,

$$x^2 - \tfrac{11}{2}x = -\tfrac{5}{2}.$$

Adding $\left(\frac{11}{4}\right)^2$ to both sides,

$$x^2 - \tfrac{11}{2}x + \left(\tfrac{11}{4}\right)^2 = \tfrac{121}{16} - \tfrac{5}{2}$$

$$i.e.,\quad \left(x - \tfrac{11}{4}\right)^2 = \tfrac{81}{16},$$

$$\therefore\quad x - \tfrac{11}{4} = \pm\tfrac{9}{4},$$

$$\therefore\quad x = \tfrac{11}{4} \pm \tfrac{9}{4} = 5 \text{ or } \tfrac{1}{2}.$$

3. $87 - 98x = 30x - 16x^2.$

4. $17x^2 - 85x + 216 = 65x - 8x^2.$

5. $\dfrac{x^2+8}{11} = 5x - x^2 - 5.$

6. $4(x^2 + 23x - 24) = 29x^2 - 8x + 1.$

7. $(3x-1)(x-4) + (x-2)(2x-3)$
$= 4x(x-3) - 5.$

The left-hand side

$$= (3x^2 - 13x + 4) + (2x^2 - 7x + 6)$$
$$= 5x^2 - 20x + 10.$$

Hence, we have $5x^2 - 20x + 10 = 4x^2 - 12x - 5,$

$$\therefore \quad x^2 - 8x = -15 \quad \text{(by transposition)},$$
$$\therefore \quad x^2 - 8x + (4)^2 = 16 - 15,$$
$$\text{or,} \quad (x-4)^2 = 1,$$
$$\therefore \quad x - 4 = \pm 1,$$
$$\therefore \quad x = 4 \pm 1$$
$$= 5, \text{ or } 3.$$

8. $(2x-5)(3x-7) - (x-1)(4x-5) = x^2 - 3(x+14).$

9. $(3x-11)(x-2) + (2x-3)(x+4) + 13x$
$$= 10(2x-1)^2 + 12.$$

10. $(x - \frac{1}{2})(x - \frac{1}{3}) + (x - \frac{1}{3})(x - \frac{1}{4}) = (x - \frac{1}{4})(x - \frac{1}{5}).$

(Calcutta University F. A. Paper, 1861)

11. $\dfrac{x}{15} + \dfrac{40}{3(10-x)} = \dfrac{3(10+x)}{95}.$

[By transposition, $\dfrac{40}{3(10-x)} = \dfrac{3(10+x)}{95} - \dfrac{x}{15} = \text{\&c.}$]

12. $\dfrac{2x}{15} + \dfrac{3x-50}{3(10+x)} = \dfrac{12x+70}{190}.$ 13. $\dfrac{x+4}{x-4} + \dfrac{x-4}{x+4} = \dfrac{10}{3}.$

14. $\dfrac{x+3}{x+2} + \dfrac{x-3}{x-2} = \dfrac{2x-3}{x-1}.$

15. $\dfrac{x-2}{x+2} + \dfrac{x+2}{x-2} = \dfrac{2(x+3)}{x-3}.$ 16. $\dfrac{x+2}{x-2} - \dfrac{x-2}{x+2} = \dfrac{5}{6}.$

✓17. $\dfrac{x+6}{x+7} - \dfrac{x+1}{x+2} = \dfrac{1}{3x+1}.$ (Calcutta University F. A. Paper, 1878)

18. $\dfrac{2x}{x-4} + \dfrac{2x-5}{x-3} = 8\frac{1}{3}.$

[We have $\left(\dfrac{2x}{x-4} - 2\right) + \left(\dfrac{2x-5}{x-3} - 2\right) = 4\frac{1}{3}.$]

19. $\dfrac{x}{x+5} - \dfrac{11x}{11x-8} + \dfrac{7}{6-4x} = 0.$

20. $\dfrac{1}{x+a}+\dfrac{1}{x+2a}+\dfrac{1}{x+3a} = \dfrac{3}{x}.$

[We have $\left(\dfrac{1}{x+a}-\dfrac{1}{x}\right)+\left(\dfrac{1}{x+2a}-\dfrac{1}{x}\right)+\left(\dfrac{1}{x+3a}-\dfrac{1}{x}\right) = 0.$

whence $\dfrac{1}{x+a}+\dfrac{2}{x+2a}+\dfrac{3}{x+3a} = 0,$

or, $\dfrac{1}{x+a}+\dfrac{1}{x+3a} = -2\left(\dfrac{1}{x+3a}+\dfrac{1}{x+2a}\right)$

whence $\dfrac{x+2a}{x+a} = -\dfrac{2x+5a}{x+2a}$, $\therefore$ &c.]

4. The general expression for the roots of a quadratic.

N.B. The roots of any equation are those values of the unknown quantity that satisfy the equation.

As every quadratic equation can be written in the form $ax^2+bx+c = 0$ (after suitable reduction if necessary) we must regard this equation as the *general type* of all quadratics. Let us solve it.

By transposition,

$$ax^2+bx = -c.$$

Dividing both sides by a,

$$x^2+\frac{b}{a}x = -\frac{c}{a}.$$

Adding $\left(\dfrac{b}{2a}\right)^2$ to both sides,

$$x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2}-\frac{c}{a}$$

$$\text{or, } \left(x+\frac{b}{2a}\right)^2 = \frac{b^2-4ac}{4a^2},$$

$$\therefore\ x+\frac{b}{2a} = \pm\frac{\sqrt{b^2-4ac}}{2a},$$

$$\therefore\ x = \frac{-b\pm\sqrt{b^2-4ac}}{2a};$$

Thus the roots of the quadratic $ax^2+bx+c = 0$, are $\dfrac{-b+\sqrt{b^2-4ac}}{2a}$ and $\dfrac{-b-\sqrt{b^2-4ac}}{2a}$, and therefore we must regard the expression $\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$ as the *general expression* sought.

By the application of this formula we can find out the roots of a quadratic equation without going through the process explained in Art. 3.

Example 1. Write down the roots of $2x^2-13x+15 = 0$.

Comparing this with the equation $ax^2+bx+c = 0$, we have $a = 2$, $b = -13$, $c = 15$.

Hence, the roots of this given equation are

$$= \frac{-(-13)\pm\sqrt{(-13)^2-4\times2\times15}}{2\times2}$$

$$= \frac{13\pm\sqrt{169-120}}{4}$$

$$= \frac{13\pm\sqrt{49}}{4} = \frac{13\pm7}{4}.$$

That is, $x = 5$, or $\frac{3}{2}$.

Example 2. Write down the roots of $-3x^2 = 11x-4$.

Bringing all the terms to one side, we have

$$-3x^2-11x+4 = 0.$$

Here, $a = -3$, $b = -11$, $c = 4$.

$$\text{Hence, } x = \frac{-(-11)\pm\sqrt{(-11)^2-4\times(-3)\times4}}{2\times(-3)}$$

$$= \frac{11\pm\sqrt{121+48}}{-6}$$

$$= \frac{11\pm\sqrt{169}}{-6}$$

$$= \frac{11\pm13}{-6} = -4, \text{ or } \tfrac{1}{3}.$$

Exercise (28).

Write down the roots of the following equations :—

1. $3x^2 - 17x + 25 = 0$. 2. $x^2 + 9x + 20 = 0$.

3. $6x^2 = 20 - 7x$. 4. $-9x^2 + 25 = 6x - 10$.

5. $8x^2 = 14x + 15$. 6. $-3x^2 + 20x = 25$.

7. $5 + x - 4x^2 = 0$.

5. Sreedharacharyya's Method of solving a Quadratic. —Reduce the equation to the form $px^2 + qx = r$; multiply both sides of this by $4p$(*i.e.*, by four times the co-efficient of x^2) and *then* add q^2 to both sides ; we thus get $4p^2x^2 + 4pqx + q^2 = 4pr + q^2$, the left-hand side of which is evidently a complete square being equal to $(2px + q)^2$.

Example. Solve $5x^2 - 17x + 6 = 0$.

By transposition, $5x^2 - 17x = -6$.

Multiplying both sides by 4×5,

$$4 \times (5x)^2 - 4 \times (5x) \times 17 = -120.$$

Adding $(17)^2$ to both sides, we have

$$4 \times (5x)^2 - 4 \times (5x) \times 17 + (17)^2 = 289 - 120$$

$$\text{or } (2 \times 5x - 17)^2 = 169,$$

$$\therefore 10x - 17 = \pm 13,$$

$$\therefore x = \frac{17 \pm 13}{10}$$

$$= 3, \text{ or } \tfrac{2}{5}.$$

Exercise (29)

Solve the following equations by Sridharacharyya's method :—

✓1. $2x^2 + 9x = 18$. ✓2. $15x^2 - 28 = x$.

3. $16x^2 + 100x = 3x^2 + x + 40$. 4. $x^2 + 50x = 102 - 15x - x^2$.

5. $17x^2 + 19x = 1848$. 6. $2cx^2 - acx = 3(2x - a)$.

7. $x^2 + ax = ab(3x + a) - 2x^2$.

6. Solution of a Quadratic by the method of Resolution into Factors.—Reducing a Quadratic to the form

$ax^2+bx+c = 0$, if we know the factors of which the left-hand side is the product, then by equating to zero either of these factors, we get a solution of the Quadratic.

Example 1. Solve $x^2-5x+6 = 0$.

Evidently the left-hand side $= (x-2)(x-3)$.

Hence, we have

$$(x-2)(x-3) = 0.$$

$$\therefore\ \text{Either}\ \left.\begin{aligned} x-2 &= 0\\ \text{and}\ \therefore\ x &= 2\end{aligned}\right\}\quad \text{or,}\ \left.\begin{aligned} x-3 &= 0\\ \text{and}\ \therefore\ x &= 3\end{aligned}\right\}.$$

Thus 2 and 3 are the roots of the equation, as the student can easily verify.

Example 2. Solve $2x^2-10x = 3x-15$.

We have $2x(x-5) = 3(x-5)$ ....(1)

$$\therefore\ (2x-3)(x-5) = 0.$$

$$\text{Hence, either}\ \left.\begin{aligned} 2x-3 &= 0\\ \text{and}\ \therefore\ x &= \tfrac{3}{2}\end{aligned}\right\}\quad \text{or,}\ \left.\begin{aligned} x-5 &= 0\\ \text{and}\ \therefore\ x &= 5\end{aligned}\right\}.$$

Thus $\frac{3}{2}$ and 5 are the roots of the equation.

NOTE. The solution also at once follows from equation (1); for $x-5$ being a factor common to both sides, the equation evidently holds good when this factor is zero, *i. e*, when $x = 5$, and evidently also the equation is satisfied when $2x = 3$ or $x = \frac{3}{2}$: therefore 5 and $\frac{3}{2}$ are the roots of the equation. The student will thus observe that it is not always necessary to transpose all the terms to the left-hand side of the equation.

Example 3. Solve $(7-4\sqrt{3})x^2+(2-\sqrt{3})x = 2$.

Since $7-4\sqrt{3} = (2-\sqrt{3})^2$,

we have $(2-\sqrt{3})^2x^2+(2-\sqrt{3})x = 2$.

Hence, putting z for $(2-\sqrt{3})x$, we have

$$z^2+z-2 = 0$$

or, $(z+2)(z-1) = 0$.

$$\text{Hence, either}\ \left.\begin{aligned} z+2 &= 0\\ \text{and}\ \therefore\ z &= -2\end{aligned}\right\}\quad \text{or,}\ \left.\begin{aligned} z-1 &= 0\\ \text{and}\ \therefore\ z &= 1\end{aligned}\right\}.$$

$$\left.\begin{aligned}\text{Thus,}\ x &= \frac{-2}{2-\sqrt{3}} = -2(2+\sqrt{3})\\ \text{or,}\ &= \frac{1}{2-\sqrt{3}} = 2+\sqrt{3}\end{aligned}\right\}$$

Example 4. Solve $\dfrac{1}{(x-b)(x-c)}+\dfrac{1}{(a+c)(a+b)}$

$$=\frac{1}{(a+c)(x-c)}+\frac{1}{(a+b)(x-b)}.$$

By transposition,

$$\frac{1}{x-c}\left\{\frac{1}{x-b}-\frac{1}{a+c}\right\}=\frac{1}{a+b}\left\{\frac{1}{x-b}-\frac{1}{a+c}\right\}.$$

Therefore, either $\dfrac{1}{x-b}-\dfrac{1}{a+c}=0$ (See Note Ex. 2.)

whence $b=a+b+c$;

or $\dfrac{1}{x-c}=\dfrac{1}{a+b}$,

whence also $x=a+b+c$.

Thus the equation has got two *equal roots.*

Example 5. Solve $\dfrac{a+c(a+x)}{a+c(a-x)}+\dfrac{a+x}{x}=\dfrac{a}{a-2cx}$.

Since $\dfrac{a+c(a+x)}{a+c(a-x)}=\dfrac{a}{a+c(a-x)}+\dfrac{c(a+x)}{a+c(a-x)}$,

we have by transposition,

$$(a+x)\left\{\frac{c}{a+c(a-x)}+\frac{1}{x}\right\}=a\left\{\frac{1}{a-2cx}-\frac{1}{a+c(a-x)}\right\},$$

or, $$(a+x).\frac{a(1+c)}{x\{a+c(a-x)\}}=a.\frac{c(a+x)}{(a-2cx)\{a+c(a-x)\}},$$

or, $$\frac{(a+x)(1+c)}{x}=\frac{c(a+x)}{a-2cx}.$$

Hence, either $a+x=0$, and $\therefore\ x=-a$;

or, $\dfrac{1+c}{x}=\dfrac{c}{a-2cx}$, whence $x=\dfrac{a(1+c)}{c(3+2c)}$.

Thus $-a$ and $\dfrac{a(1+c)}{c(3+2c)}$ are the roots of the equation.

Exercise (30).

Solve the following equations :—

1. $3x^2 - 12x + 1 = 6x - 23.$
2. $4x^2 - 4x = 80.$
3. $x + 2 - \frac{6}{x+2} = 1.$
4. $x^2 + 9x - 52 = 0.$
5. $x^2 - \frac{5}{3}x - 4 = 0.$
6. $6x^2 + 5x - 4 = 0.$
7. $3(x-2)^2 = 18 + (8x + 1).$
8. $x - \frac{x^3 - 8}{x^2 + 5} = 2.$
9. $\frac{21x^3 \quad 16}{3x^2 - 4} - 7x = 5.$
10. $x^2 - (a+b)x + ab = 0.$
11. $(a-b)x^2 - (a+b)x + 2b = 0.$
12. $\frac{x^2}{a^{\frac{1}{2}} + b^{\frac{1}{2}}} - \left(a^{\frac{1}{2}} - b^{\frac{1}{2}}\right)x = \frac{1}{\left(ab^2\right)^{-\frac{1}{2}} + \left(a^2b\right)^{-\frac{1}{2}}}$
13. $\frac{2x(a-x)}{3a - 2x} = \frac{a}{4}.$
14. $\frac{16}{x} + \frac{x^{\frac{1}{2}}}{2} = \frac{6}{x^{\frac{1}{2}}}.$
15. $\frac{a}{x-a} + \frac{b}{x-b} \quad \frac{2c}{x-c}.$
16. $\frac{a - \sqrt{2ax - x^2}}{a + \sqrt{2ax - x^2}} = \frac{x}{a - x}.$

CHAPTER VII.

METHODS OF SOLUTION OF SOME IMPORTANT CLASSES OF EQUATIONS REDUCIBLE TO QUADRATICS.

1. Equations reducible to the form $px^{2n} + qx^n = r$.

Example 1. Solve $x^6 - 9x^3 + 8 = 0.$

$$\begin{aligned}
\text{We have}\quad x^6 - 9x^3 + \left(\tfrac{9}{2}\right)^2 &= -8 + \tfrac{81}{4} \\
\text{or,}\quad \left(x^3 - \tfrac{9}{2}\right)^2 &= \tfrac{49}{4}, \\
\therefore\quad x^3 - \tfrac{9}{2} &= \pm\tfrac{7}{2}, \\
\therefore\quad x^3 &= \frac{9 \pm 7}{2} \\
&= 8, \text{ or } 1\ ; \\
\text{and}\quad \therefore\quad x &= 2, \text{ or } 1.
\end{aligned}$$

Thus 2 and 1 are the roots of the equation.

Example 2. Solve $6x^{\frac{3}{4}} = 7x^{\frac{1}{4}} - 2x^{-\frac{1}{4}}$.

Multiplying by $x^{\frac{1}{4}}$,

$$6x = 7x^{\frac{1}{2}} - 2.$$

Transposing and dividing by 6,

$$x - \tfrac{7}{6}x^{\frac{1}{2}} = -\tfrac{1}{3};$$

$$\therefore \quad x - \frac{7}{6}x^{\frac{1}{2}} + \left(\frac{7}{12}\right)^2 = \frac{49}{144} - \tfrac{1}{3},$$

$$\text{or,} \quad \left(x^{\frac{1}{2}} - \frac{7}{12}\right)^2 = \frac{1}{144};$$

$$\therefore \quad x^{\frac{1}{2}} - \frac{7}{12} = \pm\frac{1}{12},$$

$$\therefore \quad x^{\frac{1}{2}} = \frac{7 \pm 1}{12} = \tfrac{2}{3}, \text{ or } \tfrac{1}{2};$$

$$\therefore \quad x = \tfrac{4}{9}, \text{ or } \tfrac{1}{4}.$$

Exercise (31).

Solve the following equations :—

1. $x^6 - 35x^3 + 216 = 0.$
2. $x^4 - 14x^2 + 40 = 0.$
3. $x^{-1} + x^{-\frac{1}{2}} = 6.$
4. $9 + x^{-4} = 10x^{-2}.$
5. $x^{-2} - 2x^{-1} = 8.$
6. $x^{-1} + ax^{-\frac{1}{2}} = 2a^2.$
7. $x^{-\frac{4}{3}} + 2 = \dfrac{x^{-1} + 8}{x^{-\frac{2}{3}} + 5}$
8. $x^{\frac{2}{n}} + 15 = 8x^{\frac{1}{n}}.$
9. $2x^{\frac{1}{2n}} - x^{\frac{1}{n}} + 8 = 0.$
10. $x^{\frac{7}{3}} = 56x^{-\frac{2}{3}} + x^{\frac{5}{6}}.$
11. $6\sqrt[5]{x^4} = \sqrt[5]{x} + \sqrt[5]{\dfrac{1}{x^2}}.$
12. $5\dfrac{3x - 1}{1 + 5x^{\frac{1}{2}}} + \dfrac{2}{x^{\frac{1}{2}}} = 3x^{\frac{1}{2}}.$
13. $3x^n\sqrt[3]{x^n} = \dfrac{2x^n}{\sqrt[3]{x^n}} = 16.$

2 Equations in which the expression equated to zero is capable of being resolved into factors involving the unknown quantity; or where expressions on both sides of the sign of equality have common factors involving the unknown quantity.

Example 1. Solve $27x^3+21x+8 = 0$.

$$\begin{aligned}\text{The left-hand side} &= 3x(9x^2-1)+8(3x+1)\\ &= (3x+1)\{3x(3x-1)+8\}\\ &= (3x+1)(9x^2-3x+8).\end{aligned}$$

Hence, we have

$$(3x+1)(9x^2-3x+8) = 0\,;$$

$$\left.\begin{array}{lrl} \therefore\ \text{either} & 3x+1 &= 0 \ldots\ldots\ldots(1)\\ \text{or,} & 9x^2-3x+8 &= 0 \ldots\ldots\ldots(2)\end{array}\right\}$$

From (1), $x = -\frac{1}{3}$.

From (2), $x = \dfrac{3\pm\sqrt{9-288}}{18} = \dfrac{3\pm\sqrt{-279}}{18}$.

Thus $-\dfrac{1}{3}$ and $\dfrac{3\pm\sqrt{-279}}{18}$ are the roots of the equation.

Example 2. Solve $x^4+2ax^3 = 2x+\dfrac{1}{a^2}$.

(Calcutta University F. A. Paper, 1874.)

By transposition, $\left(x^4-\dfrac{1}{a^2}\right)+2ax\left(x^2-\dfrac{1}{a}\right) = 0$.

or, $\left(x^2-\dfrac{1}{a}\right)\left\{\left(x^2+\dfrac{1}{a}\right)+2ax\right\} = 0$.

$$\text{Hence, either } \left.\begin{array}{l} x^2-\dfrac{1}{a} = 0\\ \text{and } \therefore\ x = \pm\dfrac{1}{\sqrt{a}}\end{array}\right\};$$

$$\text{or, } x^2+2ax+\frac{1}{a} = 0$$

$$\text{and } \therefore\ x = \frac{-2a\pm2\sqrt{a^2-\dfrac{1}{a}}}{2}$$

$$= -a\pm\sqrt{a^2-\frac{1}{a}}.$$

Example 3. Solve $8x^3 + 16x = 9$.

By transposition,

$$(8x^3 - 1) + 8(2x - 1) = 0,$$

or, $$(2x - 1)(4x^2 + 2x + 1) + 8(2x - 1) = 0,$$

or, $$(2x - 1)(4x^2 + 2x + 9) = 0;$$

$$\therefore \text{ either } \quad \left.\begin{aligned} (2x - 1) &= 0 \quad \ldots \quad (1) \\ \text{or, } \quad 4x^2 + 2x + 9 &= 0 \quad \ldots \quad (2) \end{aligned}\right\}$$

From (1), $x = \frac{1}{2}$.

From (2), $x = \dfrac{-2 \pm \sqrt{4 - 144}}{8} = \dfrac{-1 \pm \sqrt{-35}}{4}$.

Example 4. Solve $x^3 - 3x = 2$.

By transposition,

$$x(x^2 - 4) + (x - 2) = 0,$$

or, $$(x - 2)\{x(x + 2) + 1\} = 0;$$

$$\therefore \text{ either } \quad \left.\begin{aligned} x - 2 &= 0 \quad \ldots \quad \ldots \quad \ldots \quad (1) \\ \text{or, } \quad x^2 + 2x + 1 &= 0 \quad \ldots \quad \ldots \quad \ldots \quad (2) \end{aligned}\right\}$$

$$\left.\begin{aligned} \text{From (1), } \quad x &= 2 \\ \text{From (2), } \quad x &= -1 \end{aligned}\right\}$$

Thus 2 and -1 are the roots of the equation.

Example 5. Solve $2x^3 + 5x^2 - 4x - 3 = 0$.

The left-hand side

$$\begin{aligned} &= (2x^3 + 6x^2) - (x^2 + 4x + 3) \\ &= 2x^2(x + 3) - (x + 3)(x + 1) \\ &= (x + 3)(2x^2 - x - 1) \\ &= (x + 3)\{2x(x - 1) + (x - 1)\} \\ &= (x + 3)(x - 1)(2x + 1). \end{aligned}$$

Hence, we have

$$(x + 3)(x - 1)(2x + 1) = 0.$$

Hence, the roots of the equation are -3, 1, and $-\frac{1}{2}$.

Exercise (32).

Solve the following equations :—

1. $6x^3 - 5x^2 + x = 0$.
2. $x^3 + x^2 - 4x - 4 = 0$.
3. $x^2 - \dfrac{2}{3x} = 1\frac{4}{9}$.
4. $x^3 + 7x = 22$.

5. $2x^3-x^2=1$. [We have $2x^2(x-1)+(x^2-1)=0$, &c.]

6. $x^3-6x+9=0$. 7. $3x^6+8x^4-8x^2=3$.

8. $x(x^2-2)=m(x^2+2mx+2)$.

[We have $(x^3+m^3)-2(x+m)=m(x+m)^2$, &c.]

9. $x^2(x^2-3x+2)-7x(x^2-3x)+12x^2-50x+24=0$.

(Calcutta University F. A. Paper, 1875).

10. $x^3-6x^2+10x-8=0$.

[We have $x(x^2-6x+8)+2(x-4)=0$, &c.]

11. $(x^2-5)^2=(x-3)^2+(x+1)^2$.

[We have $(x-3)^2=(x^2-5)^2-(x+1)^2$, &c.]

12. $x^3+px^2+\left(p-1+\dfrac{1}{p-1}\right)x+1=0$.

13. $(p-1)^2x^3+px^2+\left(p-1+\dfrac{1}{p-1}\right)x+1=0$.

3. Equations reducible to the form $z^2+pz=q$, where z represents a simple or quadratic expression involving the unknown quantity.

Exercise (33).

Solve the following equations :—

1. $3^{2x}+9=10.3^x$.

Putting z for 3^x, we have

$$z^2-10z+9=0,$$

$$\text{or,}\quad (z-9)(z-1)=0\,;$$

$$\left.\begin{array}{ll}\therefore\ \text{either } z=9 & \ldots\quad\ldots\quad(1)\\ \text{or,}\quad z=1 & \ldots\quad\ldots\quad(2)\end{array}\right\}$$

$$\left.\begin{array}{l}\text{From (1), } 3^x=9=3^2,\quad\therefore\ x=2\\ \text{From (2), } 3^x=1=3^0,\quad\therefore\ x=0\end{array}\right\}.$$

Thus 2 and 0 are the roots of the equation.

2. $2^{x+1}+4^x=80$.

$$\left.\begin{array}{l}\text{We have}\qquad\qquad 2^{x+1}=2.2^x\\ \text{and } 4^x=(2^2)^x=2^{2x}=(2^x)^2\end{array}\right\}.$$

Hence, putting z for 2^x, we have

$$\begin{aligned} 2z + z^2 &= 80, \\ \text{or, } (z+1)^2 &= 81, \\ \therefore \quad z &= -1 \pm 9 \\ &= 8 \quad \ldots \quad \ldots \quad (1) \\ \text{or, } &= -10 \quad \ldots \quad \ldots \quad (2) \end{aligned}$$

From (1), $2^x = 8 = 2^3$, $\therefore x = 3$.

From (2), $2^x = -10$, which gives no real value for x.

3. $5^x + 5^{2-x} = 26$. **4.** $3^{1+2x} + 3^{1-2x} = 10$.

5. $2^{2x+8} + 1 = 32.2^x$. **6.** $2^{2x+3} - 57 = 65(2^x - 1)$.

7. $7^{1+x} + 7^{-x} = 8$. **8.** $\sqrt{3^x} + \frac{3}{\sqrt{3^x}} = 4$.

9. $3x^2 - 4x + \sqrt{3x^4 - 4x - 6} = 18$.

Adding -6 to each side,

$$3x^2 - 4x - 6 + \sqrt{3x^4 - 4x - 6} = 12.$$

Putting z for $\sqrt{3x^2} - 4x - 6$, we have

$$\begin{aligned} z^2 + z &= 12, \\ \text{or, } \quad z^2 + z - 12 &= 0, \\ \text{or, } \quad (z+4)(z-3) &= 0; \\ \therefore \quad \text{either } z &= -4 \quad \ldots\ldots \quad \ldots\ldots \quad (1) \\ \text{or, } z &= 3 \quad \ldots\ldots \quad \ldots\ldots \quad (2) \end{aligned}$$

From (1),

$$\begin{aligned} \sqrt{3x^4 - 4x - 6} &= -4, \\ \text{or, } \quad 3x^2 - 4x - 6 &= 16, \\ \text{or, } \quad 3x^2 - 4x - 22 &= 0, \\ \therefore x &= \frac{4 \pm \sqrt{16 + 4\,66}}{6} \\ &= \frac{4 \pm 2\sqrt{4 + 66}}{6} \\ &= \frac{2 \pm \sqrt{70}}{} \end{aligned}$$

From (2),
$$\begin{aligned}\sqrt{3x^2-4x-6} &= 3,\\ \text{or,}\quad 3x^2-4x-6 &= 9,\\ \text{or,}\quad 3x^2-4x-15 &= 0,\\ \text{or,}\quad (x-3)(3x+5) &= 0,\\ \therefore\quad x &= 3, \text{ or } -\tfrac{5}{3}.\end{aligned}$$

10. $\dfrac{3x-2}{2}+\sqrt{2x^2-5x+3}=\dfrac{(x+1)^2}{3}$.

We have
$$\begin{aligned}3(3x-2)+6\sqrt{2x^2-5x+3} &= 2(x+1)^2\\ &= 2(x^2+2x+1).\end{aligned}$$

Hence, by transposition,
$$2x^2-5x-6\sqrt{2x^2-5x+3}=-8.$$

Adding 3 to both sides,
$$(2x^2-5x+3)-6\sqrt{2x^2-5x+3}=-5.$$

Putting z for $\sqrt{2x^2-5x+3}$, we have
$$\begin{aligned}z^2-6z+5 &= 0\\ \text{or,}\quad (z-1)(z-5) &= 0;\\ \therefore\quad \text{either}\quad z &= 1 \quad \ldots(1)\\ \text{or,}\quad z &= 5 \quad \ldots(2)\end{aligned}$$

From (1),
$$\begin{aligned}\sqrt{2x^2-5x+3} &= 1,\\ \therefore\quad 2x^2-5x+3 &= 1,\\ \text{or,}\quad 2x^2-5x+2 &= 0,\\ \text{or,}\quad (x-2)(2x-1) &= 0;\\ \therefore\quad x &= 2, \text{ or } \tfrac{1}{2}.\end{aligned}$$

From (2),
$$\begin{aligned}2x^2-5x+3 &= 25,\\ \text{or,}\quad 2x^2-5x-22 &= 0;\\ \therefore\quad x &= \frac{5\pm\sqrt{25+176}}{4}=\frac{5\pm\sqrt{201}}{4}.\end{aligned}$$

11. $x^2+2\sqrt{x^2+6x}=24-6x$.

12. $x^2+3=2\sqrt{x^2-2x+2}+2x$.

13. $x^2+5x+4=5\sqrt{x^2+5x+28}$.

14. $x^2+x+10\sqrt{x^2+3x+16}=2(20-x)$.

(Madras University F. A. Paper, 1888.)

15. $x^2 + 3 - \sqrt{2x^2 - 3x + 2} = \frac{3}{2}(x+1)$.

16. $8 + 9\sqrt{(3x-1)(x-2)} = 3x^2 - 7x$.

17. $x^2 - x + 3\sqrt{2x^2 - 3x + 2} = \frac{x}{2} + 7$.

18. $\frac{9}{1 + x + x^2} = 5 - x - x^2$.

19. $9x - 4x^2 + \sqrt{4x^2 - 9x + 11} = 5$.

20. $2(x^2 - 3x + 1)^2 + 5(x^2 - 3x + 1) + 3 = 0$.

(Calcutta University F. A. Paper, 1873.)

21. $(x+4)(x+1) + \sqrt{(x+5)(x-3)} = 3x + 31$.

(Calcutta University F. A. Paper, 1877.)

22. $3x + 2\sqrt{x^2 - 3x + 9} = x^2 + 6$.

(Calcutta University F. A. Paper, 1886.)

23. $(x-5)(x-7)(x+6)(x+4) = 504$.

$$\begin{aligned}\text{The left-hand side} &= \{(x-5)(x+4)\} \times \{(x-7)(x+6)\}\\ &= (x^2 - x - 20)(x^2 - x - 42).\end{aligned}$$

Hence, putting z for $x^2 - x$ we have

$$\begin{aligned} (z-20)(z-42) &= 504,\\ \text{or,}\quad z^2 - 62z + 840 &= 504,\\ \text{or,}\quad z^2 - 62z + 336 &= 0,\\ \text{or,}\quad (z-6)(z-56) &= 0;\end{aligned}$$

$$\therefore \left.\begin{aligned}\text{either } z &= 6 \;\ldots\;\ldots\; (1)\\ \text{or, } z &= 56 \;\ldots\;\ldots\; (2)\end{aligned}\right\}$$

From (1),

$$\begin{aligned} x^2 - x - 6 &= 0,\\ \text{or,}\quad (x-3)(x+2) &= 0;\\ \therefore\quad x &= 3, \text{ or } -2.\end{aligned}$$

From (2),

$$\begin{aligned} x^2 - x - 56 &= 0,\\ \text{or,}\quad (x-8)(x+7) &= 0;\\ \therefore\quad x &= 8, \text{ or } -7.\end{aligned}$$

24. $(2x-7)(x^2-9)(2x+5) = 91$.

The left-hand side

$$\begin{aligned} &= \{(2x-7)(x+3)\}\{(x-3)(2x+5)\}\\ &= (2x^2 - x - 21)(2x^2 - x - 15).\end{aligned}$$

Hence, putting z for $2x^2-x$, we have

$$\begin{aligned}
(z-21)(z-15) &= 91,\\
z^2-36z+315 &= 91,\\
\text{or,}\quad z^2-36z+224 &= 0,\\
\text{or,}\quad (z-28)(z-8) &= 0;\\
\therefore\quad \text{either}\quad z &= 28\ldots\ldots(1)\\
\text{or,}\quad z &= 8\ldots\ldots(2)
\end{aligned}$$

From (1),
$$\begin{aligned}
2x^2-x-28 &= 0,\\
\text{or,}\quad (x-4)(2x+7) &= 0;\\
\therefore\quad x &= 4,\ \text{or} -\tfrac{7}{2}.
\end{aligned}$$

From (2),
$$\begin{aligned}
2x^2-x-8 &= 0,\\
\therefore\quad x &= \frac{1\pm\sqrt{1+64}}{4}\\
&= \frac{1\pm\sqrt{65}}{4}.
\end{aligned}$$

✓25. $$\frac{1}{3x^2+11x+10}+\frac{1}{6x^2+19x+15}=\frac{2x^2+9x+10}{150}.$$

(Madras University F.A. Paper, 1880.)

Since
$$\begin{aligned}
3x^2+11x+10 &= (3x+5)(x+2),\\
6x^2+19x+15 &= (3x+5)(2x+3),\\
\text{and}\quad 2x^2+\ 9x+10 &\quad (2x+5)(x+2),
\end{aligned}$$

we have
$$\frac{1}{3x+5}\left\{\frac{1}{x+2}+\frac{1}{2x+3}\right\}\quad\frac{(2x+5)(x+2)}{150},$$

$$\text{or,}\quad \frac{1}{(x+2)(2x+3)}\quad\frac{(2x+5)(x+2)}{150}$$

$$\begin{aligned}
\text{or,}\quad (2x+3)(2x+5)(x+2)^2 &= 150,\\
\text{or,}\quad (4x^2+16x+15)(x^2+4x+4) &= 150.
\end{aligned}$$

Hence, putting z for x^2+4x, we have

$$\begin{aligned}
(4z+15)(z+4) &= 150,\\
\text{or,}\quad 4z^2+31z-90 &= 0,\\
\text{or,}\quad (4z-9)(z+10) &= 0.
\end{aligned}$$

Hence, &c.

26. $(x-7)(x-3)(x+5)(x+1)=1680.$

27. $(x+9)(x-3)(x-7)(x+5) = 385.$

28. $(x+9)(x-4)(x-8)(x+2) = 825.$

29. $x(2x+1)(x-2)(2x-3) = 63.$

30. $16x(x+1)(x+2)(x+3) = 9.$

31. $x^4-2x^3+x = 132.$

[The left-hand side $= (x^4-2x^3+x^2)-(x^2-x)$
$= (x^2-x)^2-(x^2-x)$; hence, &c.]

32. $x^4-8x^3+10x^2+24x+5 = 0.$

[The left-hand side $= (x^4-8x^3+16x^2)-(6x^2-24x)+5$.]

4. An Important Artifice.

Example. Solve $\sqrt{3x^2-7x-30}-\sqrt{2x^2-7x-5}$

$$= x-5. \quad \dots \quad \dots \quad (1)$$

We have *identically*

$$(3x^2-7x-30)-(2x^2-7x-5) = x^2-25 \quad \dots \quad \dots \quad (2)$$

(*i e.*, this relation is true for *every* value of x, and hence it is also true for the particular value which x has in the proposed equation.)

$$\therefore \frac{(3x^2-7x-30)-(2x^2-7x-5)}{\sqrt{3x^2-7x-30}-\sqrt{2x^2-7x-5}} = \frac{x^2-25}{x-5}$$

$$\text{or,} \quad \sqrt{3x^2-7x-30}+\sqrt{2x^2-7x-5} = x+5 \quad \dots \quad (3)$$

From (1) and (3), by addition,

$$2\sqrt{3x^2-7x-30} = 2x,$$

$$\therefore \quad 3x^2-7x-30 = x^2,$$

$$\text{or,} \quad 2x^2-7x-30 = 0,$$

$$\text{or,} \quad (x-6)(2x+5) = 0,$$

$$\therefore \quad x = 6, \text{ or } -\tfrac{5}{2}.$$

N. B.—We might as well have subtracted (1) from (3) and got the same result.

Exercise (34).

Solve the following equations :—

1. $\sqrt{2x^2+5x-2}-\sqrt{2x^2+5x-9} = 1.$

2. $\sqrt{3x^2+7x-1}+\sqrt{3x^2+7x-10} = 9.$

3. $\sqrt{5x^2 - 6x + 8} - \sqrt{5x^2 - 6x - 7} = 1.$

4. $\sqrt{4x^2 - 7x + 16} + \sqrt{4x^2 - 7x - 1} = 17.$

5. $\sqrt{5x^2 + 7x + 2} - \sqrt{4x^2 + 7x + 18} = x - 4.$

5. Reciprocal Equations.—In such equations if all the terms are brought to one side, the co-efficients of terms equidistant from the beginning and end are either equal and of the *same* sign, or equal and of *different* signs, (provided that in the latter case *the middle term be wanting* if the equation be of an *even* degree).

Thus the following equations are reciprocal :—

(i) $x^5 + 2x^4 + 3x^3 + 3x^2 + 2x + 1 = 0.$

(ii) $x^5 - 2x^4 + 3x^3 - 3x^2 + 2x - 1 = 0.$

(iii) $x^4 - 2x^3 + 3x^2 - 2x + 1 = 0.$

(iv) $x^4 - 2x^3 + 2x - 1 = 0$

NOTE. The last equation would *not* be called reciprocal *if the co-efficient of* x^2 *were not zero.* Such equations remain unaltered when x is changed into $\frac{1}{x}$ and *hence* the name.

The method of solving such equations will be illustrated by the following examples.

Example 1. Solve $x^4 + x^3 - 4x^2 + x + 1 = 0.$

We have $(x^4 + 1) + (x^3 + x) - 4x^2 = 0.$

Dividing both sides by x^2,

$$\left(x^2 + \frac{1}{x^2}\right) + \left(x + \frac{1}{x}\right) - 4 = 0,$$

$$\text{or,} \quad \left\{\left(x + \frac{1}{x}\right)^2 - 2\right\} + \left(x + \frac{1}{x}\right) - 4 = 0.$$

Hence. putting z for $x + \frac{1}{x}$, we have

$$z^2 + z - 6 = 0,$$

$$\text{or,} \quad (z + 3)(z - 2) = 0;$$

$$\left.\begin{aligned} \therefore \text{ either } z &= -3 \quad \ldots \quad \ldots \quad (1) \\ \text{or, } z &= 2 \quad \ldots \quad \ldots \quad (2) \end{aligned}\right\}$$

From (1), $$x+\frac{1}{x} = -3$$

or, $$x^2+3x+1 = 0;$$

$$\therefore\ x = \frac{-3\pm\sqrt{9-4}}{2} = \frac{-3\pm\sqrt{5}}{2}.$$

From (2), $$x+\frac{1}{x} = 2$$

or, $$x^2-2x+1 = 0,$$

or, $$(x-1)^2 = 0;$$

$$\therefore\ x = 1.$$

Example 2. Solve $x^8-2x^6+3x^4-2x^2+1 = 0$.

We have $$(x^8+1)-2(x^6+x^2)+3x^4 = 0.$$

Dividing both sides by x^4,

$$\left(x^4+\frac{1}{x^4}\right)-2\left(x^2+\frac{1}{x^2}\right)+3 = 0.$$

Let $$z = x+\frac{1}{x};$$

then $$x^2+\frac{1}{x^2} = \left(x+\frac{1}{x}\right)^2-2$$

$$= z^2-2 \quad \ldots\ldots \quad \ldots\ldots \quad (1)$$

and $$x^4+\frac{1}{x^4} = \left(x^2+\frac{1}{x^2}\right)^2-2$$

$$= (z^2-2)^2-2$$

$$= z^4-4z^2+2 \quad \ldots\ldots \quad \ldots\ldots \quad (2)$$

Hence, by (1) and (2), the equation becomes

$$(z^4-4z^2+2)-2(z^2-2)+3 = 0,$$

or, $$z^4-6z^2+9 = 0,$$

or, $$(z^2-3)^2 = 0;$$

$$\therefore\ z^2 = 3,$$

$$\therefore\ z = \pm\sqrt{3}.$$

Hence, 1st. $$x+\frac{1}{x} = \sqrt{3},$$

or, $$x^2-\sqrt{3}x+1 = 0;$$

$$\therefore \quad x = \frac{\sqrt{3} \pm \sqrt{3-4}}{2}$$

$$= \frac{\sqrt{3} \pm \sqrt{-1}}{2}.$$

2ndly, $$x + \frac{1}{x} = -\sqrt{3},$$

or, $$x^2 + \sqrt{3}x + 1 = 0,$$

$$\therefore \quad x = \frac{-\sqrt{3} \pm \sqrt{3-4}}{2}$$

$$= \frac{-\sqrt{3} \pm \sqrt{-1}}{2}.$$

Example 3. Solve $10x^4 - 63x^3 + 52x^2 + 63x + 10 = 0$.

This equation though *not reciprocal* can yet be solved by a similar method.

We have $10(x^4 + 1) - 63(x^3 - x) + 52x^2 = 0$.

Dividing by x^2,

$$10\left(x^2 + \frac{1}{x^2}\right) - 63\left(x - \frac{1}{x}\right) + 52 = 0.$$

Put z for $x - \frac{1}{x}$, then $z^2 + 2 = x^2 + \frac{1}{x^2}$;

hence, $$10(z^2 + 2) - 63z + 52 = 0,$$

or, $$10z^2 - 63z + 72 = 0,$$

or, $$(2z - 3)(5z - 24) = 0;$$

$$\therefore \quad \text{either } z = \tfrac{3}{2} \quad \ldots \quad \ldots \quad (1)$$

$$\text{or, } z = \tfrac{24}{5} \quad \ldots \quad \ldots \quad (2)$$

From (1), $$x - \frac{1}{x} = \tfrac{3}{2},$$

or, $$2x^2 - 3x - 2 = 0,$$

or, $$(x - 2)(2x + 1) = 0,$$

$$\therefore \quad x = 2, \text{ or } -\tfrac{1}{2}$$

From (2), $$x - \frac{1}{x} = \tfrac{24}{5},$$

or, $$5x^2 - 24x - 5 = 0,$$

or, $$(x - 5)(5x + 1) = 0,$$

$$\therefore \quad x = 5, \text{ or } -\tfrac{1}{5}$$

Exercise (35).

Solve the following equations :—

1. $x^4 - 3x^3 - 2x^2 - 3x + 1 = 0.$
2. $x^4 - x^3 + \frac{5}{4}x^2 - x + 1 = 0.$
3. $x^6 - x^5 + x^4 - x^2 + x - 1 = 0$

[The left-hand side $= (x^6 - 1) - x(x^4 - 1) + x^2(x^2 - 1)$
$= (x^2 - 1)\{(x^4 + x^2 + 1) - x(x^2 + 1) + x^2\}$
$= (x^2 - 1)(x^4 - x^3 + 2x^2 - x + 1).$]

4. $x^4 + 8x^2 + 1 = 5x(x^2 + 1).$
5. $4x^4 - 16x^3 + 23x^2 - 16x + 4 = 0.$
(Calcutta University F. A. Paper, 1882.)
6. $x^4 + \frac{53}{4}x^2 + 1 = 7x(x^2 + 1).$
7. $x^4 - 5x^3 + 15x + 9 = 0.$
8. $4x^4 - 16x^3 + 7x^2 + 16x + 4 = 0.$
9. $6x^4 - 25x^3 + 12x^2 + 25x + 6 = 0.$
10. $x^2 + \frac{1}{x^2} + 2\left(x + \frac{1}{x}\right) = \frac{142}{9}.$
11. $x^4 + ax^3 + bx^2 + cx + \frac{c^2}{a^2} = 0.$
12. $\frac{1 + x^4}{(1 + x)^4} = \frac{1}{2}.$

Miscellaneous Equations (36).

N. B.—Solutions of the equations given below will be obtained by some one or other of the methods explained in this chapter, or by the application of some special artifice. The student is strongly recommended to exercise his own ingenuity to the utmost and try to arrive at solutions independently of the hints given here and there.

1. $2^{x^2} : 2^{2x} = 8 : 1.$
2. $\frac{a + 2x + \sqrt{a^2 - 4x^2}}{a + 2x - \sqrt{a^2 - 4x^2}} = \frac{5x}{a}.$
3. $a^{2x}(a^2 + 1) = (a^{3x} + a^x)a.$
4. $\sqrt{2x^2 + 5x - 7} + \sqrt{3(x^2 - 7x + 6)} - \sqrt{7x^2 - 6x - 1} = 0.$

[First see if the equation could be divided out by any common factor.]

5. $\sqrt{a^2+2ax-3x^2}-\sqrt{a^2+ax-6x^2} = \sqrt{2a^2+3ax-9x^2}.$

6. $\sqrt{\dfrac{x}{1-x}}+\sqrt{\dfrac{1-x}{x}} = 2\tfrac{1}{6}.$

7. $(a+x)^{\frac{2}{3}}+4(a-x)^{\frac{2}{3}} = 5(a^2-x^2)^{\frac{1}{3}}.$

[Divide both sides by $(a-x)^{\frac{2}{3}}$ and then put z for $\left(\dfrac{a+x}{a-x}\right)^{\frac{1}{3}}$]

8. $\left(x+m\right)^{\frac{2}{5}}+\left(x-m\right)^{\frac{2}{5}}=\left(n+\dfrac{1}{n}\right)\left(x^2-m^2\right)^{\frac{1}{5}}.$

(Madras University F. A. Paper, 1884.)

9. $x^4-2x^3+x = 380.$

[We have $(x^2-x)^2-(x^2-x) = 380$. Hence, putting z for x^2-x, we have $z^2-z-380 = 0$, &c.]

10. $x^2+x^{-2}+x+x^{-1} = 4.$

11. $x^4-3x^3 = 1-3x.$

12. $x+4+\left(\dfrac{x+4}{x-4}\right)^{\frac{1}{2}} = \dfrac{12}{x-4}.$

13. $x-1 = 2+\dfrac{2}{x^{\frac{1}{2}}}.$

14. $(x^2+2x)(x+4) = 2-(x+4).$

15. $\dfrac{a-x}{x}+\dfrac{x}{a-x} = \dfrac{b}{c}.$

16. $\dfrac{49x^2}{4}+\dfrac{48}{x^2}-49 = 9+\dfrac{6}{x}.$

[Add $\dfrac{1}{x^2}$ to both sides, when each side becomes a complete square.]

17. $x = \dfrac{12+8x^{\frac{1}{2}}}{x-5}.$

[We have $x^2-4x = 12+8x^{\frac{1}{2}}+x$: adding 4 to each side we get complete squares.]

18. $\dfrac{x^2}{4} = \dfrac{x-12}{x^2-18}.$

[We have $x^4-14x^2 = 4x^2+4x-48$; now add 49 to both sides.]

19. $x^4-4x^3-2x^2+12x+9 = 0.$

[We have $(x^2-2x)^2-6(x^2-2x)+9 = 0$; hence, &c.]

20. $x^4 - 6x^3 + 6x^2 + 9x + 2 = 0.$

21. $x^4 - 8x^3 + 10x^2 + 24x + 5 = 0.$

22. $$\frac{x + c + \sqrt{x^2 - c^2}}{x + c - \sqrt{x^2 - c^2}} = \frac{9(x + c)}{8c}.$$

23. $$\frac{a^2 + ax + x^2}{a^2 - ax + x^2} = \frac{a^2}{x^2}.$$

24. $$\left(\frac{x}{x - 1}\right)^2 + \left(\frac{x}{x + 1}\right)^2 = n(n - 1).$$

[The left-hand side $= \left(\dfrac{x}{x - 1} + \dfrac{x}{x + 1}\right)^2 - \dfrac{2x^2}{x^2 - 1}$

$= \left(\dfrac{2x^2}{x^2 - 1}\right)^2 - \dfrac{2x^2}{x^2 - 1}$; hence, putting z for $\dfrac{2x^2}{x^2 - 1}$, &c.]

25. $$1 + \sqrt{1 - \frac{a}{x}} = \sqrt{1 + \frac{x}{a}}.$$

[We have $\sqrt{1 + \dfrac{x}{a}} - \sqrt{1 - \dfrac{a}{x}} = 1$; now square both sides.]

26. $$\sqrt{x - \frac{1}{x}} - \sqrt{1 - \frac{1}{x}} = \frac{x - 1}{x}.$$

27. $(x - 2)(x - 3)(x - 4) = 1.2.3.$

[Since the equation is obviously satisfied if $x = 5$, we know from Art. 2 that the expression obtained by bringing all the terms to one side must have $(x - 5)$ for a factor.

Now, since the left-hand side of the equation $= (x^2 - 5x + 6)(x - 4) = x^3 - 9x^2 + 26x - 24$, the equation becomes $x^3 - 9x^2 + 26x - 30 = 0$ when all the terms are brought to one side. Hence $x^3 - 9x^2 + 26x - 30$ must have $x - 5$ for a factor, which shows how to proceed.]

28. $$\frac{x}{a} + \frac{b}{x} + \frac{b^2}{x^2} = 1 + \frac{b}{a} + \frac{b^2}{a^2}.$$

29. $(x - 1)(x - 2)(x - 3) - (6 - 1)(6 - 2)(6 - 3) = 0.$

30. $(x - 1)(x - 2)(x - 3) = 24.$

31. $$(1 + x + x^2)^2 = \frac{a + 1}{a - 1}(1 + x^2 + x^4).$$

32. $\dfrac{1 + x^4}{(1 + x)^4} = 7.$ 33. $x^3 - 1 = 0.$ 34. $x^4 + 1 = 0.$

35. $\left(\dfrac{2x+3}{2x-3}\right)^{\frac{1}{3}} + \left(\dfrac{2x-3}{2x+3}\right)^{\frac{1}{3}} = \dfrac{8}{13}\cdot\dfrac{4x^2+9}{4x^2-9}.$

[Cubing both sides, the left-hand side becomes

$$\frac{2(4x^2+9)}{4x^2-9} + \frac{24}{13}\cdot\frac{4x^2+9}{4x^2-9} \text{ or, } \frac{50}{13}\,\frac{4x^2+9}{4x^2-9},$$

$$\because \quad (a+b)^3 = a^3+b^3+3ab(a+b).]$$

36. $\dfrac{(x-2)^2}{x^2-4x} + \dfrac{2}{(x-2)^2} = 4.$

(Calcutta University F. A. Paper, 1889)

[We have $\left\{\dfrac{(x-2)^2}{x^2-4x} - 1\right\} + \dfrac{2}{(x-2)^2} = 3$

or, $\dfrac{4}{x^2-4x} + \dfrac{2}{x^2-4x+4} = 3.$ Now put z for x^2-4x.]

CHAPTER VIII.

THEORY OF QUADRATIC EQUATIONS AND QUADRATIC EXPRESSIONS.

2. A Quadratic Equation cannot have more than two roots.—Let us take the equation $ax^2+bx+c=0$ which is the general form of all quadratics. If possible, let this equation have three *different* roots α, β, γ. We then must have

$$\left.\begin{aligned} a\alpha^2+b\alpha+c &= 0 \quad \ldots \quad \ldots \quad \ldots \quad (1)\\ a\beta^2+b\beta+c &= 0 \quad \ldots \quad \ldots \quad \ldots \quad (2)\\ a\gamma^2+b\gamma+c &= 0 \quad \ldots \quad \ldots \quad \ldots \quad (3)\end{aligned}\right\}$$

From (1) and (2), by subtraction, we have

$$a(\alpha^2-\beta^2)+b(\alpha-\beta) = 0,$$

$$\text{or,} \quad (\alpha-\beta)\{a(\alpha+\beta)+b\} = 0.$$

Now, since $\alpha-\beta$ is *not* zero (α and β being *different*),

$$\therefore \quad a(\alpha+\beta)+b = 0 \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

Similarly, from (1) and (3),

$$a(\alpha+\gamma)+b = 0 \quad \ldots \quad \ldots \quad \ldots \quad (5)$$

Hence, subtracting (5) from (4), we have

$$a(\beta - \gamma) = 0 \quad \dots \quad \dots \quad \dots \quad (6)$$

which is impossible since a is not zero and also *by supposition* $\beta - \gamma$ is not zero. Hence, *the supposition is wrong,* in other words, there *cannot be three different roots* to a quadratic.

Cor. 1. The equation $ax^2 + bx + c = 0$ can be satisfied by *three different* values of x *only* when $a = 0$, $b = 0$, and $c = 0$. For, the equation is so satisfied *only* when the relation (6) holds good, and β, γ being *different,* this relation is true *only* when $a = 0$; but when $a = 0$, by (4) or (5) b also $= 0$, and therefore by (1), (2), or (3) c also $= 0$.

Cor. 2. If the equation $ax^2 + bx + c = 0$ be satisfied by three *different* values of x, it will be satisfied by *any value whatever* of x, (or, the equation will be an identity). For by Cor. 1, the equation now becomes $0.x^2 + 0.x + 0 = 0$, which evidently is true for all values of x.

Example 1. Is $2\{(x-a)(x-b) + (a-x)(a-b) + (b-x)(b-a)\} = (a-b)^2 + (x-a)^2 + (x-b)^2$ an *Identity* or an *Equation ?*

The student can easily see that this relation is satisfied when $x = a$, or $x = b$, or $x = 0$, *i. e.*, by three different values of x; hence it is satisfied by *any value whatever* of x and is therefore an identity.

Example 2. Show that, if—

$$(p-a)^3 + (p-b)^3 = 2(p-c)^3,$$
$$(q-a)^3 + (q-b)^3 = 2(q-c)^3,$$
$$(r-a)^3 + (r-b)^3 = 2(r-c)^3,$$

then will also

$$(s-a)^3 + (s-b)^3 = 2\,(s-c)^3.$$

2. Nature of the roots of a Quadratic.—If α, β denote the roots of the quadratic $ax^2 + bx + c = 0$, we have by art. 4, Chap. [illegible]I.,

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad \beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

Hence the following conclusions are evident :—

(1) If $b^2 - 4ac$ be a positive quantity, $\sqrt{b^2 - 4ac}$ is real and therefore α, β are *real* (and *unequal*).

(2) If b^2-4ac be a *perfect square*, $\sqrt{b^2-4ac}$ is *rational* and therefore α, β are *rational* (and *unequal*)

(3) If b^2-4ac be zero, $\sqrt{b^2-4ac}$ is zero, and therefore α, β are *real* and *equal.*

(4) If b^2-4ac be a negative quantity, $\sqrt{b^2-4ac}$ is imaginary and therefore α, β are *imaginary* (and *unequal.*)

Exercise (37).

1. Examine the roots of the equation $2x^2+7x-4 = 0$.

The expression under the radical sign in the roots of the given equation $= 7^2-4.2.(-4)$

$= 49+32$

$= 81$ which is a perfect square ;

$\therefore$ the roots are *rational* and *unequal.*

2. Examine the roots of the equation $2x^2-9x+8 = 0$.

The expression under the radical sign in the roots of the given equation $= (-9)^2-4.2.8$

$= 81-64$

$= 17$ whose square roots cannot be exactly found ;

$\therefore$ the roots are *real* (and *unequal*), but *not rational.*

3. Examine the roots of the equation $3x^2+4x+2 = 0$.

The expression under the radical sign in the roots of the given equation $= (4)^2-4.3.2$

$= 16-24$

$= -8$, whose square roots are impossible or *imaginary* ;

$\therefore$ the roots under examination too are *imaginary.*

4. Examine the roots of the following equations :—

(1) $3x^2+20x-19 = 0$. (2) $2x^2-8x+9 = 0$.

(3) $x^2+5x+4 = 0$. (4) $4x^2-12x+9 = 0$.

(5) $-3x^2-2x+6 = 0$. (6) $-4x^2+5x-8 = 0$.

5. Show that the equation $3x^2+7x+8=0$ cannot be satisfied by any real values of x.

6. For what values of m will the equation $x^2-2(5+2m)x+3(7+10m)=0$ have equal roots?

By the condition of the problem, the expression under the radical sign in the roots must be zero.

Hence, we must have

$$\begin{aligned} 4(5+2m)^2-4.3(7+10m) &= 0, \\ \text{or,} \quad (25+20m+4m^2)-(21+30m) &= 0, \\ \text{or,} \quad 4m^2-10m+4 &= 0, \\ \text{or,} \quad 2m^2-5m+2 &= 0, \\ \text{or,} \quad (2m-1)(m-2) &= 0; \\ \therefore \quad m &= \tfrac{1}{2}, \text{ or } 2. \end{aligned}$$

That is, the given equation has equal roots either when $m=\frac{1}{2}$, or when $m=2$.

7. If the equation $4x^2-px+9=0$ has equal roots, find p.

8. For what value of m will the equation $2x^2+8x+m=0$ have equal roots?

9. If the equation $(1+m)x^2-2(1+3m)x+(1+8m)=0$ has equal roots, find the values of m.

10. Prove that the roots of the equation $(a-b+c)x^2+4(a-b)x+(a-b-c)=0$ are real.

11. Show that the roots of $x^2+px+q=0$ will be rational, if $p=k+\frac{q}{k}$, where p, q, k are rational quantities.

The expression under the radical sign in the roots of the given equation

$$\begin{aligned} &= p^2-4q \\ &= \left(k+\frac{q}{k}\right)^2-4q \\ &= k^2-2q+\frac{q^2}{k^2} \\ &= \left(k-\frac{q}{k}\right)^2, \text{ whose square root is } \textit{rational}. \end{aligned}$$

∴ the roots of the given equation are *rational.*

12. Show that the roots of the following equations are real :—

(1) $4x^2 - 8x + (4 - a^2 - b^2) = 0.$

(2) $(a-b)x^2 + 2(a+b)x - (a-b) = 0.$

13. Prove that the roots of the equation $(a+c-b)x^2 + 2cx + (b+c-a) = 0$ are rational.

3. Applications of the results proved in Art. 2.—

Exercise (38).

1. If x be *real*, prove that $x^2 - 10x + 27$ can never be less than 2.

$$\begin{aligned} \text{Let} \quad x^2 - 10x + 27 &= m\,; \\ \text{then,} \quad x^2 - 10x + 25 &= m - 2, \\ \therefore \quad x - 5 &= \pm\sqrt{m-2}, \\ \therefore \quad x &= 5 \pm \sqrt{m-2}. \end{aligned}$$

Now, since x is *real*, $m - 2$ *cannot be negative*, and $\therefore$ m can never be less than 2. That is, whatever real value may be given to x, the corresponding value of the given expression will never be less than 2.

Otherwise :—

$$\begin{aligned} \text{The given expression} &= (x^2 - 10x + 25) + 2 \\ &= (x-5)^2 + 2. \end{aligned}$$

Now, since x is *real*, $(x-5)^2$ cannot be negative. Hence the given expression can never be less than 2.

2. If x be real, prove that—

(1) $4x^2 - 12x + 17$ cannot be less than 8.

(2) $x^2 - 9x + 21$,, ,, ,, ,, ,, $\frac{3}{4}$.

(3) $5x^2 - 7x + 4$,, ,, ,, ,, ,, $1\frac{11}{20}$.

3. Find the least (or *minimum*) value of $2x^2 - 7x + 6$ for real values of x.

$$\begin{aligned} \text{Let} \quad 2x^2 - 7x + 6 &= m\,; \\ \text{then} \quad 2x^2 - 7x + (6 - m) &= 0, \end{aligned}$$

$$\therefore \quad x = \frac{7 \pm \sqrt{49 - 8(6-m)}}{4}.$$

Hence in order that x may be real,

$49 - 8(6 - m)$ must not be negative,
or, $1 + 8m$ must not be negative,
or, $8(\frac{1}{8} + m)$ must not be negative.

Hence, since this expression becomes negative when m is less than $-\frac{1}{8}$, the least value m can have is $-\frac{1}{8}$.

Thus, whatever real values be given to x, the value of the given expression will never be less than $-\frac{1}{8}$

N. B.—The student must observe that when the given expression has its minimum value, the corresponding value of x is $\frac{7}{4}$, because in this case the expression under the radical sign is zero.

Otherwise :—

$$\text{The given expression} = 2\left(x^2 - \frac{7}{2}x\right) + 6$$

$$= 2\left(x^2 - \frac{7}{2}x + \frac{49}{16}\right) - \frac{49}{8} + 6$$

$$= 2\left(x - \frac{7}{4}\right)^2 - \frac{1}{8}.$$

Now, since x is real, $\left(x - \frac{7}{4}\right)^2$ and $\therefore$ $2\left(x - \frac{7}{4}\right)^2$ cannot be negative.

Hence, whatever real value x may have, the given expression can ***never*** be ***less than*** $-\frac{1}{8}$; *i.e.*, $-\frac{1}{8}$ is its least value.

It is also evident from the above relation that the given expression has its least value when $x = \frac{7}{4}$.

4. Find the minimum values of the following expressions for real values of x :—

(1) $4x^2 - 9x + 5$.
(2) $3x^2 - 5x + 4$.
(3) $2x^2 - 13x + 22$.

5. In the above examples find also the values of x corresponding to the minimum values of the expressions.

6. Shew that whatever real value may be given to x, the value of the expression $3 + 5x - 2x^2$ will never be greater than $6\frac{1}{8}$.

Let $$3+5x-2x^2 = m;$$
then, $$2x^2-5x+(m-3) = 0,$$
$$\therefore \quad x = \frac{5 \pm \sqrt{25-8(m-3)}}{4}.$$

Now, in order that x may be real the expression $25-8(m-3)$ must be positive or zero, *i.e.*, $49-8m$ must be positive or zero. Hence $8m$ cannot be greater than 49, or m cannot be greater than $\frac{49}{8}$.

That is, none of the different values of the given expression, due to different values of x, will be greater than $6\frac{1}{8}$.

N. B.—When the given expression has its greatest value (*i.e.*, $6\frac{1}{8}$), the expression under the radical sign is zero, and therefore the corresponding value of x is $\frac{5}{4}$.

Otherwise **:—**

$$\text{The given expression} = 3-2\left(x^2-\frac{5}{2}x\right)$$
$$= 3-2\left(x^2-\frac{5}{2}x+\frac{25}{16}\right)+\frac{25}{8}$$
$$= \frac{49}{8}-2\left(x-\frac{5}{4}\right)^2$$
$$= \frac{49}{8}+\left\{-2\left(x-\frac{5}{4}\right)^2\right\}.$$

Now, since x is real, $\left(x-\frac{5}{4}\right)^2$ cannot be negative, and therefore the expression within the braces can*not* be *positive*.

Hence, whatever real value x may have, the given expression can *never* be *greater* than $\frac{49}{8}$, *i.e.*, $\frac{49}{8}$ or $6\frac{1}{8}$ is its maximum value

It is also evident from the above relation that the given expression has its maximum value when $x = \frac{5}{4}$.

7. Find the greatest (or *maximum*) values of the following expressions for real values of x.

(1) $6x-x^2-1$.
(2) $5+8x-8x^2$.
(3) $5+4x-4x^2$.

8. In the above three cases find also the values of x corresponding to the maxima values of the expressions.

9. Find the greatest value of $\dfrac{x^2+14x+9}{x^2+2x+3}$ for real values of x.

Let $$\frac{x^2+14x+9}{x^2+2x+3} = m;$$

then, $$(m-1)x^2+2(m-7)x+3(m-3) = 0;$$

$$\therefore\quad x = \frac{2(7-m)\pm 2\sqrt{(m-7)^2-3(m-1)(m-3)}}{2(m-1)}.$$

The expression under the radical sign

$$= (m^2-14m+49)-3(m^2-4m+3)$$
$$= -2(m^2+m-20)$$
$$= -2(m+5)(m-4).$$

Now, since this must be *positive* or *zero*, *m cannot be greater than*, but *may be equal to*, 4. Hence 4 is the greatest value of the given expression.

10. In the last example find also the least (or *minimum*) value of the given expression.

Since $-2(m+5)(m-4)$ must be *positive* or *zero*, evidently *m cannot be less than*, but *may be equal to*, -5; (for when m has any value less than -5, say -6, both $(m+5)$ and $(m-4)$ are negative, and thus the above expression, being the product of *three* negative quantities, becomes also negative); hence -5 is the least value of the given expression.

N. B.—The student should carefully observe from the above results that whatever real value may be given to x, the value of the given expression will *never be greater than* 4 *and never less than* -5.

11. If x be *real*, prove that $\dfrac{x^2+34x-71}{x^2+2x-7}$ can have all numerical values except such as lie between 5 and 9.

Let $$\frac{x^2+34x-71}{x^2+2x-7} = m;$$

then, $$(m-1)x^2+2(m-17)x-(7m-71) = 0,$$

$$\therefore\quad x = \frac{2(17-m)\pm 2\sqrt{(m-17)^2+(m-1)(7m-71)}}{2(m-1)}.$$

The expression under the radical sign

$$\begin{aligned} &= (m^2 - 34m + 289) + (7m^2 - 78m + 71) \\ &= 8m^2 - 112m + 360 \\ &= 8(m^2 - 14m + 45) \\ &= 8(m - 5)(m - 9). \end{aligned}$$

Now, when m is less than 5, both $(m - 5)$ and $(m - 9)$ are negative, and therefore their product is positive ; also when m is greater than 9, both the factors are positive, and therefore their product is positive ; hence, for real values of x the given expression can have any value from 5 downwards or from 9 upwards.

But this product becomes negative when m has *any value between* 5 and 9 (for then $m - 5$ is positive, but $m - 9$ is negative), and therefore for *real* values of x the given expression can have *no value between* 5 *and* 9.

12. If x is real, prove that the expression $\dfrac{x^2 + 2x - 11}{2(x - 3)}$ can have all numerical values except such as lie between 2 and 6.

13. If x is real the value of $\dfrac{(x - 1)(x + 3)}{(x - 2)(x + 4)}$ does not lie between $\frac{4}{9}$ and 1.

(Madras University F. A. Paper, 1884.)

14. The expression $\dfrac{1}{x + 1} + \dfrac{1}{3x + 1} - \dfrac{1}{(x + 1)(3x + 1)}$ cannot lie between the values of 1 and 4, for any real value of x.

(Madras University F. A. Paper, 1880.)

15. If x be real, prove that $\dfrac{x^2 - 2x + 21}{6x - 14}$ cannot lie between 2 and $-\frac{10}{9}$.

$$\text{Let} \quad \frac{x^2 - 2x + 21}{6x - 14} = m \ ;$$

$$\text{then,} \quad x^2 - 2(1 + 3m)x + 7(3 + 2m) = 0 \ ;$$

$$\therefore \quad x = \frac{2(1 + 3m) \pm 2\sqrt{(1 + 3m)^2 - 7(3 + 2m)}}{2}.$$

The expression under the radical sign

$$\begin{aligned} &= (1 + 6m + 9m^2) - (21 + 14m) \\ &= 9m^2 - 8m - 20 \\ &= (9m + 10)(m - 2) \\ &= 9(m + \tfrac{10}{9})(m - 2). \end{aligned}$$

This becomes *negative* when m lies between $-\frac{10}{9}$ and 2. For when m is positive and less than 2, $m+\frac{10}{9}$ is positive, but $m-2$ is negative and therefore their product, negative ; also when m is negative and greater than $-\frac{10}{9}$(*i.e.*, *numerically less than* $\frac{10}{9}$), $m+\frac{10}{9}$ is positive, but $m-2$ is negative, and therefore their product, negative. Hence, for real values of x the given expression can have no value between $-\frac{10}{9}$ and 2.

16. If x be real, prove that $\frac{11x^2+12x+6}{x^2+4x+2}$ cannot lie between -5 and 3.

17. Prove that $\frac{x}{x^2-5x+9}$ must lie between 1 and $-\frac{1}{11}$, for all real values of x.

18. In the equation $x^2-px+q^2=0$, if x be *real*, prove that p cannot lie between $+2q$ and $-2q$.

(Calcutta University F. A. Paper, 1877.)

19. If x be real, prove the expression $\frac{x^2+8x+80}{2x+8}$ admits of all values except such as lie between -8 and 8.

20. If p be greater than unity, then for all real values of x, the expression $\frac{x^2-2x+p^2}{x^2+2x+p^2}$ lies between $\frac{p-1}{p+1}$ and $\frac{p+1}{p-1}$.

21. Prove that if the equation $x^2+4y^2-8x+12=0$ is to be satisfied by *real* values of x and y, x must lie between 2 and 6, and y must lie between -1 and 1.

First, let us express y in terms of x.

From the given equation we have

$$\begin{aligned} 4y^2 &= -(x^2-8x+12) \\ &= -(x-2)(x-6), \end{aligned}$$

$$\therefore \quad 2y = \pm\sqrt{-(x-2)(x-6)}.$$

Since y is real, $(x-2)(x-6)$ must be negative, and this is possible only when x is greater than 2 but less than 6.

Hence x must lie between 2 and 6.

Secondly, let us express x in terms of y.

From the given equation we have

$$x^2 - 8x + 4(y^2+3) = 0;$$

$$\therefore\quad x = \frac{8 \pm \sqrt{64 - 16(y^2+3)}}{2}$$

$$= 4 \pm 2\sqrt{1-y^2}$$

$$= 4 \pm 2\sqrt{-(y+1)(y-1)}.$$

Hence, arguing as before, y must lie between 1 and -1.

22. If $4x^2 + 9y^2 - 36y = 0$ for real values of x and y, find the limits between which x and y must lie.

23. Prove that the equation $4x^2 + 9y^2 - 48x - 72y + 252 = 0$ will be satisfied only by such real values of x and y as respectively lie between the limits (3, 9) and (2, 6).

24. Prove that in the equation $9x^2 - 25y^2 - 108x + 99 = 0$ for *any* real value of y, x will be real but that the corresponding values of x will *never lie between* the limits 1 and 11.

4. Relations between the Roots and the co-efficients.—If α, β denote the roots of the quadratic $ax^2 + bx + c = 0$, to prove that

$$\alpha + \beta = -\frac{b}{a}, \text{ and } \alpha\beta = \frac{c}{a}.$$

Solving the equation as in Art. 4, Chap. VI., we have

$$\frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

and

$$\beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

Hence, by addition,

$$\alpha + \beta = \frac{-2b}{2a} = -\frac{b}{a};$$

and by multiplication,

$$\alpha\beta = \frac{(-b - \sqrt{b^2 - 4ac})(-b - \sqrt{b^2 - 4ac})}{4a^2}$$

$$= \frac{(-b)^2 - (b^2 - 4ac)}{4a^2}$$

$$= \frac{4ac}{4a^2} = \frac{c}{a}.$$

Cor. Since the equation $ax^2+bx+c=0$ can also be written in the form $x^2+\frac{b}{a}x+\frac{c}{a}=0$, we conclude that in a quadratic equation of the form $x^2+px+q=0$ (*i.e.*, where *the co-efficient of* x^2 *is unity and the terms are all on one side*),

(i) the sum of the roots is equal to the co-efficient of x with its sign changed ;

(ii) the product of the roots is equal to the last term (*i.e.*, the term independent of x or the *absolute term*, as it is called).

5. Applications of the preceding Results.—

1. To find the equation of which the roots are given.

Suppose α, β are the given roots and let $x^2-px+q=0$ be the equation sought.

Then since the sum of the roots is equal to the co-efficient of the second term with its sign changed, we must have

$$p=\alpha+\beta,$$

and since the product of the roots = the *absolute term*, we must have $$q=\alpha\beta.$$

Hence the equation sought is

$$x^2-(\alpha+\beta)x+\alpha\beta=0 \quad \ldots \quad (A)$$

$$\text{or, } (x-\alpha)(x-\beta)=0 \quad \ldots \quad (B)$$

Thus we can easily form quadratics whose roots are given.

Example 1. Form the equation whose roots are 4 and -5.

By (B), the equation sought is

$$(x-4)(x+5)=0,$$

$$\text{or, } x^2+x-20=0.$$

Example 2. Form the quadratic whose roots are $3+\sqrt{5}$ and $3-\sqrt{5}$

Since $(3+\sqrt{5})+(3-\sqrt{5})=6$,

and $(3+\sqrt{5})\times(3-\sqrt{5})=4$,

$\therefore$ by (A), the equation sought is

$$x^2-6x+4=0.$$

Exercise (39).

Form the quadratics whose roots are :—

1. 3 and 2.
2. 4 and -7.
3. 8 and -13.
4. 17 and $\frac{2}{5}$.
5. 3 and $-\frac{3}{5}$.
6. $5+\sqrt{2}$ and $5-\sqrt{2}$.
7. $4+\sqrt{7}$ and $4-\sqrt{7}$.
8. $7 \pm 2\sqrt{5}$.
9. $\pm 3\sqrt{5}-7$.
10. $\dfrac{m-n}{m+n}$ and $-\dfrac{m+n}{m-n}$
11. $\frac{1}{2}\left(1+\sqrt{3}+\sqrt{2\sqrt{3}}\right)$ and $\frac{1}{2}\left(1+\sqrt{3}-\sqrt{2\sqrt{3}}\right)$.

(Madras University F. A. Paper, 1881.)

II. Given the sum and product of any two quantities, we know the quantities at once by solving a quadratic.

Example. Given $p+q=5$ and $pq=6$, find p and q.

From the given relations, evidently p and q are the roots of the quadratic $x^2-5x+6=0$.

But the roots of this quadratic $= \dfrac{5 \pm \sqrt{25-24}}{2}$

$$= \frac{5 \pm 1}{2} = 3, \text{ or } 2.$$

Hence, either $\left.\begin{aligned} p &= 3 \\ q &= 2 \end{aligned}\right\}$ or, $\left.\begin{aligned} p &= 2 \\ q &= 3 \end{aligned}\right\}$.

Exercise (40).

Find the values of p, q from the relations given below :—

1. $p+q=7$, $pq=12$.
2. $p+q=2$, $pq=-15$.
3. $p+q=-6$, $pq=8$.
4. $p+q=6$, $pq=2$

III. If one root of a quadratic be known by inspection or otherwise, the other root can be immediately found.

Example 1. Solve $\dfrac{a+c}{x+a}+\dfrac{b+c}{x+b}=\dfrac{2(a+b+c)}{x+a+b}$.

Here obviously $x=c$ is a solution.

Clearing of fractions, we have

$$(a+b)x^2+\{a^2+b^2-c(a+b)\}x-c(a^2+b^2) = 0.$$

Therefore the product of the roots $= \dfrac{-c(a^2+b^2)}{a+b}$.

Hence, c being one root, the other root must be $-\dfrac{(a^2+b^2)}{a+b}$.

Example 2. Solve the equation

$$\sqrt{x^2+ax-1}+\sqrt{x^2+bx-1} = \sqrt{a}+\sqrt{b} \quad \ldots \quad \ldots \quad (1).$$

Here obviously $x = 1$ is a solution.

Identically we have

$$(x^2+ax-1)-(x^2+bx-1) = (a-b)x \quad \ldots \quad \ldots \quad (2).$$

Hence, dividing (2) by (1), we have

$$\sqrt{x^2+ax-1}-\sqrt{x^2+bx-1} = (\sqrt{a}-\sqrt{b})x \quad \ldots \quad \ldots \quad (3).$$

From (1) and (3), by addition,

$$2\sqrt{x^2+ax-1} = (\sqrt{a}-\sqrt{b})x+(\sqrt{a}+\sqrt{b}).$$

Squaring both sides,

$$4(x^2+ax-1) = (\sqrt{a}-\sqrt{b})^2x^2+2(a-b)x+(\sqrt{a}+\sqrt{b})^2$$

or, $\{(\sqrt{a}-\sqrt{b})^2-4\}x^2-2(a+b)x+\{(\sqrt{a}+\sqrt{b})^2+4\} = 0.$

Hence, the product of the roots $= \dfrac{(\sqrt{a}+\sqrt{b})^2+4}{(\sqrt{a}-\sqrt{b})^2-4}$;

but one root $= 1$, $\therefore$ the other root

$$= \frac{(\sqrt{a}+\sqrt{b})^2+4}{(\sqrt{a}-\sqrt{b})^2-4}.$$

Exercise (41).

Solve the following equations :—

1. $(x-2)(x-4) = 5.3.$

2. $(x-a+b)(x+c-a) = bc.$

3. $(x-3)(x-4) = (a-3)(a-4).$

4. $\sqrt{x^2+2x-1}+\sqrt{x^2+x+1} = \sqrt{2}+\sqrt{3}.$

IV. Values of expressions directly or indirectly involving the sum and product of the roots of a quadratic can be easily found in terms of the co-efficients.

Example 1. If α, β be the roots of the equation $x^2 - px + q = 0$, find the values of $\alpha - \beta$ and $\alpha^3 + \beta^3$.

Since $\alpha + \beta = p$ and $\alpha\beta = q$,

we have
$$(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta$$
$$= p^2 - 4q.$$

Hence, if α be greater than β,
$$\alpha - \beta = \sqrt{p^2 - 4q}.$$

Again,
$$\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$$
$$= p^3 - 3pq$$
$$= p(p^2 - 3q).$$

Example 2. If α, β be the roots of the equation—

$ax^2 + bx + c = 0$, find the values of $\frac{1}{\alpha^2} + \frac{1}{\beta^2}$ and $\left(\frac{\alpha}{\beta} - \frac{\beta}{\alpha}\right)^2$.

Since $\alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$,

we have
$$\frac{1}{\alpha^2} + \frac{1}{\beta^2} = \left(\frac{1}{\alpha} + \frac{1}{\beta}\right)^2 - \frac{2}{\alpha\beta}$$
$$= \left(\frac{\alpha + \beta}{\alpha\beta}\right)^2 - \frac{2}{\alpha\beta}$$
$$= \left(\frac{-b}{c}\right)^2 - \frac{2a}{c}$$
$$= \frac{b^2 - 2ac}{c^2}.$$

Again,
$$\left(\frac{\alpha}{\beta} - \frac{\beta}{\alpha}\right)^2 = \frac{(\alpha^2 - \beta^2)^2}{\alpha^2\beta^2} = \frac{(\alpha + \beta)^2(\alpha - \beta)^2}{\alpha^2\beta^2}$$
$$= \frac{(\alpha + \beta)^2\{(\alpha + \beta)^2 - 4\alpha\beta\}}{\alpha^2\beta^2}$$
$$= \frac{\frac{b^2}{a^2}\left(\frac{b^2}{a^2} - \frac{4c}{a}\right)}{\frac{c^2}{a^2}}$$
$$= \frac{b^2(b^2 - 4ac)}{a^2c^2}.$$

Example 3. If $x = 2 + 7\sqrt{-1}$, find the value of
$x^3 - 2x^2 + 45x + 114$.

Let us form the equation whose roots are $2 \pm 7\sqrt{-1}$;
the sum of the roots $= 4$,
and the product of the roots $= 4 + 49 = 53$;
hence the desired equation is $x^2 - 4x + 53 = 0$.

Thus the expression $x^2 - 4x + 53$ vanishes when x has the given value.

Hence, the given expression, which is $x^3 - 2x^2 + 45x + 114$,

$$= x(x^2 - 4x + 53) + 2(x^2 - 4x + 53) + 8$$
$$= (x \times 0) + (2 \times 0) + 8$$
$$= 8.$$

Example 4. If α and β be the roots of the equation— $x^2 + 2ax + b = 0$, form a quadratic equation with rational co-efficients, one of whose roots is $\alpha + \beta + \sqrt{\alpha^2 + \beta^2}$.

(Calcutta University F. A. Paper, 1878.)

We have $\alpha + \beta = -2a$ and $\alpha\beta = b$.

Now since one of the roots of the equation sought is $\alpha + \beta + \sqrt{\alpha^2 + \beta^2}$, the other root must clearly be $\alpha + \beta - \sqrt{\alpha^2 + \beta^2}$; for otherwise neither the sum of the roots nor their product will be rational.

Hence, the sum of the roots $= 2(\alpha + \beta) = -4a$,

and their product $= (\alpha + \beta)^2 - (\alpha^2 + \beta^2)$

$$= 2\alpha\beta = 2b.$$

Hence, the equation sought is

$$x^2 + 4ax + 2b = 0.$$

Example 5. If a be a root of the equation $4x^2 + 2x - 1 = 0$, prove that $4a^3 - 3a$ is the other root.

(Bombay University F. A. Paper, 1889.)

Since a is a root of the given equation, we must have

$$4a^2 + 2a - 1 = 0,$$

and $\therefore\ 4a^2 = 1 - 2a$.

Hence,
$$\begin{aligned}4a^3 - 3a &= a(1-2a) - 3a\\ &= -2a - 2a^2\\ &= -2a - \tfrac{1}{2}(1-2a)\\ &= -a - \tfrac{1}{2}.\end{aligned}$$

Hence,
$$\begin{aligned}a + (4a^3 - 3a) &= a + (-a - \tfrac{1}{2})\\ &= -\tfrac{1}{2},\end{aligned}$$

and
$$\begin{aligned}a(4a^3 - 3a) &= a.(-a-\tfrac{1}{2})\\ &= -a^2 - \tfrac{1}{2}a\\ &= -\tfrac{1}{4}(1-2a) - \tfrac{1}{2}a\\ &= -\tfrac{1}{4}:\end{aligned}$$

which shews that a and $4a^3 - 3a$ are the two roots of the equation $4x^2 + 2x - 1 = 0$.

Exercise (42).

1. If α, β be the roots of $x^2 + px + q = 0$, find the equation whose roots are $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$.

(Calcutta University F. A. Paper, 1887.)

2. If α, β be the roots of $x^2 - px + q = 0$, form the equation whose roots are $\frac{2}{\alpha}$, $\frac{2}{\beta}$ and prove that $\frac{1}{\alpha^3} + \frac{1}{\beta^3} = \frac{p^3}{q^3} - \frac{3p}{q^2}$.

(Calcutta University F. A. Parer, 1886.)

3. If α, β be the roots of $x^2 + px + q = 0$, form the equation whose roots are $\alpha^2 + \alpha\beta$ and $\beta^2 + \alpha\beta$.

(Bombay University F. A. Paper, 1887.)

4. If r be the ratio of the roots of the equation $ax^2 + bx + c = 0$, shew that $\frac{(r+1)^2}{r} = \frac{b^2}{ac}$.

(Calcutta University F. A. Paper, 1883.)

5. Find the difference of the roots of $x^2 - 42x + 117 = 0$.

6. If x_1, x_2 represent the two values of x which satisfy the equation $ax^2 + bx + c = 0$, prove that $\frac{x_1}{x_2} + \frac{x_2}{x_1} = \frac{b^2 - 2ac}{ac}$; and verify this formula by the equation $2x^2 - 7x + 3 = 0$.

7. If α, β be the roots of the equation $ax^2+bx+c=0$, find the value of :—

(1) $\alpha^4\beta^7+\alpha^7\beta^4$. (2) $\alpha^2\beta^{-1}+\alpha^{-1}\beta^2$. (3) $\alpha^2\beta^{-2}+\alpha^{-2}\beta^2$.

8. If x_1, x_2 be the roots of the equation $x^2+mx+m^2+n^2=0$, prove that $x_1^4+x_1^2x_2^2+x_2^4 = n^2(2m^2+3n^2)$.

(Bombay University P. E. Paper, 1890).

9. Find the value of $x^3-4x^2-9x+97$, when $x=4+\sqrt{-7}$.

10. Find the value of $x^4+4x^3+6x^2+4x+9$,

when $x = \sqrt{-2}-1$

(Calcutta University F. A. Paper, 1886.)

[The given expression $= (x+1)^4+8 =$ &c.]

11. Find the value of $x^4-2x^3+x^2+4x+8$,

when $x = \frac{1}{2}(3+\sqrt{-7})$.

12. Find the value of $16x^3-16x^2-11x+173$,

when $x = \frac{1}{4}(6+7\sqrt{-1})$.

13. If one root of the equation $ax^2+bx+c=0$ be the square of the other, prove that $b^3+a^2c+ac^2 = 3abc$.

(Calcutta University F. A. Paper, 1879).

[Since $\alpha = \beta^2$, we have $\beta^2+\beta = -\frac{b}{a}$.. (1) and $\beta^2\beta = \frac{c}{a}$ ..(2).

From (2) $\beta = \left(\frac{c}{a}\right)^{\frac{1}{3}}$; hence from (1) $\left(\frac{c}{a}\right)^{\frac{2}{3}}+\left(\frac{c}{a}\right)^{\frac{1}{3}} = -\frac{b}{a}$,

$\therefore \left(\frac{c}{a}\right)^2+\frac{c}{a}+3\frac{c}{a}\left(-\frac{b}{a}\right) = -\frac{b^3}{a^3}$ &c.]

14. If $\alpha \pm \sqrt{\beta}$ be the roots of the equation $x^2+px+q=0$, prove that $\frac{1}{\alpha} \pm \frac{1}{\sqrt{\beta}}$ will be the roots of the equation—

$(p^2-4q)(p^2x^2+4px) = 16q$.

(Calcutta University F. A. Paper, 1880.)

15. If the ratio of the roots of the equation $x^2+px+q=0$ be equal to that of the roots of $x^2+p_1x+q_1=0$, shew that $p^2q_1 = p_1^2q$.

(Calcutta University F. A. Paper, 1885.)

[$\because \frac{\alpha}{\beta} = \frac{\alpha_1}{\beta_1}$ $\therefore \frac{\alpha}{\alpha_1} = \frac{\beta}{\beta_1}$; hence, $\frac{\alpha+\beta}{\alpha_1+\beta_1} = \left(\frac{\alpha\beta}{\alpha_1\beta_1}\right)^{\frac{1}{2}}$.]

6. Distinction between a Quadratic Expression and a Quadratic Equation.—A Quadratic Expression *when equated to zero* gives rise to a Quadratic Equation. Thus $ax^2 + bx + c$ is a *quadratic expression*, the value of which depends upon that of x so that for different values of x the corresponding values of the expression are also different. But $ax^2 + bx + c = 0$ is a *quadratic equation*, where x must have one of *two definite values* and for *any other* value of x the given relation will not hold.

(i) To show that the *quadratic expression* $ax^2 + bx + c = a(x - \alpha)(x - \beta)$ where α, β are the roots of the *quadratic equation* $ax^2 + bx + c = 0$.

Since α, β are the roots of the equation $ax^2 + bx + c = 0$,

$$\therefore \quad \alpha + \beta = -\frac{b}{a} \text{ and } \alpha\beta = \frac{c}{a}.$$

$$\begin{aligned} \text{Hence,} \quad ax^2 + bx + c &= a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) \\ &= a\{x^2 - (\alpha + \beta)x + \alpha\beta\} \\ &= a(x - \alpha)(x - \beta). \end{aligned}$$

This furnishes us with a ready means of resolving a quardatic expression into the product of two linear factors.

N. B.—Expressions containing only the first power of x are called *linear*.

Example 1. Resolve the expression $3x^2 - 7x + 3$ into linear factors.

The roots of the equation $3x^2 - 7x + 3 = 0$, are

$$= \frac{7 \pm \sqrt{49 - 36}}{6} = \frac{7 \pm \sqrt{13}}{6}.$$

Hence, the expression $3x^2 - 7x + 3$

$$= 3\left(x - \frac{7 + \sqrt{13}}{6}\right)\left(x - \frac{7 - \sqrt{13}}{6}\right)$$

Example 2. Resolve the expression $17x^2 - 6x - 7$ into the product of linear factors.

The roots of the equation $17x^2 - 6x - 7 = 0$, are

$$= \frac{6 \pm 2\sqrt{9 + 119}}{34} = \frac{3 \pm \sqrt{128}}{17} = \frac{3 + 8\sqrt{2}}{17} \text{ and } \frac{3 - 8\sqrt{2}}{17}.$$

Hence, the expression $17x^2 - 6x - 7$

$$= 17\left(x - \frac{3 + 8\sqrt{2}}{17}\right)\left(x - \frac{3 - 8\sqrt{2}}{17}\right).$$

(ii) To show that the *expression* $ax^2 + bx + c$ is divisible by $x - h$ *only when* h is a root of the equation $ax^2 + bx + c = 0$.

Let us divide $ax^2 + bx + c$ by $x - h$:—

$$\begin{array}{l} x - h\big)\,ax^2 + bx + c\,\big(ax + (ah + b) \\ \qquad\;\; ax^2 - ahx \\ \hline \qquad\qquad (ah + b)x + c \\ \qquad\qquad (ah + b)x - h(ah + b) \\ \hline \qquad\qquad\qquad ah^2 + bh + c \end{array}$$

Thus the remainder is $ah^2 + bh + c$; and hence $ax^2 + bx + c$ is divisible by $x - h$ *only when* $ah^2 + bh + c = 0$, *i.e., only when* h is a root of the equation $ax^2 + bx + c = 0$.

Example. Examine if the expression $x^2 + 7x - 18$ be divisible by $x - 2$.

If we put 2 for x in the given expression, it becomes

$$2^2 + 7.2 - 18 \text{ or zero ;}$$

$\therefore$ 2 is a root of the equation $x^2 + 7x - 18 = 0$.

Hence the expression $x^2 + 7x - 18$ is divisible by $x - 2$.

Exercise (43).

Resolve the following quadratic expressions into the product of linear factors :—

1. $4x^2 - 5x - 21$. **2.** $x^2 + 22x + 85$. **3.** $x^2 - 12x - 640$.

4. $3x^2 + 8x - 1$. **5.** $5x^2 + 12x + 2$.

Examine if the expression

6. $3x^2 - 4x - 15$ be divisible by $x - 3$.

7. $x^2 + 2x - 4$,, ,, ,, $x - 1$.

8. $2x^2 + x - 10$,, ,, ,, $x - 2$.

9. $x^2 - 5x - 48$,, ,, ,, $x - 8$.

10. $x^2 + 6x - 24$,, ,, ,, $x - 3$.

11. What values of x will make $2x^2 + x - 15$ positive ?

[The given expression $= 2(x-\frac{5}{2})(x+3)$. If $x>\frac{5}{2}$ both the factors are positive and therefore the given expression is positive. If x be less than -3 (*i.e.*, negative and *numerically greater than* 3) both the factors are negative and $\therefore$ the product is positive. If x be less than $\frac{5}{2}$ and greater than -3, then $x-\frac{5}{2}$ is negative, but $x+3$ is positive and $\therefore$ their product is negative. Hence all values of x excepting those which lie between $\frac{5}{2}$ and -3 (*i.e.*, *between the roots of the equation* $2x^2+x-15=0$) will make the given expression positive.]

12. Show that the expression $10-x-3x^2$ is positive for those values of x only that lie between the roots of the equation $10-x-3x^2 = 0$.

13. What values of x will make $8x^2-18x+9$ negative?

7. Some Important Theorems and examples.

(i) To show that for all real values of x the expression ax^2+bx+c has the same sign as a, except when the roots of the corresponding equation ($ax^2+bx+c = 0$) are real and different, and x lies between them.

Since ax^2+bx+c

$$= a\left\{x^2+\frac{b}{a}x+\frac{c}{a}\right\}$$

$$= a\left\{\left(x^2+\frac{b}{a}x+\frac{b^2}{4a^2}\right)+\left(\frac{c}{a}-\frac{b^2}{4a^2}\right)\right\}$$

$$= a\left\{\left(x+\frac{b}{2a}\right)^2-\left(\frac{b^2-4ac}{4a^2}\right)\right\},$$

$\therefore$ it has, or has not, the same sign as a, according as

$$\left\{\left(x+\frac{b}{2a}\right)^2-\left(\frac{b^2-4ac}{4a^2}\right)\right\} \text{ is positive or negative.}$$

Now $\left(x+\dfrac{b}{2a}\right)^2$, being a square, is always positive whatever real value x may have; the only uncertainty then is about the sign of $\dfrac{b^2-4ac}{4a^2}$. Thus three cases will arise :—

Case I. When the roots of the equation $ax^2+bx+c = 0$ are *impossible* (or *imaginary*).

If this be so, then b^2-4ac, and $\therefore$ $\dfrac{b^2-4ac}{4a^2}$, is negative;

$$\therefore \quad \left\{\left(x+\frac{b}{2a}\right)^2-\left(\frac{b^2-4ac}{4a^2}\right)\right\} \text{ is positive.}$$

Hence, in this case ax^2+bx+c has the same sign as a.

Case II. When the roots of the equation $ax^2+bx+c=0$ are *real* and *equal*.

Here $b^2-4ac=0$; $\therefore$ $\left\{\left(x+\dfrac{b}{2a}\right)^2-\left(\dfrac{b^2-4ac}{4a^2}\right)\right\}$ becomes $\left(x+\dfrac{b}{2a}\right)^2$ and is therefore positive.

Hence, in this case also ax^2+bx+c has the same sign as a.

Case III. When the roots of the equation $ax^2+bx+c=0$ are *real* and *unequal*.

Here then b^2-4ac, and $\therefore$ $\dfrac{b^2-4ac}{4a^2}$ is positive.

Consequently $\left\{\left(x+\dfrac{b}{2a}\right)^2-\left(\dfrac{b^2-4ac}{4a^2}\right)\right\}$ will be positive only

if $\left(x+\dfrac{b}{2a}\right)^2$ be greater than $\dfrac{b^2-4ac}{4a^2}$,

i.e., if $\left(x+\dfrac{b}{2a}\right)$ be either greater than $+\dfrac{\sqrt{b^2-4ac}}{2a}$,

or less than $-\dfrac{\sqrt{b^2-4ac}}{2a}$;

i.e., if x be either greater than $\dfrac{-b+\sqrt{b^2-4ac}}{2a}$,

or less than $\dfrac{-b-\sqrt{b^2-4ac}}{2a}$;

i.e., if x be either greater than the greater root or less than the lesser root of the equation $ax^2+bx+c=0$.

Hence, in this case ax^2+bx+c has the same sign as a except when the values of x are intermediate between the roots of the equation $ax^2+bx+c=0$.

Cor. It is evident from cases I and II that ax^2+bx+c is *always* (*i.e.*, for all real values of x) positive *if a be positive* and b^2-4ac, *negative or zero.*

(ii) To find the condition that the expression $ax^2 + 2hxy + by^2 + 2gx + 2fy + c$ may be resolved into two linear factors (*i.e.*, factors containing only the first powers of x and y.)

Assume that the given expression

$$= (lx + my + n)(l'x + m'y + n')$$

which $= ll'x^2 + (lm' + l'm)xy + mm'y^2 + (ln' + l'n)x + (mn' + m'n)y + nn'$.

Hence,
$$\left.\begin{aligned} ll' &= a \\ mm' &= b \\ nn' &= c \end{aligned}\right\}, \qquad \left.\begin{aligned} lm' + l'm &= 2h \\ ln' + l'n &= 2g \\ mn' + m'n &= 2f \end{aligned}\right\}.$$

Hence,
$$\begin{aligned} 8fgh &= (mn' + m'n)(ln' + l'n)(lm' + l'm) \\ &= (mn' + m'n)\{ll'(mn' + m'n) + m'n'l^2 + mnl'^2\} \\ &= ll'(mn' + m'n)^2 + mm'(n'^2l^2 + n^2l'^2) \\ &\qquad + nn'(m^2l'^2 + m'^2l^2) \\ &= ll'(mn' + m'n)^2 + mm'\{(n'l + nl')^2 - 2nn'll'\} \\ &\qquad + nn'\{(ml' + m'l)^2 - 2mm'll'\} \\ &= a.4f^2 + b\,4g^2 + c.4h^2 - 4abc\ ; \end{aligned}$$

whence $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$, which is the required condition.

(iii) To find the condition that the equations $ax^2 + bx + c = 0$ and $ax^2 + bx + c' = 0$ may have a common root.

Let α be the common root.

Then both the equations are satisfied by α, and we have

$$\left.\begin{aligned} a\alpha^2 + b\alpha + c &= 0 \\ a'\alpha^2 + b'\alpha + c' &= 0 \end{aligned}\right\}$$

Hence, by cross multiplication,

$$\frac{\alpha^2}{bc' - b'c} = \frac{\alpha}{ca' - c'a} = \frac{1}{ab' - a'b}\ ; \qquad \dots \qquad \dots \quad \text{(A)}$$

$$\therefore \qquad \frac{\alpha^2}{(ca' - c'a)^2} = \frac{\alpha^2}{(bc' - b'c)} \cdot \frac{1}{ab' - a'b}\ ;$$

$\therefore$ $(ca' - c'a)^2 = (bc' - b'c)(ab' - a'b)$, which is the condition required.

***Cor.* 1.** From (A) it is evident that the common root α

$$= \frac{ca' - c'a}{ab' - a'b} \text{ or } = \frac{bc' - b'c}{ca' - c'a}.$$

***Cor.* 2.** Hence the remaining roots of the equations are respectively $\left(\frac{c}{a} \div \frac{ca' - c'a}{ab' - a'b}\right)$, and $\left(\frac{c'}{a'} \div \frac{ca' - c'a}{ab' - a'b}\right)$, *i e.*, $\frac{c(ab' - a'b)}{a(ca' - c'a)}$ and $\frac{c'(ab' - a'b)}{a'(ca' - c'a)}$; or the remaining roots are

$$\frac{c(ca' - c'a)}{a(bc' - b'c)} \text{ and } \frac{c'(ca' - c'a)}{a'(bc' - b'c)}.$$

Example 1. Prove that if the equation $x^2 + bx + ca = 0$, $x^2 + cx + ab = 0$, have a common root, their other roots will satisfy the equation $x^2 + ax + bc = 0$.

(Calcutta University F. A. Paper, 1892.)

Let α be the common root; then we have

$$\alpha^2 + b\alpha + ca = 0,$$

$$\text{and} \quad \alpha^2 + c\alpha + ab = 0.$$

Hence, by cross multiplication,

$$\frac{\alpha^2}{a(b^2 - c^2)} = \frac{\alpha}{-a(b - c)} = \frac{1}{-(b - c)};$$

$$\text{or,} \quad \frac{\alpha^2}{a(b + c)} = \frac{\alpha}{-a} = \frac{1}{-1}.$$

Hence, $\alpha = a$; ... ... (1)

$$\text{also} \quad \frac{\alpha^2}{a(b + c)} \cdot \left(\frac{1}{-1}\right) = \left(\frac{\alpha}{-a}\right)^2,$$

and $\therefore$ $-a(b + c) = a^2$,

or, $a + b + c = 0$. ... ... (2)

From (1) we know that the remaining roots of the given equations are respectively c and b. Hence they must satisfy the equation $x^2 - (b + c)x + bc = 0$.

Example 2. If the equations $ax^3 + 3bx^2 + 3cx + d = 0$ and $ax^2 + 2bx + c = 0$ have a common root, prove that

$$(bc - ad)^2 = 4(ac - b^2)(bd - c^2).$$

(Calcutta University F. A. Paper, 1889.)

Let α denote the common root of the given equations; then we have

$$a\alpha^3 + 3b\alpha^2 + 3c\alpha + d = 0 \quad \ldots \quad (1)$$

$$\text{and} \quad a\alpha^2 + 2b\alpha + c = 0 \quad \ldots \quad (2)$$

Hence, we must have

$$(a\alpha^3 + 3b\alpha^2 + 3c\alpha + d) - \alpha(a\alpha^2 + 2b\alpha + c) = 0$$

$$\text{or,} \quad b\alpha^2 + 2c\alpha + d = 0 \quad \ldots \quad (3)$$

Hence, from (2) and (3), by cross multiplication,

$$\frac{\alpha^2}{2(bd - c^2)} = \frac{\alpha}{bc - ad} = \frac{1}{2(ac - b^2)};$$

$$\therefore \quad \frac{\alpha^2}{(bc - ad)^2} = \frac{\alpha^2}{2(bd - c^2) \cdot 2(ac - b^2)}.$$

Hence, $(bc - ad)^2 = 4(ac - b^2)(bd - c^2)$, which is the required condition.

Example 3. Find the condition that the expression $lx^2 + mxy + ny^2$, and $l'x^2 + m'xy + n'y^2$ may have a common linear factor.

Suppose $\quad lx^2 + mxy + ny^2 = l(x - ay)(x - by)$,

and $\quad l'x^2 + m'xy + n'y^2 = l'(x - ay)(x - b'y)$, so that $x - ay$ is the linear factor common to the given expressions.

Now since $l(x - ay)(x - by)$ becomes zero, when $x = ay$;

$\therefore \quad lx^2 + mxy + ny^2$ also becomes zero, when $x = ay$.

Hence we have

$$l(ay)^2 + m(ay)y + ny^2 = 0,$$

$$\text{or,} \quad la^2 + ma + n = 0; \quad \ldots \quad \ldots \quad (1)$$

$$\text{and similarly,} \quad l'a^2 + m'a + n' = 0. \quad \ldots \quad \ldots \quad (2)$$

Hence, from (1) and (2), by cross multiplication,

$$\frac{a^2}{mn' - m'n} = \frac{a}{nl' - n'l} = \frac{1}{lm' - l'm},$$

$$\text{whence} \quad \frac{a^2}{(nl' - n'l)^2} = \frac{a^2}{(mn' - m'n)(lm' - l'm)},$$

$$\text{and} \quad \therefore \quad (nl' - n'l)^2 = (mn' - m'n)(lm' - l'm),$$

which is the condition required.

Example 4. Find the condition that the expressions $ax^2 + 2hxy + by^2$, and $a'x^2 + 2h'xy + b'y^2$ may be respectively divisible by factors of the form $y - mx$ and $my + x$.

Suppose $ax^2 + 2hxy + by^2 = b(y - mx)(y - nx)$
and $a'x^2 + 2h'xy + b'y^2 = a'(x + my)(x - n'y)$.

Now evidently the first given expression vanishes when $y = mx$,

$$\therefore\ ax^2 + 2hx.mx + bm^2x^2 = 0,$$

$$\text{or,} \quad a + 2hm + bm^2 = 0.$$

Evidently also the 2nd given expression vanishes when $x = -my$; hence

$$a'(-my)^2 + 2h'(-my).y + b'y^2 = 0,$$

$$\text{or,} \quad a'm^2 - 2h'm + b' = 0.$$

Thus we have $\quad bm^2 + 2hm + a = 0,$
and $\quad a'm^2 - 2h'm + b' = 0.$

Hence, by cross multiplication,

$$\frac{m^2}{2(hb' + h'a)} = \frac{m}{aa' - bb'} = \frac{1}{-2(bh' + a'h)};$$

whence $\displaystyle \frac{m^2}{(aa' - bb')^2} = \frac{m^2}{2(hb' + h'a) \cdot -2(bh' + a'h)},$

and $\therefore\ (aa' - bb')^2 = -4(hb' + h'a)(bh' + a'h),$
or, $\quad (aa' - bb')^2 + 4(hb' + h'a)(bh' + a'h) = 0$;
which is the condition required.

Example 5. Prove that if x is real the expression $\dfrac{(x-a)(x-c)}{x-b}$ is capable of assuming all values if a, b, c are in ascending order of magnitude.

(Madras University F. A. Paper, 1889)

Let $\quad \dfrac{(x-a)(x-c)}{x-b} = y.$

Then we have

$$x^2 - (a + c)x + ac = y(x - b),$$

$$\text{or,} \quad x^2 - (a + c + y)x + (ac + by) = 0;$$

$$\therefore\ x = \frac{(a + c + y) \pm \sqrt{(a + c + y)^2 - 4(ac + by)}}{2}.$$

The expression under the radical sign

$$= y^2 + 2(a + c - 2b)y + (a - c)^2$$

$$= \{y + (a + c - 2b)\}^2 + \{(a - c)^2 - (a + c - 2b)^2\}$$

$$= \{y + (a + c - 2b)\}^2 + \{(a - c) + (a + c - 2b)\}\{(a - c) - (a + c - 2b)\}$$

$$= \{y + (a + c - 2b)\}^2 + 4(a - b)(b - c),$$

and is therefore positive for *all* values of y if $a < b$ and $b < c$.

Thus it is clear that whatever real value may be attributed to the given expression the corresponding values of x will be real if a, b, c be in ascending order of magnitude ; which was to be proved.

Exercise (44).

1. For what values of m will $x^2 + 8xy - 4y^2 + 2my - 5$ be equivalent to the product of two linear factors ?

2. Find the values of m which will make $2x^2 + mxy + 3y^2 - 5y - 2$ equivalent to the product of two linear factors.

3. Show that the expression $5x^2 - 30x + 47$ is always positive whatever real values x may have.

4. Show that the expression

$$\frac{(x^2 - 4)(x^2 + 3x + 2)(x^2 - x - 2) + 10}{x^2 + 5x + 7}$$

is positive for all real values of x.

(Madras University F. A. Paper, 1881).

5. If the equations $x^2 + px + q = 0$, $x^2 + p'x + q' = 0$ have a common root, shew that it must be either $\frac{pq' - p'q}{q - q'}$ or, $\frac{q - q'}{p' - p}$.

6. Prove that for real values of x the expression $\frac{x + a}{x^2 + bx + c^2}$ will always lie between two fixed finite limits if $a^2 + c^2 > ab$ and $b^2 < 4c^2$.

7. Prove that if $a^2 + c^2 > ab$ and also $b^2 > 4c^2$, then for all real values of x there are two limits between which the above expression *cannot lie.*

8. Prove that the same expression is capable of all values if $b^2 > 4c^2$ and $a^2 + c^2 = ab$ or $< ab$.

9. If x be real shew that the expression $\frac{m^2}{1 + x} - \frac{n^2}{1 - x}$ can have any real value. (Madras University F. A. Paper, 1883).

10. Find the condition that the expressions $ax^2+bxy+cy^2$ and $a'x^2-b'xy+c'y^2$ may be respectively divisible by expressions of the form $x+py$ and $px+y$.

11. If $x-a$ is a factor of $a_1x^2+2b_1x+c_1$

and $x+a$ of $a_2x^2+2b_2x+c_2$, prove that

$$(c_2a_1-c_1a_2)^2+4(a_1b_2+a_2b_1)(b_1c_2+b_2c_1) = 0.$$

(Madras University F. A. Paper, 1890.)

12. If the equations $ax^3+bx^2+cx+d = 0$ and $a'x^3+b'x^2+c'x+d' = 0$ have a common root, find it.

CHAPTER IX.

SIMULTANEOUS EQUATIONS INVOLVING QUADRATICS.

1. Simple cases.—Most of the equations treated of in this article admit of being solved by methods similar to those for solving Simultaneous Equations of the first degree, but other artifices may sometimes be profitably employed as will be illustrated by the following examples.

Example 1. Solve

$$\left.\begin{aligned} 5x-y &= 3 \quad \ldots \quad \ldots \quad (1) \\ y^2-6x^2 &= 25 \quad \ldots \quad \ldots \quad (2) \end{aligned}\right\}$$

From (1), $y = 5x-3$ ... ... (3)

Substituting this value of y in (2), we have

$$(5x-3)^2-6x^2 = 25,$$

$$\text{or,}\quad 19x^2-30x-16 = 0,$$

$$\text{or,}\quad (x-2)(19x+8) = 0;$$

$$\therefore\quad x = 2, \text{ or } -\tfrac{8}{19}.$$

$$\text{Hence from (3),}\quad y = 7, \text{ or } -\tfrac{97}{19}.$$

$$\text{Thus we have}\quad \left.\begin{aligned} x &= 2 \\ y &= 7 \end{aligned}\right\} \quad \text{or,} \quad \left.\begin{aligned} x &= -\tfrac{8}{19} \\ y &= -\tfrac{97}{19} \end{aligned}\right\}.$$

Example 2. Solve

$$\left.\begin{array}{rcl} 3x+4y &=& 18 \quad \dots \quad \dots \quad \dots \quad (1) \\ \dfrac{1}{x}+\dfrac{1}{y} &=& \dfrac{5}{6} \quad \dots \quad \dots \quad \dots \quad (2) \end{array}\right\}$$

$$\text{From (1)}, \quad y = \frac{18-3x}{4} \quad \dots \quad \dots \quad (3)$$

Substituting this value of y in (2), we have

$$\frac{1}{x}+\frac{4}{18-3x} = \frac{5}{6},$$

$$\begin{array}{lrcl} \text{or,} & 6(18-3x)+24x &=& 5x(18-3x), \\ \text{or,} & 15x^2-84x+6.18 &=& 0, \\ \text{or,} & 5x^2-28x+36 &=& 0, \\ \text{or,} & (x-2)(5x-18) &=& 0\,; \end{array}$$

$$\therefore \quad x = 2, \text{ or } \tfrac{18}{5}.$$

$$\text{Hence from (3),} \quad y = \frac{18-6}{4} = 3,$$

$$\text{or,} \quad = \frac{18-\frac{54}{5}}{4} = \frac{36}{5\times4} = \tfrac{9}{5}.$$

$$\text{Thus we have} \quad \left.\begin{array}{l} x = 2 \\ y = 3 \end{array}\right\} \quad \text{or,} \quad \left.\begin{array}{l} x = \frac{18}{5} \\ y = \frac{9}{5} \end{array}\right\}.$$

Example 3. Solve

$$\left.\begin{array}{rcl} x-y &=& 2 \quad \dots \quad \dots \quad (1) \\ xy &=& 15 \quad \dots \quad \dots \quad (2) \end{array}\right\}$$

$$\begin{array}{lrcl} \text{We have} & x^2+2xy+y^2 &=& (x-y)^2+4xy \\ & &=& 4+60, \\ \text{or,} & (x+y)^2 &=& 64\,; \end{array}$$

$$\left.\begin{array}{lrcl} \therefore & x+y &=& 8 \quad \dots \quad \dots \quad (3) \\ \text{or,} & x+y &=& -8 \quad \dots \quad \dots \quad (4) \end{array}\right\}$$

From (3) and (1), we have

$$\left.\begin{array}{rcl} x+y &=& 8 \\ x-y &=& 2 \end{array}\right\}$$

Hence, adding and subtracting, we have

$$\left.\begin{array}{rrcl} & 2x &=& 10 \text{ or, } x = 5 \\ \text{and} & 2y &=& 6 \text{ or, } y = 3 \end{array}\right\}$$

Again, from (4) and (1), we have

$$\left.\begin{array}{rcl} x+y &=& -8 \\ x-y &=& 2 \end{array}\right\}$$

Hence, adding and subtracting, we have

$$\left.\begin{aligned} 2x &= -\ 6 \text{ or, } x = -3 \\ \text{and}\quad 2y &= -10 \text{ or, } y = -5 \end{aligned}\right\}$$

Thus we have $\left.\begin{aligned} x &= 5 \\ y &= 3 \end{aligned}\right\}$ or, $\left.\begin{aligned} x &= -3 \\ y &= -5 \end{aligned}\right\}$.

Example 4. Solve

$$\left.\begin{aligned} 3x-5y &= 2 \quad \dots \quad \dots \quad (1) \\ xy &= 8 \quad \dots \quad \dots \quad (2) \end{aligned}\right\}$$

We have

$$\begin{aligned} (3x+5y)^2 &= (3x-5y)^2+60xy \\ &= 4+480, \\ &= 484\ ; \end{aligned}$$

$$\left.\begin{aligned} \therefore\quad 3x+5y &= 22 \quad \dots \quad \dots \quad (3) \\ \text{or,}\quad 3x+5y &= -22 \quad \dots \quad \dots \quad (4) \end{aligned}\right\}$$

Now, from (1) and (3), we have

$$\left.\begin{aligned} 3x+5y &= 22 \\ 3x-5y &= 2 \end{aligned}\right\}$$

$\therefore$ by adding and subtracting, we have

$$\left.\begin{aligned} 6x &= 24 \text{ or, } x = 4 \\ \text{and}\quad 10y &= 20 \text{ or, } y = 2 \end{aligned}\right\}$$

Again from (1) and (4), we have

$$\left.\begin{aligned} 3x+5y &= -22 \\ 3x-5y &= 2 \end{aligned}\right\}$$

$\therefore$ by adding and subtracting, we have

$$\left.\begin{aligned} 6x &= -20 \text{ or, } x = -\tfrac{10}{3} \\ \text{and}\quad 10y &= -24 \text{ or, } y = -\tfrac{12}{5} \end{aligned}\right\}$$

Thus we have $\left.\begin{aligned} x &= 4 \\ y &= 2 \end{aligned}\right\}$ or, $\left.\begin{aligned} x &= -\tfrac{10}{3} \\ y &= -\tfrac{12}{5} \end{aligned}\right\}$.

Example 5. Solve

$$\left.\begin{aligned} 2x+3y &= 18 \quad \dots \quad \dots \quad \dots \quad (1) \\ xy &= 12 \quad \dots \quad \dots \quad \dots \quad (2) \end{aligned}\right\}$$

We have

$$\begin{aligned} (2x-3y)^2 &= (2x+3y)^2-24xy \\ &= (18)^2-24\times 12 \\ &= 6^2(3^2-4\times 2) \\ &= 36\ ; \end{aligned}$$

$$\left.\begin{aligned} \therefore\quad 2x-3y &= 6 \quad \dots \quad \dots \quad \dots \quad (3) \\ \text{or,}\quad 2x-3y &= -6 \quad \dots \quad \dots \quad \dots \quad (4) \end{aligned}\right\}$$

From (1) and (3), we have

$$\left.\begin{aligned} 2x+3y &= 18 \\ 2x-3y &= 6 \end{aligned}\right\}$$

∴ by adding and subtracting, we have

$$\left.\begin{aligned} 4x &= 24 \text{ or, } x = 6 \\ \text{and}\quad 6y &= 12 \text{ or, } y = 2 \end{aligned}\right\}$$

Also from (1) and (4), we have

$$\left.\begin{aligned} 2x+3y &= 18 \\ 2x-3y &= -6 \end{aligned}\right\}$$

∴ by adding and subtracting, we have

$$\left.\begin{aligned} 4x &= 12 \text{ or, } x = 3 \\ \text{and}\quad 6y &= 24 \text{ or, } y = 4 \end{aligned}\right\}$$

Thus we have $\left.\begin{aligned} x &= 6 \\ y &= 2 \end{aligned}\right\}$ or, $\left.\begin{aligned} x &= 5 \\ y &= 4 \end{aligned}\right\}$

Example 6. Solve $\left.\begin{aligned} x^2+y^2 &= 13 \quad \ldots \quad \ldots \quad (1) \\ xy &= 6 \quad \ldots \quad \ldots \quad (2) \end{aligned}\right\}$

We have

$$x^2+y^2 = 13$$

$$\text{and}\quad 2xy = 12;$$

∴ by addition,

$$\begin{aligned} x^2+2xy+y^2 &= 25, \\ \text{or,}\quad (x+y)^2 &= 25, \\ \therefore\quad x+y &= \pm 5; \end{aligned}$$

and by subtraction,

$$\begin{aligned} x^2-2xy+y^2 &= 1, \\ \text{or,}\quad (x-y)^2 &= 1, \\ \therefore\quad x-y &= \pm 1. \end{aligned}$$

Now we have the following four cases to consider :—

$$\left.\begin{aligned} x+y &= 5 \\ x-y &= 1 \end{aligned}\right\},\quad \left.\begin{aligned} x+y &= 5 \\ x-y &= -1 \end{aligned}\right\},\quad \left.\begin{aligned} x+y &= -5 \\ x-y &= 1 \end{aligned}\right\},\quad \left.\begin{aligned} x+y &= -5 \\ x-y &= -1 \end{aligned}\right\}.$$

Hence, corresponding to these we have the following four sets of values for x and y :—

$$\left.\begin{aligned} x &= 3 \\ y &= 2 \end{aligned}\right\},\quad \left.\begin{aligned} x &= 2 \\ y &= 3 \end{aligned}\right\},\quad \left.\begin{aligned} x &= -2 \\ y &= -3 \end{aligned}\right\},\quad \left.\begin{aligned} x &= -3 \\ y &= -2 \end{aligned}\right\}.$$

Example 7. Solve $\left.\begin{aligned} x^2+y^2 &= 25 \quad \ldots \quad \ldots \quad \ldots \quad (1) \\ x+y &= 7 \quad \ldots \quad \ldots \quad \ldots \quad (2) \end{aligned}\right\}.$

Subtracting (1) from the square of (2), we have

$$\begin{aligned} 2xy &= 49 - 25 \\ &= 24 \quad \ldots \quad \ldots \qquad (3) \end{aligned}$$

Hence from (1) and (3), by subtraction, we have

$$\begin{aligned} (x-y)^2 &= 1, \\ \therefore \quad x - y &= +1 \quad \ldots \quad \ldots \quad \ldots \quad (4) \\ \text{or,} \quad x - y &= -1 \quad \ldots \quad \ldots \quad \ldots \quad (5) \end{aligned}$$

Now from (2) and (4),

$$\left.\begin{aligned} x + y &= 7 \\ x - y &= 1 \end{aligned}\right\}, \qquad \therefore \quad \left.\begin{aligned} x &= 4 \\ y &= 3 \end{aligned}\right\}$$

Also from (2) and (5),

$$\left.\begin{aligned} x + y &= 7 \\ x - y &= -1 \end{aligned}\right\}, \qquad \therefore \quad \left.\begin{aligned} x &= 3 \\ y &= 4 \end{aligned}\right\}$$

Thus we have two sets of values :—

$$\left.\begin{aligned} x &= 4 \\ y &= 3 \end{aligned}\right\}, \qquad \left.\begin{aligned} x &= 3 \\ y &= 4 \end{aligned}\right\}$$

Example 8. Solve

$$\left.\begin{aligned} \frac{x^2}{y} + \frac{y^2}{x} &= 18 \quad \ldots \quad \ldots \quad \ldots \quad (1) \\ x + y &= 12 \quad \ldots \quad \ldots \quad \ldots \quad (2) \end{aligned}\right\}$$

Since from (1), we have $x^3 + y^3 = 18xy$;

$$\begin{aligned} \therefore \text{ from (2),} \qquad 1728 &= x^3 + y^3 + 3xy(x+y) \\ &= 18xy + 36xy \\ &= 54xy; \\ \therefore \quad xy &= 32 \quad \ldots \quad \ldots \qquad (3) \end{aligned}$$

Now since $(x-y)^2 = (x+y)^2 - 4xy$,

$\therefore$ from (2) and (3),

$$\begin{aligned} (x-y)^2 &= 144 - 128 \\ &= 16; \\ \therefore \quad x - y &= +4 \quad \ldots \quad \ldots \quad \ldots \quad (4) \\ \text{or,} \quad x - y &= -4 \quad \ldots \quad \ldots \quad \ldots \quad (5) \end{aligned}$$

Now from (2) and (4),

$$\left.\begin{aligned} x + y &= 12 \\ x - y &= 4 \end{aligned}\right\}, \qquad \therefore \quad \left.\begin{aligned} x &= 8 \\ y &= 4 \end{aligned}\right\}.$$

Also from (2) and (5),

$$\left.\begin{aligned}x+y&=12\\x-y&=-4\end{aligned}\right\},\qquad\therefore\ \left.\begin{aligned}x&=4\\y&=8\end{aligned}\right\}.$$

Thus we have two sets of values :—

$$\left.\begin{aligned}x&=8\\y&=4\end{aligned}\right\},\qquad\left.\begin{aligned}x&=4\\y&=8\end{aligned}\right\}.$$

Exercise (45).

Solve the following equations :—

1. $\left.\begin{aligned}4x-3y&=1\\12xy+13y^2&=25\end{aligned}\right\}$

2. $\left.\begin{aligned}2x+3y&=37\\\frac{1}{x}+\frac{1}{y}&=\frac{14}{45}\end{aligned}\right\}$

3. $\left.\begin{aligned}x+y&=30\\xy&=216\end{aligned}\right\}$

4. $\left.\begin{aligned}x-y&=18\\xy&=1363\end{aligned}\right\}$

5. $\left.\begin{aligned}x^2+y^2&=13\\x+y&=5\end{aligned}\right\}$

6. $\left.\begin{aligned}x^2+y^2&=185\\x-y&=3\end{aligned}\right\}$

7. $\left.\begin{aligned}x^2+y^2&=65\\x-y&=3\end{aligned}\right\}$

8. $\left.\begin{aligned}\frac{1}{x}+\frac{1}{y}&=\frac{7}{12}\\xy&=12\end{aligned}\right\}$

9. $\left.\begin{aligned}x^2+2y^2&=34\\x+y&=7\end{aligned}\right\}$

10. $\left.\begin{aligned}ax+by&=2\\abxy&=1\end{aligned}\right\}$

11. $\left.\begin{aligned}x^2+y^2&=58\\xy&=21\end{aligned}\right\}$

12. $\left.\begin{aligned}x^2+y^2&=89\\xy&=40\end{aligned}\right\}$

13. $\left.\begin{aligned}x^2+y^2&=74\\xy&=35\end{aligned}\right\}$

14. $\left.\begin{aligned}3x-2y&=7\\xy&=20\end{aligned}\right\}$

15. $\left.\begin{aligned}x^3+y^3&=637\\x+y&=13\end{aligned}\right\}$

16. $\left.\begin{aligned}x^3-y^3&=218\\x-y&=2\end{aligned}\right\}$

17. $\left.\begin{aligned}x^4+x^2y^2+y^4&=2613\\x^2+xy+y^2&=67\end{aligned}\right\}$

[$x^4+x^2y^2+y^4=\{(x^2+y^2)+xy\}\{(x^2+y^2)-xy\}$; hence the value of x^2-xy+y^2 is known, which combined with the second of the given equations gives $x^2+y^2=53$ and $xy=14$.]

18. $\left.\begin{aligned}x^4+x^2y^2+y^4&=9211\\x^2-xy+y^2&=61\end{aligned}\right\}$

19. $\left.\begin{aligned}x^3+y^3&=4914\\x+y&=18\end{aligned}\right\}$

20. $\left.\begin{aligned}x^2-xy+y^2&=7\\x^4+x^2y^2+y^4&=133\end{aligned}\right\}$

21. $\left.\begin{aligned}x^2+xy+y^2&=49\\x^4+x^2y^2+y^4&=931\end{aligned}\right\}$

22. $$\left.\begin{aligned}\frac{x}{y}+\frac{y}{x}&=2\tfrac{1}{2}\\ x+y&=6\end{aligned}\right\}$$

23. $$\left.\begin{aligned}\frac{x}{y}+\frac{y}{x}&=2\tfrac{16}{21}\\ x-y&=4\end{aligned}\right\}$$

24. $$\left.\begin{aligned}\frac{1}{x^2}+\frac{1}{y^2}&=\frac{481}{576}\\ \frac{1}{x}+\frac{1}{y}&=\frac{29}{24}\end{aligned}\right\}$$

25. $$\left.\begin{aligned}\frac{1}{x^3}+\frac{1}{y^3}&=1\frac{1}{125}\\ \frac{1}{x}+\frac{1}{y}&=1\frac{1}{5}\end{aligned}\right\}$$

26. $$\left.\begin{aligned}x+y+\sqrt{xy}&=14\\ x^2+y^2+xy&=84\end{aligned}\right\}$$

27. $$\left.\begin{aligned}x+y-\sqrt{xy}&=7\\ x^2+y^2+xy&=133\end{aligned}\right\}$$

28. $$\left.\begin{aligned}x^3+y^3&=126\\ x^2-xy+y^2&=21\end{aligned}\right\}$$

29. $$\left.\begin{aligned}\sqrt{\frac{x}{y}}+\sqrt{\frac{y}{x}}&=\frac{5}{2}\\ x+y&=10\end{aligned}\right\}$$

30. $$\left.\begin{aligned}x+y&=1072\\ x^{\frac{1}{3}}+y^{\frac{1}{3}}&=16\end{aligned}\right\}$$

$$\left[\left(x^{\frac{1}{3}}+y^{\frac{1}{3}}\right)^3=x+y+3x^{\frac{1}{3}}y^{\frac{1}{3}}\left(x^{\frac{1}{3}}+y^{\frac{1}{3}}\right)=x+y+48x^{\frac{1}{3}}y^{\frac{1}{3}}.\right]$$

31. $$\left.\begin{aligned}x+y&=65\\ x^{\frac{1}{3}}+y^{\frac{1}{3}}&=5\end{aligned}\right\}$$

32. $$\left.\begin{aligned}\frac{x^2}{y}+\frac{y^2}{x}&=\frac{9}{2}\\ \frac{3}{x+y}&=1\end{aligned}\right\}$$

33. $$\left.\begin{aligned}\frac{1}{x^3}-\frac{1}{y^3}&=91\\ \frac{1}{x}-\frac{1}{y}&=1\end{aligned}\right\}$$

34. $$\left.\begin{aligned}\frac{1}{x^2}+\frac{1}{y^2}&=13\\ \frac{1}{x}-\frac{1}{y}&=1\end{aligned}\right\}$$

(Calcutta University F. A. Paper, 1879)

2. Homogeneous Equations.—Equations in which the sum of the indices of the powers of x and y in each term is the same, can very conveniently be solved by the artifice illustrated below :—

Example 1. Solve $$\left.\begin{aligned}2x^2-3xy+y^2&=24\\ 3x^2-5xy+2y^2&=33\end{aligned}\right\}$$

[Here the equations are homogeneous and of the second degree, the sum of the indices of the powers of x and y in each term being 2.]

We have $$x^2\left(2-3.\frac{y}{x}+\frac{y^2}{x^2}\right)=24 \quad \ldots \quad (1)$$

$$x^2\left(3-5.\frac{y}{x}+2.\frac{y^2}{x^2}\right)=33 \quad \ldots \quad (2)$$

Hence, putting m for $\frac{y}{x}$, we have

$$\frac{2-3m+m^2}{3-5m+2m^2}=\frac{24}{33}=\frac{8}{11};$$

$$\begin{aligned} \therefore \quad 22-33m+11m^2 &= 24-40m+16m^2, \\ \text{or,} \quad 5m^2-7m+2 &= 0, \\ \text{or,} \quad (m-1)(5m-2) &= 0, \\ \therefore \quad m &= 1, \text{ or } \tfrac{2}{5}. \end{aligned}$$

Now we have two cases to consider corresponding to the two values of m :—

(i) Taking $m=1$ and substituting in (1) we get $x^2\times 0 = 24$, which is inconsistent with x having any finite value and so we must reject this case.

(ii) Taking $m=\frac{2}{5}$ and substituting in (1) we have

$$\begin{aligned} x^2\left(2-\tfrac{6}{5}+\tfrac{4}{25}\right) &= 24, \\ \text{or,} \quad x^2\times\tfrac{24}{25} &= 24, \\ \therefore \quad x^2 &= 25; \\ \therefore \quad x &= \pm 5; \\ \therefore \quad y &= \tfrac{2}{5}(\pm 5) = \pm 2. \end{aligned}$$

Thus we have $$\left.\begin{aligned} x&=5 \\ y&=2 \end{aligned}\right\} \text{ or, } \left.\begin{aligned} x&=-5 \\ y&=-2 \end{aligned}\right\}.$$

Example 2. Solve $$\left.\begin{aligned} 3x^2+xy+y^2 &= 15 \\ 31xy-3x^2-5y^2 &= 45 \end{aligned}\right\}$$

Putting m for $\frac{y}{x}$ we have

$$x^2(3+m+m^2) = 15 \quad \ldots \quad \ldots \quad (1)$$
$$x^2(31m-3-5m^2) = 45 \quad \ldots \quad \ldots \quad (2)$$

$\therefore$ by division $$\frac{3+m+m^2}{31m-3-5m^2}=\frac{1}{3};$$

$$\begin{aligned}
\therefore\quad 9+3m+3m^2 &= 31m-3-5m^2,\\
\text{or,}\quad 8m^2-28m+12 &= 0,\\
\text{or,}\quad 2m^2-7m+3 &= 0,\\
\text{or,}\quad (m-3)(2m-1) &= 0\,;\\
\therefore\quad m &= 3, \text{ or } \tfrac{1}{2}.
\end{aligned}$$

(i) Taking $m = 3$ and substituting in (1) we have

$$x^2 = \tfrac{15}{15} = 1, \quad \left.\begin{aligned}\therefore\quad x &= \pm 1\\ \text{and}\ \therefore\quad y &= \pm 3\end{aligned}\right\}$$

(ii) Taking $m = \frac{1}{2}$ and substituting in (1) we have

$$x^2 = \frac{15}{3\frac{3}{4}} = 4, \quad \therefore\quad x = \pm 2,$$

$$\begin{aligned}\text{and}\ \therefore\quad y &= \tfrac{1}{2}(\pm 2)\\ &= \pm 1.\end{aligned}$$

Thus we have four sets of values :—

$$\left.\begin{aligned}x&=1\\y&=3\end{aligned}\right\},\quad \left.\begin{aligned}x&=-1\\y&=-3\end{aligned}\right\},\quad \left.\begin{aligned}x&=2\\y&=1\end{aligned}\right\},\quad \left.\begin{aligned}x&=-2\\y&=-1\end{aligned}\right\}.$$

Exercise (46).

Solve the following equations :—

1. $\left.\begin{aligned}x^2+xy &= 15\\ xy-y^2 &= 2\end{aligned}\right\}$

2. $\left.\begin{aligned}x^2+xy+4y^2 &= 6\\ 3x^2+8y^2 &= 14\end{aligned}\right\}$

3. $\left.\begin{aligned}2x^2+3xy &= 26\\ 3y^2+2xy &= 39\end{aligned}\right\}$

4. $\left.\begin{aligned}x^2+xy+y^2 &= 3\tfrac{1}{4}\\ 2x^2-3xy+2y^2 &= 2\tfrac{3}{4}\end{aligned}\right\}$

5. $\left.\begin{aligned}x^2+xy+y^2 &= 52\\ xy-x^2 &= 8\end{aligned}\right\}$

6. $\left.\begin{aligned}3x^2+165 &= 16xy\\ 7xy+3y^2 &= 132\end{aligned}\right\}$

7. $\left.\begin{aligned}x^2+y^2-3 &= 3xy\\ 2x^2-6+y^2 &= 0\end{aligned}\right\}$

8. $\left.\begin{aligned}2x^2+3xy+y^2 &= 20\\ 5x^2+4y^2 &= 41\end{aligned}\right\}$

9. $\left.\begin{aligned}x^2+xy &= 12\\ xy-2y^2 &= 1\end{aligned}\right\}$

10. $\left.\begin{aligned}x^2-xy+y^2 &= 21\\ y^2-2xy+15 &= 0\end{aligned}\right\}$

11. $\left.\begin{aligned}x^2-2xy &= 21\\ xy+y^2 &= 18\end{aligned}\right\}$

12. $\left.\begin{aligned}x^3+y^3 &= 91\\ x^2y+xy^2 &= 84\end{aligned}\right\}$

13. $\left.\begin{aligned}x^3-y^3 &= 127\\ x^2y-xy^2 &= 42\end{aligned}\right\}$

14. $\left.\begin{aligned}3x^2+2y^2 &= 50\\ xy-3y^2 &= 1\end{aligned}\right\}$

(Calcutta University F. A. Paper, 1877.)

15. $\left.\begin{aligned} 3x^2 - 4xy + 5y^2 &= 33 \\ 4x^2 - xy &= 10 \end{aligned}\right\}$ (Cal. University F. A. Paper, 1878).

16. $\left.\begin{aligned} 4x^2 + 7y^2 &= 148 \\ 12(x^2 + y^2) &= 25xy \end{aligned}\right\}$ (Punj. University I. E. Paper, 1887.)

3. Equations symmetrical with respect to x and y :—In such equations the unknown quantities are *similarly* involved, so that if x be put for y and y for x no change whatever is made in the equations. Thus the equations $x + y = 5$ and $xy - x - y = 6$ are both symmetrical with respect to x and y.

Some equations of this class as well as a few others can be very well solved by the application of the artifice illustrated below :—

Example 1. Solve $\left.\begin{aligned} x^4 + y^4 &= 706 \quad \ldots \ldots \ldots \quad (1) \\ x + y &= 8 \quad \ldots \ldots \ldots \quad (2) \end{aligned}\right\}$

[Here the equations are both symmetrical with respect to x and y.]

Assume $x = u + v$ and $y = u - v$, then from (2), we have

$$2u = 8, \quad \text{or} \quad u = 4.$$

Hence, we have $\left.\begin{aligned} x &= 4 + v \\ \text{and} \quad y &= 4 - v \end{aligned}\right\}.$

Substituting these values of x, y in (1), we have

$$\begin{aligned} (4+v)^4 + (4-v)^4 &= 706, \\ \text{or,} \quad 2(4^4 + 6.4^2v^2 + v^4) &= 706, \\ \text{or,} \quad 256 + 96v^2 + v^4 &= 353, \\ \text{or,} \quad v^4 + 96v^2 - 97 &= 0, \\ \text{or.} \quad (v^2 - 1)(v^2 + 97) &= 0; \\ \therefore \quad v^2 &= 1 \text{ or } -97, \\ \therefore \quad v &= \pm 1 \text{ or } \pm \sqrt{-97}. \end{aligned}$$

Hence, $\left.\begin{aligned} x &= 4 \pm 1 \\ &= 5, \text{ or } 3 \\ \text{and} \quad y &= 4 \mp 1 \\ &= 3, \text{ or } 5 \end{aligned}\right\}$ or, $\left.\begin{aligned} x &= 4 \pm \sqrt{-97} \\ \text{and} \quad y &= 4 \mp \sqrt{-97} \end{aligned}\right\}$

Thus we have four solutions :—

$$\left.\begin{aligned} x &= 5 \\ y &= 3 \end{aligned}\right\}, \left.\begin{aligned} x &= 3 \\ y &= 5 \end{aligned}\right\}, \left.\begin{aligned} x &= 4 + \sqrt{-97} \\ y &= 4 - \sqrt{-97} \end{aligned}\right\}, \left.\begin{aligned} x &= 4 - \sqrt{-97} \\ y &= 4 + \sqrt{-97} \end{aligned}\right\}.$$

Example 2. Solve
$$\left.\begin{aligned} x^5 - y^5 &= 3093 \quad \ldots \quad \ldots \quad (1) \\ x - y &= 3 \quad \ldots \quad \ldots \quad (2) \end{aligned}\right\}$$

[Here the equations though not symmetrical with respect to x and y can yet be solved by the method above illustrated.]

Assume $x = u + v$ and y

then from (2), $2v = 3$, or $v = \frac{3}{2}$.

Hence, we have $\left.\begin{aligned} x &= u + \tfrac{3}{2} \\ \text{and} \quad y &= u - \tfrac{3}{2} \end{aligned}\right\}$.

Substituting these values of x and y in (1), we have

$$(u + \tfrac{3}{2})^5 - (u - \tfrac{3}{2})^5 = 3093,$$

$$\text{or,} \quad 2\{5.u^4.\tfrac{3}{2} + 10.u^2.(\tfrac{3}{2})^3 + (\tfrac{3}{2})^5\} = 3093,$$

$$\text{or,} \quad 5u^4 + \frac{90u^2}{4} + \frac{81}{16} = 1031,$$

$$\text{or,} \quad 80u^4 + 360u^2 + 81 = 16496,$$

$$\text{or,} \quad 80u^4 + 360u^2 - 16415 = 0,$$

$$\text{or,} \quad 16u^4 + 72u^2 - 3283 = 0,$$

$$\text{or,} \quad (4u^2 - 49)(4u^2 + 67) = 0,$$

$$\therefore \quad u^2 = \tfrac{49}{4}, \text{ or } -\tfrac{67}{4},$$

$$\therefore \quad u = \pm\tfrac{7}{2}, \text{ or } \pm\tfrac{1}{2}\sqrt{-67}.$$

$$\text{Hence,}\ \left.\begin{aligned} x &= \pm\tfrac{7}{2} + \tfrac{3}{2} \\ &= 5 \text{ or } -2, \\ \text{and}\quad y &= \pm\tfrac{7}{2} - \tfrac{3}{2} \\ &= 2 \text{ or } -5. \end{aligned}\right\} \quad \text{or,} \quad \left.\begin{aligned} x &= \pm\tfrac{1}{2}\sqrt{-67} + \tfrac{3}{2}, \\ \text{and}\quad y &= \pm\tfrac{1}{2}\sqrt{-67} - \tfrac{3}{2}. \end{aligned}\right\}$$

Thus we have four solutions :—

$$\left.\begin{aligned} x &= 5 \\ y &= 2 \end{aligned}\right\}, \quad \left.\begin{aligned} x &= -2 \\ y &= -5 \end{aligned}\right\}, \quad \left.\begin{aligned} x &= \tfrac{1}{2}\sqrt{-67} + \tfrac{3}{2} \\ y &= \tfrac{1}{2}\sqrt{-67} - \tfrac{3}{2} \end{aligned}\right\}, \quad \left.\begin{aligned} x &= -\tfrac{1}{2}\sqrt{-67} + \tfrac{3}{2} \\ y &= -\tfrac{1}{2}\sqrt{-67} - \tfrac{3}{2} \end{aligned}\right\}.$$

Example 3. Solve
$$\frac{bx}{y+b} + \frac{ay}{x+a} = \frac{a+b}{2} \quad \ldots \quad \ldots \quad (1)$$

$$\frac{x}{a} + \frac{y}{b} = 2 \quad \ldots \quad \ldots \quad (2)$$

Assume $\frac{x}{a} \quad u + v$, and $\frac{y}{b} = u - v$,

then from (2), $2u = 2$, or $u = 1$.

$$\text{Hence, } \frac{x}{a} = 1 + v$$

$$\text{and } \frac{y}{b} = 1 - v$$

Substituting these values of $\frac{x}{a}$ and $\frac{y}{b}$ in (1), which is

$$\frac{x}{\frac{y}{b}+1} + \frac{y}{\frac{x}{a}+1} = \frac{a+b}{2}, \text{ we have}$$

$$\frac{a(1+v)}{2-v} + \frac{b(1-v)}{2+v} = \frac{a+b}{2}$$

$$\text{or, } a\left\{\frac{1+v}{2-v} - \frac{1}{2}\right\} = b\left\{\frac{1}{2} - \frac{1-v}{2+v}\right\}$$

$$\text{or, } a.\frac{3v}{2-v} = b.\frac{3v}{2+v},$$

$$\therefore \text{ either } v = 0, \text{ or } \frac{a}{2-v} = \frac{b}{2+v},$$

$$\text{and } \therefore v = \frac{2(b-a)}{a+b}.$$

(i) Taking $v = 0$, we have $\frac{x}{a} = 1$, $\frac{y}{b} = 1$

$$\text{and } x = a, \quad y = b.$$

(ii) Taking $v = \frac{2(b-a)}{a+b}$, we have

$$\frac{x}{a} = 1 + \frac{2(b-a)}{a+b} = \frac{3b-a}{a+b}, \text{ and } \frac{y}{b} = 1 - \frac{2(b-a)}{a+b} = \frac{3a-b}{a+b};$$

$$\therefore x = \frac{3ab-a^2}{a+b}, \quad y = \frac{3ab-b^2}{a+b}$$

Thus we have two solutions :—

$$\left.\begin{aligned} x &= a \\ y &= b \end{aligned}\right\}, \qquad \begin{aligned} x &= \frac{3ab-a^2}{a+b} \\ y &= \frac{3ab-b^2}{a+b} \end{aligned}$$

Example 4. If $\sqrt{ax}+\sqrt{by} = \frac{1}{2}(x+y) = a+b$, find x and y.

Assume $x = u+v$ and $y = u-v$;

then $2u = x+y = 2(a+b)$ and $\therefore\ u = a+b$.

Hence, we have $\left.\begin{aligned} x &= (a+b)+v \\ y &= (a+b)-v \end{aligned}\right\}$.

Now since $\sqrt{ax}+\sqrt{by} = (a+b)$,

$\therefore\ ax+by+2\sqrt{ab}\sqrt{xy} = (a+b)^2$;

hence, by the substitution of the above values of x and y, we have –

$$(a+b)^2+(a-b)v+2\sqrt{ab}\sqrt{(a+b)^2-v^2} = (a+b)^2,$$

$$\therefore\ (a-b)v = -2\sqrt{ab}\sqrt{(a+b)^2-v^2},$$

$$\therefore\ (a-b)^2v^2 = 4ab\{(a+b)^2-v^2\},$$

$$\therefore\ v^2(a+b)^2 = 4ab(a+b)^2,$$

$$\therefore\ v = \pm 2\sqrt{ab}.$$

Hence and $\left.\begin{aligned} x &= a+b\pm 2\sqrt{ab} = (\sqrt{a}\pm\sqrt{b})^2 \\ y &= a+b\mp 2\sqrt{ab} = (\sqrt{a}\mp\sqrt{b})^2 \end{aligned}\right\}$.

Exercise (47).

Solve the following equations :—

1. $\left.\begin{aligned} x^4+y^4 &= 82 \\ x+y &= 4 \end{aligned}\right\}$

2. $\left.\begin{aligned} \frac{a}{x}+\frac{b}{y} &= 4 \\ \frac{x}{a}+\frac{y}{b} &= 1 \end{aligned}\right\}$

3. $\left.\begin{aligned} x^4+y^4 &= 272 \\ x-y &= 2 \end{aligned}\right\}$

4. $\left.\begin{aligned} x^4+y^4 &= 257 \\ x+y &= 5 \end{aligned}\right\}$

5. $\left.\begin{aligned} x^4-y^4 &= 14560 \\ x-y &= 8 \end{aligned}\right\}$

6. $\left.\begin{aligned} x^5-y^5 &= 992 \\ x-y &= 2 \end{aligned}\right\}$

7. $\left.\begin{aligned} x^5+y^5 &= 33 \\ x+y &= 3 \end{aligned}\right\}$

8. $\left.\begin{aligned} x^4+y^4 &= 14x^2y^2 \\ x+y &= a \end{aligned}\right\}$

9. $\left.\begin{aligned} \frac{a}{a+x}+\frac{b}{b+y} &= 1 \\ x+y &= a+b \end{aligned}\right\}$

10. $\begin{aligned} (x^2+y^2)(x^3+y^3) &= 455 \\ x+y &= 5 \end{aligned}$

4. The method of cross multiplication.—

If there be two equations such as $a_1x+b_1y+c_1z = 0$ and $a_2x+b_2y+c_2z = 0$, we can find the relation between x, y, z in the following way :—Multiply the 1st equation by c_2 and the second by c_1 and subtract the latter product from the former, we thus get $x(a_1c_2-a_2c_1)+y(b_1c_2-b_2c_1) = 0$,

or, $\dfrac{x}{b_1c_2-b_2c_1} = \dfrac{y}{c_1a_2-c_2a_1}$; similarly, multiplying the 1st equation by a_2 and the second by a_1, and subtracting as before,

we get $$\frac{y}{c_1a_2-c_2a_1} = \frac{z}{a_1b_2-a_2b_1}.$$

Thus we have $$\frac{x}{b_1c_2-b_2c_1} = \frac{y}{c_1a_2-c_2a_1} = \frac{z}{a_1b_2-a_2b_1}.$$

This result can be easily remembered, for writing down the equations one above the other,

$$\left.\begin{aligned} a_1x+b_1y+c_1z &= 0 \\ a_2x+b_2y+c_2z &= 0 \end{aligned}\right\}, \text{ we find that}$$

(i) the quantity under x = co-efficient of y in the 1st equation × co-efficient of z in the 2nd *minus* co-efficient of y in the 2nd × co-efficient of z in the 1st ;

(ii) the quantity under y = co-efficient of z in the 1st equation × co-efficient of x in the 2nd *minus* co-efficient of z in the 2nd × co-efficient of x in the 1st ;

(iii) the quantity under z = co-efficient of x in the 1st equation × co-efficient of y in the second *minus* co-efficient of x in the second × co-efficient of y in the 1st.

Cor. In the above equations if we put $z = 1$,

we have $$\frac{x}{b_1c_2-b_2c_1} = \frac{y}{c_1a_2-c_2a_1} = \frac{1}{a_1b_2-a_2b_1},$$

which gives us the solution of the equations

$$a_1x+b_1y+c_1 = 0 \text{ and } a_2x+b_2y+c_2 = 0.$$

The above results should be committed to memory as a ready application of them will save the student much time and trouble in solving a certain class of equations involving three unknown quantities ; the following examples will afford illustrations :—

Example 1. Solve $$\left.\begin{aligned} 2x+y-2z &= 0 \quad \ldots \quad (1) \\ 7x+6y-9z &= 0 \quad \ldots \quad (2) \\ x^3+y^3+z^3 &= 1728 \quad \ldots \quad (3) \end{aligned}\right\}$$

From (1) and (2), by cross multiplication, we have

$$\frac{x}{1(-9)-6.(-2)} = \frac{y}{(-2).7-(-9).2} = \frac{z}{2.6-7.1},$$

$$\text{or,} \qquad \frac{x}{3} = \frac{y}{4} = \frac{z}{5}.$$

Suppose each of these ratios $= k$; then we have $x = 3k$, $y = 4k$, $z = 5k$.

Substituting these values in (3), we have

$$\begin{aligned} k^3(27+64+125) &= 1728, \\ \text{or,} \qquad 216k^3 &= 1728, \\ \therefore \qquad k^3 &= 8, \\ \therefore \qquad k &= 2. \end{aligned}$$

Hence $\qquad x = 6,\ y = 8,\ z = 10.$

Example 2. Solve

$$\begin{aligned} x+y+z &= a+b+c \qquad (1) \\ x^2+y^2+z^2 &= a^2+b^2+c^2 \ldots \ (2) \\ \frac{x}{a}+\frac{y}{b}+\frac{z}{c} &= 3 \qquad \ldots \qquad \ldots \ (3) \end{aligned}$$

(Calcutta University F. A. Paper, 1880.)

From (3), $\qquad \left(\frac{x}{a}-1\right)+\left(\frac{y}{b}-1\right)+\left(\frac{z}{c}-1\right) = 0.$

From (1), $\qquad (x-a)+(y-b)+(z-c) = 0,$

$$\text{or,} \qquad a\left(\frac{x}{a}-1\right)+b\left(\frac{y}{b}-1\right)+c\left(\frac{z}{c}-1\right) = 0.$$

Hence, by cross multiplication,

$$\frac{\frac{x}{a}-1}{c-b} = \frac{\frac{y}{b}-1}{a-c} = \frac{\frac{z}{c}-1}{b-a} = k \text{ (say)};$$

$$\begin{aligned} \text{whence} \qquad \frac{x}{a}-1 &= k(c-b), \quad \therefore \quad x-a = a(c-b)k, \\ \frac{y}{b}-1 &= k(a-c), \quad \therefore \quad y-b = b(a-c)k, \\ \frac{z}{c}-1 &= k(b-a), \quad \therefore \quad z-c = c(b-a)k. \end{aligned}$$

Now, from (2), we have $\qquad (x^2-a^2)+(y^2-b^2)+(z^2-c^2) = 0,$

or, $(x-a)(x-a+2a)+(y-b)(y-b+2b)+(z-c)(z-c+2c) = 0;$

substituting in this equation the values of $x-a$, $y-b$, $z-c$, found above, we have

$$\begin{aligned}\{a^2(c-b)^2k^2 + 2a^2(c-b)k\} + \{b^2(a-c)^2k^2 + 2b^2(a-c)k\}&\\ + \{c^2(b-a)^2k^2 + 2c^2(b-a)k\} &= 0\end{aligned}$$

$$\begin{aligned}\text{or,}\quad k^2\{a^2(b-c)^2 + b^2(c-a)^2 + c^2(a-b)^2\}&\\ = 2k\{a^2(b-c) + b^2(c-a) + c^2(a-b)\}&;\end{aligned}$$

$$\begin{aligned}\therefore\ \text{either } k &= 0,\\ \text{or,}\quad k &= 2\,\frac{a^2(b-c) + b^2(c-a) + c^2(a-b)}{a^2(b-c)^2 + b^2(c-a)^2 + c^2(a-b)^2}\\ &= P \text{ (suppose).}\end{aligned}$$

(i) Taking the 1st value of k, we have

$$\left.\begin{aligned}\frac{x}{a} - 1 &= 0, \quad \therefore\ x = a\\ \frac{y}{b} - 1 &= 0, \quad \therefore\ y = b\\ \frac{z}{c} - 1 &= 0, \quad \therefore\ z = c\end{aligned}\right\}.$$

(ii) Taking the 2nd value, we have

$$\begin{aligned}\frac{x}{a} - 1 &= P(c-b), \quad \therefore\ x = a\{1 + P(c-b)\};\\ \frac{y}{b} - 1 &= P(a-c), \quad \therefore\ y = b\{1 + P(a-c)\};\\ \frac{z}{c} - 1 &= P(b-a), \quad \therefore\ z = c\{1 + P(b-a)\}.\end{aligned}$$

Exercise (48).

Solve the following equations:—

1. $\left.\begin{aligned}5x - 4y + z &= 0\\ 2x + 5y - 4z &= 0\\ x^2 - 2y^2 + z^2 &= 0\end{aligned}\right\}$

2. $\left.\begin{aligned}3x - 4y + 7z &\quad 0\\ 2x - y - 2z &\quad 0\\ 3x^3 - y^3 + z^3 &= 18\end{aligned}\right\}$

3. $\left.\begin{aligned}ax + by + cz &= 0\\ bcx + cay + abz &= 0\\ xyz + abc(a^3x + b^3y + c^3z) &= 0\end{aligned}\right\}$

4. $\left.\begin{aligned}3x + y - 2z &= 0\\ 4x - y - 3z &= 0\\ x^3 + y^3 + z^3 &= 467\end{aligned}\right\}$

5. $\left.\begin{aligned}x^2 + 2y^2 + 3z^2 &= 23\\ 3x + y - 5z &= 0\\ 7x - 3y - 9z &= 0\end{aligned}\right\}$

(Calcutta University F. A. Paper, 1873.)

6. $\left.\begin{aligned} 9x + y - 8z &= 0 \\ 4x - 8y + 7z &= 0 \\ yz + zx + xy &= 47 \end{aligned}\right\}$

7. $\left.\begin{aligned} 3x^2 - 2y^2 + 5z^2 &= 0 \\ 7x^2 - 3y^2 - 15z^2 &= 0 \\ 5x - 4y + 7z &= 6 \end{aligned}\right\}$

8. $\left.\begin{aligned} \frac{x-a}{b-c} + \frac{y-b}{c-a} + \frac{z-c}{a-b} &= 0 \\ ax + by + cz &= 0 \\ \frac{x}{a} + \frac{y}{b} + \frac{z}{c} &= 0 \end{aligned}\right\}$ (Calcutta University F. A. Paper, 1875.)

5. **Miscellaneous Artifices.**—The student is strongly recommended to try each equation himself before looking into its solution.

Example 1. Solve $\left.\begin{aligned} 4x^2 + 5y &= 6 + 20xy - 25y^2 + 2x \quad &(1) \\ 7x - 11y &= 17 \quad \ldots \quad &(2) \end{aligned}\right\}$

From (1), we have, by transposition,

$$4x^2 - 20xy + 25y^2 = 2x - 5y + 6$$

or, $$(2x - 5y)^2 - (2x - 5y) - 6 = 0$$

or, $$\{(2x - 5y) - 3\}\{(2x - 5y) + 2\} = 0,$$

$$\therefore \text{ either } \left.\begin{aligned} 2x - 5y &= 3 \\ \text{or, } 2x - 5y &= -2 \end{aligned}\right\}$$

(i) Taking the 1st of these values, we have

$$2x - 5y - 3 = 0,$$

and from (2), $7x - 11y - 17 = 0$;

$\therefore$ by cross multiplication,

$$\frac{x}{85 - 33} = \frac{y}{-21 + 34} = \frac{1}{-22 + 35}$$

or, $$\frac{x}{52} = \frac{y}{13} = \frac{1}{13};$$

$\therefore$ $x = 4$, and $y = 1$.

(ii) Taking the 2nd value of $2x - 5y$, we have

$$2x - 5y + 2 = 0,$$

and from (2), $7x - 11y - 17 = 0$;

$\therefore$ by cross multiplication,

$$\frac{x}{85 + 22} = \frac{y}{14 + 34} = \frac{1}{-22 + 35}$$

$$\text{or,}\quad \frac{x}{107} = \frac{y}{78} = \frac{1}{13};$$

$$\therefore \quad x = \frac{107}{13},\ y = \frac{48}{13}.$$

Thus we have two solutions :—

$$\left.\begin{aligned} x &= 4 \\ y &= 1 \end{aligned}\right\};\ x = \frac{107}{13},\ y = \frac{48}{13}.$$

Example 2. Solve

$$\left.\begin{aligned} \frac{x^3+y^3}{(x+y)^2} + \frac{x^3-y^3}{(x-y)^2} &= \frac{43x}{8} \quad \ldots (1) \\ 5x - 7y &= 4 \quad \ldots (2) \end{aligned}\right\}$$

Since $x^3 + y^3 = (x+y)^3 - 3xy(x+y)$

and $x^3 - y^3 = (x-y)^3 + 3xy(x-y)$,

∴ from (1), we have

$$\left\{(x+y) - \frac{3xy}{x+y}\right\} + \left\{(x-y) + \frac{3xy}{x-y}\right\} = \frac{43x}{8}$$

$$\text{or,}\quad 3xy\left\{\frac{1}{x-y} - \frac{1}{x+y}\right\} = \frac{43x}{8} - 2x = \frac{27x}{8}$$

$$\text{or,}\quad 3xy.\frac{2y}{x^2-y^2} = \frac{27x}{8}$$

$$\text{or,}\quad x.\frac{2y^2}{x^2-y^2} = \frac{9}{8}x;$$

∴ either $x = 0$, and ∴ from (2), $y = -$

or, $9x^2 = 25y^2$,

∴ $3x = \pm 5y$.

(i) Taking the upper sign, we have

$$\frac{x}{5} = \frac{y}{3} \text{ and } \therefore \text{ each of these ratios} = \frac{5x-7y}{25-21} = 1, \text{ by (2).}$$

$$\therefore \quad x = 5,\ y = 3.$$

(ii) Taking the lower sign, we have

$$\frac{x}{-5} = \frac{y}{3} \text{ and } \therefore \text{ each of them} = \frac{5x-7y}{-25-21}$$

$$\frac{4}{-46} \quad [\text{by (2)}]$$

$$\frac{2}{-23};$$

$$\therefore \quad x = \frac{10}{23},\ y = -\frac{6}{23}.$$

Thus we have three solutions : —

$$\left.\begin{array}{l} x = 0 \\ y = -\frac{4}{7} \end{array}\right\},\quad \left.\begin{array}{l} x = 5 \\ y = 3 \end{array}\right\},\quad \left.\begin{array}{l} x = \frac{10}{23} \\ y = -\frac{6}{23} \end{array}\right\}.$$

Example 3. Solve

$$\left.\begin{array}{lr} x^2+y^2 = 7+xy & \ldots\ \ldots\ (1) \\ x^3+y^3 = 6xy-1 & \ldots\ \ldots\ (2) \end{array}\right\}$$

From (1), we have

$$\begin{aligned} 7 &= x^2 - xy + y^2\ ; \\ \therefore\ 7(x+y) &= x^3+y^3 \\ &= 6xy-1, \text{ by } (2) \ldots\ \ldots\ \ldots\ (3) \end{aligned}$$

Again from (1), adding $2xy$ to both sides,

$$\begin{aligned} 7+3xy &= (x+y)^2, \quad \ldots\ \ldots\ \ldots\ (4) \\ \therefore\ 6xy-1 &= 2(x+y)^2-15. \end{aligned}$$

Hence, from (3),

$$2(x+y)^2-15 = 7(x+y).$$

Hence, $\{(x+y)-5\}\{2(x+y)+3\} = 0$;

$$\therefore \left.\begin{array}{ll} \text{either} & x+y = 5 \\ \text{or,} & x+y = -\frac{3}{2} \end{array}\right\}.$$

Taking the 1st value, we have from (4),

$$\begin{aligned} 3xy &= 25-7 \\ &= 18, \quad \therefore\ xy = 6. \end{aligned}$$

Thus we have $x+y = 5$ and $xy = 6$;

$$\therefore\ (x-y)^2 = (x+y)^2-4xy = 1,\ \therefore\ x-y = \pm 1.$$

Now since $\left.\begin{array}{l} x+y = 5 \\ x-y = \pm 1 \end{array}\right\}$ and, $\therefore \left.\begin{array}{l} x = 3,\ 2 \\ y = 2,\ 3 \end{array}\right\}$.

Similarly, we get two other solutions by taking the other value of $x+y$.

Example 4. Solve $$\left.\begin{aligned}\left(3-\frac{6y}{x+y}\right)^2+\left(3+\frac{6y}{x-y}\right)^2 &= 82 \ldots(1)\\ 3x+7y &= 26 \ldots(2)\end{aligned}\right\}$$

From (1), $$\left\{\frac{3(x-y)}{x+y}\right\}^2+\left\{\frac{3(x+y)}{x-y}\right\}^2 = 82.$$

Put $\dfrac{x+y}{x-y} = z$; then we have

$$9\left(\frac{1}{z^2}+z^2\right) = 82,$$

$$\text{or,}\quad 9z^4-82z^2+9 = 0,$$

$$\text{or,}\quad (z^2-9)(9z^2-1) = 0\,;$$

$$\therefore\quad z^2 = 9, \text{ or } \tfrac{1}{9}.$$

Thus we have four values for z :—

$$z = 3,\ -3,\ \tfrac{1}{3} \text{ or } -\tfrac{1}{3}.$$

(i) Taking the 1st value, we have

$$\frac{x+y}{x-y} = 3\,;$$

$$\therefore\quad \frac{x}{y} = \frac{4}{2} = 2 \qquad \text{(Componendo and dividendo)}$$

$$\therefore\quad \frac{x}{2} = \frac{y}{1} \text{ and } \therefore \text{ each of these ratios}$$

$$= \frac{3x+7y}{6+7} = \frac{26}{13}\quad [\text{by } (2)]$$

$$= 2\,;\ \therefore\ \left.\begin{aligned} x &= 4\\ \text{and}\quad y &= 2\end{aligned}\right\}.$$

(ii) Taking the 2nd value, we have

$$\frac{x+y}{x-y} = \frac{3}{-1}\,;$$

$\therefore$ by componendo and dividendo,

$$\frac{x}{y} = \frac{2}{4} = \frac{1}{2}\,;$$

$$\therefore\quad \frac{x}{1} = \frac{y}{2} \text{ and } \therefore \text{ each of them} = \frac{3x+7y}{3+14} = \frac{26}{17}\,;\ \therefore\ \left.\begin{aligned} x &= \tfrac{26}{17}\\ \text{and } y &= \tfrac{52}{17}.\end{aligned}\right\}$$

(iii) Taking the 3rd value, we have

$$\frac{x+y}{x-y} = \frac{1}{3};$$

$\therefore$ by componendo and dividendo,

$$\frac{x}{y} = \frac{4}{-2} = \frac{2}{-1};$$

$$\therefore \quad \frac{x}{2} = \frac{y}{-1} \text{ and } \therefore \text{ each of them}$$

$$= \frac{3x+7y}{6-7} = \frac{26}{-1}, \quad \therefore \quad \left.\begin{aligned} x &= -52 \\ \text{and } y &= 26 \end{aligned}\right\}.$$

(iv) Taking the 4th value, we have

$$\frac{x+y}{x-y} = \frac{1}{-3};$$

$\therefore$ by componendo and dividendo,

$$\frac{x}{y} = \frac{-2}{4} = \frac{-1}{2};$$

$$\therefore \quad \frac{x}{-1} = \frac{y}{2} \text{ and } \therefore \text{ each of them}$$

$$= \frac{3x+7y}{-3+14} = \frac{26}{11}, \quad \therefore \quad \left.\begin{aligned} x &= -\tfrac{26}{11} \\ \text{and } y &= \tfrac{52}{11} \end{aligned}\right\}$$

Thus we have four solutions :—

$$\left.\begin{aligned} x &= 4 \\ y &= 2 \end{aligned}\right\}, \quad \left.\begin{aligned} x &= \frac{26}{17} \\ y &= \frac{52}{17} \end{aligned}\right\}, \quad \left.\begin{aligned} x &= -52 \\ y &= 26 \end{aligned}\right\}, \quad \left.\begin{aligned} x &= -\frac{26}{11} \\ y &= \frac{52}{11} \end{aligned}\right\}.$$

Example 5. Solve

$$\left.\begin{aligned} \frac{1}{x} + \frac{1}{y} + \frac{1}{z} &= 9 \quad \dots \quad \dots \quad (1) \\ \frac{2}{x} + \frac{3}{y} &= 13 \quad \dots \quad \dots \quad (2) \\ 8x + 3y &= 5 \quad \dots \quad \dots \quad (3) \end{aligned}\right\}$$

From (2), $\dfrac{3x+2y}{xy} = 13.$

Multiplying this by (3), we have

$$\frac{(3x+2y)(8x+3y)}{xy} = 65.$$

Now putting $y = mx$, we have

$$\frac{(3+2m)(8+3m)}{m} = 65,$$

$$\text{or,} \quad 6m^2+25m+24 = 65m;$$

$\therefore$ by transposing and dividing by 2,

$$3m^2-20m+12 = 0,$$

$$\text{or,} \quad (3m-2)(m-6) = 0;$$

$$\therefore \quad m = \tfrac{2}{3}, \text{ or } 6.$$

(i) Taking the 1st value, we have

$$\frac{y}{x} = \frac{2}{3}, \quad \therefore \frac{x}{3} = \frac{y}{2}, \text{ and } \therefore \text{ each of them}$$

$$= \frac{8x+3y}{24+6} = \frac{5}{30}\left[\text{from (3)}\right] = \tfrac{1}{6};$$

$$\therefore \quad x = \tfrac{1}{2} \text{ and } y = \tfrac{1}{3}.$$

Hence from (1), $\dfrac{1}{z} = 4, \quad \therefore \quad z = \tfrac{1}{4}.$

(ii) Taking the 2nd value of m, we have

$$\frac{y}{x} = 6, \quad \therefore \frac{x}{1} = \frac{y}{6}, \text{ and } \therefore \text{ each of them}$$

$$= \frac{8x+3y}{8+18} = \frac{5}{26}\left[\text{from (3)}\right]$$

$$\therefore \quad x = \tfrac{5}{26} \text{ and } y = \tfrac{15}{13}.$$

Hence, from (1), $\dfrac{1}{z} = 9 - \left(\dfrac{26}{5} + \dfrac{13}{15}\right) = 9 - \dfrac{91}{15}$

$$= \tfrac{44}{15}, \quad \therefore \quad z = \tfrac{15}{44}.$$

Thus we have two solutions :—

$$\left.\begin{array}{l} x = \tfrac{1}{2} \\ y = \tfrac{1}{3} \\ z = \tfrac{1}{4} \end{array}\right\}, \qquad \begin{array}{l} x = \tfrac{5}{26} \\ y = \\ z = \end{array}$$

Example 6. Solve
$$\left.\begin{aligned} x(x+y+z) &= a^2 \ldots (1)\\ y(x+y+z) &= b^2 \ldots (2)\\ z(x+y+z) &= c^2 \ldots (3)\end{aligned}\right\}$$

Adding the equations, we have
$$(x+y+z)(x+y+z) = a^2+b^2+c^2;$$
$$\therefore \quad x+y+z = \pm\sqrt{a^2+b^2+c^2}.$$
$$\text{Hence, from (1), } x = \pm\frac{a^2}{\sqrt{a^2+b^2+c^2}},$$
$$\text{,, (2), } y = \pm\frac{b^2}{\sqrt{a^2+b^2+c^2}},$$
$$\text{,, (3), } z = \pm\frac{c^2}{\sqrt{a^2+b^2+c^2}}.$$

Example 7. Solve
$$\left.\begin{aligned} x(y+z) &= a\\ y(z+x) &= b\\ z(x+y) &= c\end{aligned}\right\}$$

We have
$$\begin{aligned} a &= xy+zx,\\ b &= yz+xy,\\ c &= zx+yz.\end{aligned}$$
Hence,
$$\begin{aligned} a+b-c &= (xy+xz)+(yz+xy)-(xz+yz),\\ &= 2xy \quad \ldots\ \ldots\ \ldots\ \ldots \quad (1)\\ \text{Similarly, } b+c-a &= 2yz \quad \ldots\ \ldots\ \ldots\ \ldots \quad (2)\\ \text{and } c+a-b &= 2zx \quad \ldots\ \ldots\ \ldots\ \ldots \quad (3)\end{aligned}$$

Now multiplying together (1), (2), (3) we have
$$8x^2y^2z^2 = (a+b-c)(b+c-a)(c+a-b);$$
$$\therefore \quad 2xyz = \pm\sqrt{\frac{(a+b-c)(b+c-a)(c+a-b)}{2}}.$$
$$\text{Hence, from (2), } x = \pm\sqrt{\frac{(a+b-c)(c+a-b)}{2(b+c-a)}},$$
$$\text{,, (3), } y = \pm\sqrt{\frac{(a+b-c)(b+c-a)}{2(c+a-b)}},$$
$$\text{,, (1), } z = \pm\sqrt{\frac{(b+c-a)(c+a-b)}{2(a+b-c)}}.$$

Example 8. Solve

$$\left.\begin{array}{lll} xy = a(x+y) & \dots\ \dots & (1) \\ xz = b(x+z) & \dots\ \dots & (2) \\ yz = c(y+z) & \dots\ \dots & (3) \end{array}\right\}.$$

From (1), $\frac{1}{a} = \frac{x+y}{xy} = \frac{1}{y} + \frac{1}{x}$,

" (2), $\frac{1}{b} = \frac{x+z}{xz} = \frac{1}{z} + \frac{1}{x}$,

" (3), $\frac{1}{c} = \frac{y+z}{yz} = \frac{1}{z} + \frac{1}{y}$.

Hence, $\frac{1}{a} + \frac{1}{b} - \frac{1}{c} = \frac{2}{x}$, $\therefore\ x = \frac{2abc}{bc+ca-ab}$;

$\frac{1}{b} + \frac{1}{c} - \frac{1}{a} = \frac{2}{z}$, $\therefore\ z = \frac{2abc}{ca+ab-bc}$;

and $\frac{1}{c} + \frac{1}{a} - \frac{1}{b} = \frac{2}{y}$, $\therefore\ y = \frac{2abc}{ab+bc-ca}$.

Example 9. Solve $xyz = a^2(y+z) = b^2(z+x) = c^2(x+y)$.

We have

$xyz = a^2(y+z)$, $\therefore\ \frac{1}{a^2} = \frac{1}{xz} + \frac{1}{xy}$;

$xyz = b^2(z+x)$, $\therefore\ \frac{1}{b^2} = \frac{1}{xy} + \frac{1}{yz}$;

$xyz = c^2(x+y)$, $\therefore\ \frac{1}{c^2} = \frac{1}{yz} + \frac{1}{xz}$.

Hence, $\frac{1}{a^2} + \frac{1}{b^2} - \frac{1}{c^2} = \frac{2}{xy}$ ... ... (1)

$\frac{1}{b^2} + \frac{1}{c^2} - \frac{1}{a^2} = \frac{2}{yz}$ ... ... (2)

and $\frac{1}{c^2} + \frac{1}{a^2} - \frac{1}{b^2} = \frac{2}{xz}$ ... ... (3)

Now, dividing (2) by the product of (1) and (3), we have

$$\frac{x^2}{2} = \frac{\dfrac{1}{b^2}+\dfrac{1}{c^2}-\dfrac{1}{a^2}}{\left(\dfrac{1}{a^2}+\dfrac{1}{b^2}-\dfrac{1}{c^2}\right)\left(\dfrac{1}{c^2}+\dfrac{1}{a^2}-\dfrac{1}{b^2}\right)};$$

$$\therefore\quad x = \pm\sqrt{\frac{2\left(\dfrac{1}{b^2}+\dfrac{1}{c^2}-\dfrac{1}{a^2}\right)}{\left(\dfrac{1}{a^2}+\dfrac{1}{b^2}-\dfrac{1}{c^2}\right)\left(\dfrac{1}{c^2}+\dfrac{1}{a^2}-\dfrac{1}{b^2}\right)}}.$$

Similarly, y and z can be found.

Example 10. Solve $xyz = a(y^2+z^2) = b(z^2+x^2) = c(x^2+y^2)$.

We have

$$\frac{1}{a} = \frac{y}{zx}+\frac{z}{xy},$$

$$\frac{1}{b} = \frac{z}{xy}+\frac{x}{yz},$$

$$\frac{1}{c} = \frac{x}{yz}+\frac{y}{zx}.$$

Hence, $$\frac{1}{b}+\frac{1}{c}-\frac{1}{a} = \frac{2x}{yz} \quad \dots\dots\dots\dots \quad (1)$$

$$\frac{1}{c}+\frac{1}{a}-\frac{1}{b} = \frac{2y}{zx} \quad \dots\dots\dots\dots \quad (2)$$

and $$\frac{1}{a}+\frac{1}{b}-\frac{1}{c} = \frac{2z}{xy} \quad \dots\dots\dots\dots \quad (3)$$

Now, multiplying together (2) and (3), we have

$$\frac{4}{x^2} = \left(\frac{1}{c}+\frac{1}{a}-\frac{1}{b}\right)\left(\frac{1}{a}+\frac{1}{b}-\frac{1}{c}\right),$$

$$\therefore\quad x = \pm\frac{2}{\sqrt{\left(\dfrac{1}{c}+\dfrac{1}{a}-\dfrac{1}{b}\right)\left(\dfrac{1}{a}+\dfrac{1}{b}-\dfrac{1}{c}\right)}}.$$

Similarly, y and z can be found.

Example 11. Solve $$\left.\begin{aligned}(x+y)(x+z) &= 40\\ (y+z)(y+x) &= 35\\ (z+x)(z+y) &= 56\end{aligned}\right\}.$$

Putting u, v, w for $y+z$, $z+x$, $x+y$ respectively, we have

$$wv = 40,\ uw = 35,\ vu = 56\ \ ..\qquad (1)$$

Multiplying together these equations, we have

$$u^2v^2w^2 = 40\times 35\times 56 = 5^2\times 8^2\times 7^2,$$

$$\therefore\quad uvw = \pm(5\times 8\times 7)$$

Hence from (1),

$$\left.\begin{aligned}u &= \frac{5\times 8\times 7}{40} = 7\\ v &= \frac{5\times 8\times 7}{35} = 8\\ w &= \frac{5\times 8\times 7}{56} = 5\end{aligned}\right\}\ \text{or,}\quad \left.\begin{aligned}u &= -7\\ v &= -8\\ w &= -5\end{aligned}\right\}.$$

Taking the 1st set of values, we have

$$\left.\begin{aligned}y+z &= 7\\ z+x &= 8\\ x+y &= 5\end{aligned}\right\},\qquad \therefore\ \left.\begin{aligned}2x &= 8+5-7 = 6, &\text{or, } x &= 3\\ 2y &= 5+7-8 = 4, &\text{or, } y &= 2\\ 2z &= 7+8-5 = 10, &\text{or, } z &= 5\end{aligned}\right\}$$

Similarly, for the other set of values of u, v, w, we have

$$x = -3,\ y = -2,\ z = -5.$$

Example 12. Solve $$\left.\begin{aligned}x^2+y^2+z^2 &= 84\ \ \dots\ \dots\ (1)\\ x+y+z &= 14\ \ \dots\ \dots\ (2)\\ xz &= y^2\ \ \dots\ \dots\ (3)\end{aligned}\right\}$$

(Calcutta University F. A. Paper, 1884.)

Since $(x+y+z)^2 = x^2+y^2+z^2+2(xy+yz+zx)$, we have from (1) and (2),

$$(14)^2 = 84+2(xy+yz+zx),$$

$$\therefore\quad xy+yz+zx = \tfrac{1}{2}(196-84) = 56.$$

Hence, from (3), $\quad xy+yz+y^2 = 56,$

or, $\quad y(x+z+y) = 56$;

$\therefore$ by (2), $\quad y = \frac{56}{14} = 4.$

Hence, from (2), $x + z = 10,$

and from (3), $xz = 16;$

$$\therefore \quad (x-z)^2 = (x+z)^2 - 4xz$$
$$= 100 - 64 = 36;$$
$$\therefore \quad x - z = \pm\ 6.$$

Thus we have and $\left.\begin{aligned} x+z &= 10 \\ x-z &= \pm\ 6 \end{aligned}\right\}$

Hence, $\left.\begin{aligned} x &= 8 \\ z &= 2 \end{aligned}\right\}$ or, $\left.\begin{aligned} x &= 2 \\ z &= 8 \end{aligned}\right\}$

Thus the solutions are :—

$$\left.\begin{aligned} x &= 8 \\ y &= 4 \\ z &= 2 \end{aligned}\right\} \quad \text{and} \quad \left.\begin{aligned} x &= 2 \\ y &= 4 \\ z &= 8 \end{aligned}\right\}$$

Otherwise :—

From (1), $(x+z)^2 - 2xz + y^2 = 84;$

$\therefore$ from (3), $(x+z)^2 - y^2 = 54;$

also from (2), $(x+z) + y = 14.$ ... ... (4)

Hence, $(x+z) - y = 6,$

which combined with (4) gives the values of y and $x + z$.

Example 13. Solve

$$\left.\begin{aligned} x^2 + xy + y^2 &= 37 \quad \ldots \quad \ldots \quad \ldots \quad (1) \\ y^2 + yz + z^2 &= 19 \quad \ldots \quad \ldots \quad \ldots \quad (2) \\ z^2 + zx + x^2 &= 28 \quad \ldots \quad \ldots \quad \ldots \quad (3) \end{aligned}\right\}$$

Subtracting (2) from (1),

$$(x^2 - z^2) + y(x - z) = 18,$$
$$\text{or,} \quad (x-z)(x+z+y) = 18. \quad \ldots \quad \ldots \quad (4)$$

Again subtracting (3) from (1),

$$(y^2 - z^2) + x(y - z) = 9,$$
$$\text{or,} \quad (y-z)(y+z+x) = 9. \quad \ldots \quad \ldots \quad (5)$$

Hence, from (4) and (5), by division—

$$\frac{x-z}{y-z} = 2$$
$$\text{whence} \quad y = \tfrac{1}{2}(x+z). \quad \ldots \quad (6)$$

Substituting this value of y in (2), we have

$$\frac{(x+z)^2}{4} + \frac{z}{2}(x+z) + z^2 = 19,$$

$$\text{or,} \quad x^2 + 4xz + 7z^2 = 76\,;$$

$$\text{also from (3),} \quad x^2 + xz + z^2 = 28\,;$$

$$\therefore \quad \frac{x^2 + 4xz + 7z^2}{x^2 + xz + z^2} = \frac{19}{7},$$

$$\text{or,} \quad \frac{1 + 4m + 7m^2}{1 + m + m^2} = \frac{19}{7}, \text{ putting } m \text{ for } \frac{z}{x}\,;$$

$$\therefore \quad 30m^2 + 9m - 12 = 0,$$

$$\text{or,} \quad 10m^2 + 3m - 4 = 0,$$

$$\text{or,} \quad (2m-1)(5m+4) = 0\,;$$

$$\therefore \quad m = \tfrac{1}{2}, \text{ or } -\tfrac{4}{5}.$$

(i) Taking $m = \frac{1}{2}$, we have from (3)

$$x^2 = \frac{28 \times 4}{7} = 16\,;$$

$$\left.\begin{aligned} \therefore \quad x &= \pm 4 \\ \therefore \quad z &= \pm 2 \\ \text{and} \quad \therefore \quad \text{from (6)} \quad y &= \pm 3 \end{aligned}\right\}$$

(ii) Taking $m = -\frac{4}{5}$ we have $\frac{x}{5} = \frac{z}{-4} = k$ (suppose)

Hence, from (3), $k^2(16 - 20 + 25) = 28$;

$$\therefore \quad k^2 = \tfrac{4}{3}, \text{ or } k = \pm \frac{2}{\sqrt{3}}.$$

$$\text{Hence} \quad x = \pm \frac{10}{\sqrt{3}} = \pm \frac{10}{3}\sqrt{3},$$

$$\text{and} \quad z = \mp \tfrac{8}{3}\sqrt{3},$$

$$\text{and} \quad \therefore \quad \text{from (6)}, \; y = \tfrac{1}{2}\left(\pm \tfrac{10}{3}\sqrt{3} \mp \tfrac{8}{3}\sqrt{3}\right) = \pm \tfrac{1}{3}\sqrt{3}.$$

Hence the solutions are :—

$$\left.\begin{aligned} x &= 4 \\ y &= 3 \\ z &= 2 \end{aligned}\right\}, \quad \left.\begin{aligned} x &= -4 \\ y &= -3 \\ z &= -2 \end{aligned}\right\}, \quad \left.\begin{aligned} x &= \tfrac{10}{3}\sqrt{3} \\ y &= \tfrac{1}{3}\sqrt{3} \\ z &= -\tfrac{8}{3}\sqrt{3} \end{aligned}\right\}, \quad \left.\begin{aligned} x &= -\tfrac{10}{3}\sqrt{3} \\ y &= -\tfrac{1}{3}\sqrt{3} \\ z &= \tfrac{8}{3}\sqrt{3} \end{aligned}\right\}.$$

Example 14. Solve $x+y+z = \frac{1}{x}+\frac{1}{y}+\frac{1}{z} = \frac{7}{2}$, $xyz = 1$.

Since $\frac{1}{x}+\frac{1}{y}+\frac{1}{z} = \frac{7}{2}$,

$$\therefore \quad yz+zx+xy = \tfrac{7}{2}xyz = \tfrac{7}{2}.$$

Thus we have

$$\left.\begin{aligned} x+y+z &= \tfrac{7}{2} & \dots && \dots && (1)\\ yz+zx+xy &= \tfrac{7}{2} & \dots && \dots && (2)\\ xyz &= 1 & \dots && \dots && (3) \end{aligned}\right\}$$

From (2), we have

$$yz + x(y+z) = \tfrac{7}{2};$$

Hence, from (3) and (1),

$$\frac{1}{x} + x\left(\tfrac{7}{2} - x\right) = \tfrac{7}{2};$$

$$\begin{aligned} \therefore \quad 2x^3 - 7x^2 + 7x - 2 &= 0,\\ \text{or,} \quad 2(x^3-1) - 7x(x-1) &= 0,\\ \text{or,} \quad (x-1)(2x^2-5x+2) &= 0,\\ \text{or,} \quad (x-1)(x-2)(2x-1) &= 0;\\ \therefore \quad x &= 1, \text{ or } 2, \text{ or } \tfrac{1}{2}. \end{aligned}$$

(i) Taking $x = 1$, we have from (1)

and from (3), $\left.\begin{aligned} y+z &= \tfrac{5}{2}\\ yz &= 1 \end{aligned}\right\}$

whence $\left.\begin{aligned} y &= 2\\ z &= \tfrac{1}{2} \end{aligned}\right\}$ or, $\left.\begin{aligned} y &= \tfrac{1}{2}\\ z &= 2 \end{aligned}\right\}$

(ii) Taking $x = 2$, we have from (1)

and from (3), $\left.\begin{aligned} y+z &= \tfrac{3}{2}\\ yz &= \tfrac{1}{2} \end{aligned}\right\}$

whence $\left.\begin{aligned} y &= 1\\ z &= \tfrac{1}{2} \end{aligned}\right\}$ or, $\left.\begin{aligned} y &= \tfrac{1}{2}\\ z &= 1 \end{aligned}\right\}$

(iii) Taking $x = \tfrac{1}{2}$, we have from (1)

and from (3), $\left.\begin{aligned} y+z &= 3\\ yz &= 2 \end{aligned}\right\}$

whence $\left.\begin{aligned} y &= 2\\ z &= 1 \end{aligned}\right\}$ or, $\left.\begin{aligned} y &= 1\\ z &= 2 \end{aligned}\right\}$

Thus we have six solutions.

Example 15. Solve
$$\left.\begin{aligned} x^2 - yz &= a^2 \quad \ldots \quad (1)\\ y^2 - zx &= b^2 \quad \ldots \quad (2)\\ z^2 - xy &= c^2 \quad \ldots \quad (3)\end{aligned}\right\}$$

Multiplying (1) by y, (2) by z and (3) by x, we have
$$\begin{aligned} a^2y &= x^2y - y^2z,\\ b^2z &= y^2z - z^2x,\\ c^2x &= z^2x - x^2y.\end{aligned}$$

Hence, by addition,
$$c^2x + a^2y + b^2z = 0 \quad \ldots \quad \ldots \quad (4)$$

Again, multiplying (1) by z, (2) by x and (3) by y, we have
$$\begin{aligned} a^2z &= x^2z - z^2y,\\ b^2x &= y^2x - x^2z,\\ c^2y &= z^2y - y^2x.\end{aligned}$$

Hence, by addition,
$$b^2x + c^2y + a^2z = 0 \quad \ldots \quad \ldots \quad (5)$$

Now from (4) and (5), by cross multiplication, we have
$$\frac{x}{a^4 - b^2c^2} = \frac{y}{b^4 - c^2a^2} = \frac{z}{c^4 - a^2b^2}.$$

Let each of these ratios $= k$;

then
$$\left.\begin{aligned} x &= k(a^4 - b^2c^2)\\ y &= k(b^4 - c^2a^2)\\ z &= k(c^4 - a^2b^2)\end{aligned}\right\}$$

Substituting these values of x, y, z in (1), we have
$$k^2\{(a^4 - b^2c^2)^2 - (b^4 - c^2a^2)(c^4 - a^2b^2)\} = a^2,$$
$$\text{or,}\quad k^2(a^6 + b^6 + c^6 - 3a^2b^2c^2) = 1;$$
$$k = \pm\frac{1}{\sqrt{a^6 + b^6 + c^6 - 3a^2b^2c^2}}.$$

Hence,
$$x = \pm\frac{a^4 - b^2c^2}{\sqrt{a^6 + b^6 + c^6 - 3a^2b^2c^2}};$$

and we have similar expressions for y and z.

Example 16. Solve
$$\left.\begin{aligned} x + y + z &= 9 \;.\;.\;.\;.\; (1)\\ x^2 + y^2 + z^2 &= 29 \;.\;.\;.\;.\; (2)\\ x^3 + y^3 + z^3 &= 99 \;.\;.\;.\;.\; (3)\end{aligned}\right\}$$

Since $2(xy+yz+zx)$

$$= (x+y+z)^2 - (x^2+y^2+z^2),$$

we have from (1) and (2),

$$xy+yz+zx = \tfrac{1}{2}(81-29)$$
$$= 26 \quad . \; . \; . \; . \quad (4)$$

Again, since $x^3+y^3+z^3-3xyz$

$$= (x+y+z)(x^2+y^2+z^2-yz-zx-xy),$$

[a well-known Algebraical identity

we have from (1), (2), (3) and (4),

$$3xyz = 99 - 9 \times (29-26)$$
$$= 72\,;$$
$$\therefore \quad xyz = 24.$$

Thus we have

$$\left.\begin{array}{rcll} x+y+z &=& 9 & \ldots\;\ldots\;\ldots\;(\alpha) \\ xy+yz+zx &=& 26 & \ldots\;\ldots\;\ldots\;(\beta) \\ xyz &=& 24 & \ldots\;\ldots\;\ldots\;(\gamma) \end{array}\right\}$$

From (β) we have

$$yz + x(y+z) = 26\,;$$

hence, by (α) and (γ),

$$\frac{24}{x} + x(9-x) = 26\,;$$

$$\therefore \quad x^3 - 9x^2 + 26x - 24 = 0,$$

or, $(x-2)(x^2-7x+12) = 0,$

or, $(x-2)(x-3)(x-4) = 0\,;$

$$\therefore \quad x = 2, \text{ or } 3, \text{ or } 4.$$

(i) Taking $x = 2$, we have

from (α), and from (γ),

$$\left.\begin{array}{rcl} y+z &=& 7 \\ yz &=& 12 \end{array}\right\}$$

$$\text{whence } \left.\begin{array}{rcl} y &=& 3 \\ z &=& 4 \end{array}\right\} \quad \text{or, } \left.\begin{array}{rcl} y && \\ z &=& 3 \end{array}\right\}$$

(ii) Taking $x = 3$, we have as before

$$y+z = 6, \text{ and } yz = 8\,;$$

$$\text{whence } \left.\begin{array}{rcl} y &=& 2 \\ z &=& 4 \end{array}\right\} \quad \text{or, } \left.\begin{array}{rcl} y &=& 4 \\ z &=& 2 \end{array}\right\}$$

(iii) Taking $x = 4$, we have

$$y + z = 5, \text{ and } yz = 6;$$

$$\text{whence } \left.\begin{aligned} y &= 2 \\ z &= 3 \end{aligned}\right\} \quad \text{or,} \quad \left.\begin{aligned} y &= 3 \\ z &= 2 \end{aligned}\right\}$$

Thus we have six solutions altogether.

Exercise (49).

Solve the following equations :—

1. $\left.\begin{aligned} x + y &= x^2 \\ 3y - x &= y^2 \end{aligned}\right\}$

2. $\left.\begin{aligned} 4(x + y) &= 3xy \\ x + y + x^2 + y^2 &= 26 \end{aligned}\right\}$

(Madras University F. A. Paper, 1887.)

3. $\left.\begin{aligned} \frac{x^2}{y} + \frac{y^2}{x} &= 9 \\ \frac{1}{x} + \frac{1}{y} &= \frac{3}{4} \end{aligned}\right\}$

4. $\left.\begin{aligned} x + \sqrt{x^2 - y^2} &= 8 \\ x - y &= 1 \end{aligned}\right\}$

5. $\left.\begin{aligned} \sqrt{\frac{x}{y}} + \sqrt{\frac{y}{x}} &= \frac{7}{\sqrt{xy}} + 1 \\ \sqrt{x^3 y} + \sqrt{y^3 x} &= 78 \end{aligned}\right\}$

6. $\left.\begin{aligned} 7x^2 - 8xy &= 159 \\ 5x + 2y &= 7 \end{aligned}\right\}$

7. $\left.\begin{aligned} 5xy &= 84 - x^2y^2 \\ x - y &= 6 \end{aligned}\right\}$

8. $\left.\begin{aligned} x + y + xy &= 11 \\ x^2y + xy^2 &= 30 \end{aligned}\right\}$

9. $\left.\begin{aligned} x + 3(x + y)^{\frac{1}{2}} &= 18 - y \\ x^2 - y^2 &= 9 \end{aligned}\right\}$

10. $\left.\begin{aligned} x^3 + y^3 &= 351 \\ xy &= 14 \end{aligned}\right\}$

11. $\left.\begin{aligned} \frac{2x + y}{3x - y} + \frac{x - y}{x + y} &= 2\tfrac{8}{15} \\ 7x + 5y &= 29 \end{aligned}\right\}$ [Reducing the 1st equation we have $3x = 2y$ or, $13x = -19y$.]

12. $\left.\begin{aligned} (x + y)^{\frac{2}{3}} + 2(x - y)^{\frac{2}{3}} &= 3(x^2 - y^2)^{\frac{1}{3}} \\ 3x - 2y &= 13 \end{aligned}\right\}$

[Divide each term of the 1st equation by $(x - y)^{\frac{2}{3}}$ and then put z for $\left(\frac{x + y}{x - y}\right)^{\frac{1}{3}}$; we thus get a quadratic in z.]

13. $$\left.\begin{aligned}\sqrt{x+y}+\sqrt{x-y} &= 4\\ x^2-y^2 &= 9\end{aligned}\right\}$$

14. $$\left.\begin{aligned}x^2+4y^2-15x &= 10(3y-8)\\ xy &= 6\end{aligned}\right\}$$

15. $$\left.\begin{aligned}2x^2-xy+y^2 &= 2y\\ 2x^2+4xy &= 5y\end{aligned}\right\}$$

16. $$\left.\begin{aligned}\frac{x^2}{y^2}+\frac{y}{x}+\frac{x}{y} &= \frac{27}{4}-\frac{y^2}{x^2}\\ x-y &= 2\end{aligned}\right\}$$ (Calcutta University F. A. Paper, 1865).

[The 1st equation may be reduced to a quadratic in $\left(\frac{x}{y}+\frac{y}{x}\right)$]

17. $x+y+3\sqrt{x+y} = x^2+y^2 = 10.$

(Punjab University I. E. Paper, 1887.)

18. $$\left.\begin{aligned}x+\frac{1}{y} &= a\\ y+\frac{1}{x} &= b\end{aligned}\right\}$$ (Calcutta University F. A. Paper, 1884.)

19. $$\left.\begin{aligned}6x+5y &= \frac{6}{x}+\frac{5}{y}+29\tfrac{1}{3}\\ 3x+4y &= \frac{3}{x}+\frac{4}{y}+18\tfrac{2}{3}\end{aligned}\right\}$$ (Madras University F. A. Paper, 1888.)

$\left[\text{We have } x-\frac{1}{x} = y-\frac{1}{y} = 2\tfrac{2}{3}\right]$

20. $$\left.\begin{aligned}\sqrt{x}-\sqrt{y} &= 2\sqrt{xy}\\ x+y &= 20\end{aligned}\right\}$$

21. $$\left.\begin{aligned}x^2+y^2+2(x+y) &= 11\\ 3xy &= 2(x+y)\end{aligned}\right\}$$

22. $x^2+a^2 = y^2+b^2 = (x+y)^2+(a-b)^2.$

[We have $(x+y)^2-x^2 = a^2-(a-b)^2$ and $(x+y)^2-y^2 = b^2-(a-b)^2$ which are Homogeneous; hence find a relation between x and y.]

23. $$\left.\begin{aligned}x(9-xy) &= y(xy-36)\\ xy(3x+12y-xy) &= 108(x+y-3)\end{aligned}\right\}$$ (Madras University F. A. Paper, 1882.)

[From (1), $xy(x+y) = 9(x+4y)$, and from (2), $xy\{xy-3(x+4y)\}$ $= 108\{3-(x+y)\}$. Hence, $x^2y^2\left\{1-\frac{x+y}{3}\right\} = 108\{3-(x+y)\}$; whence, $x+y = 3$, or $x^2y^2 = 3.108 = 9.36 = 18^2$.]

24. $$\left.\begin{aligned}(x^2+y^2)\frac{y}{x} &= 8\tfrac{2}{3}\\ (x^2-y^2)\frac{x}{y} &= 7\tfrac{1}{2}\end{aligned}\right\}$$

[Dividing one by the other and putting $y = mx$ we find one value of $m = \frac{2}{3}$; hence from $x^4 - y^4 = 8\frac{2}{3} \cdot 7\frac{1}{2}$, x is determined.]

25. $$\left.\begin{aligned}x^4 &= mx + ny\\ y^4 &= my + nx\end{aligned}\right\}$$ find only one solution.

26. $$\left.\begin{aligned}\frac{x}{y} - \frac{y}{x} &= \frac{x+y}{x^2+y^2}\\ \frac{x^2}{y^2} - \frac{y^2}{x^2} &= \frac{x-y}{y^2}\end{aligned}\right\}$$

[Dividing the 2nd equation by the 1st,

$$\frac{x}{y} + \frac{y}{x} = \frac{x-y}{x+y} \cdot \frac{x^2+y^2}{y^2}.$$

From this $\frac{x}{y} = 1 \pm \sqrt{2}$.

Now from the 1st equation $\frac{x}{y} - \frac{y}{x} = \dfrac{\frac{x}{y}+1}{y\left(\frac{x^2}{y^2}+1\right)}$, which gives y.]

27. $$\frac{ax}{a+x} + \frac{by}{b+y} = \frac{(a+b)c}{a+b+c};\quad x+y = c.$$

[Substituting $x+y$ for c in the 1st equation and transposing we have

$$x\left(\frac{a}{a+x} - \frac{a+b}{a+b+x+y}\right) = y\left(\frac{a+b}{a+b+x+y} - \frac{b}{b+y}\right),$$

whence $ay - bx = 0$, or $\frac{x}{a} = \frac{y}{b}$.]

28. $$\left.\begin{aligned}\sqrt{(x^2+a^2)(y^2+b^2)} + \sqrt{(x^2+b^2)(y^2+a^2)} &= (a+b)^2\\ x+y &= a+b\end{aligned}\right\}$$

[We have $\sqrt{(x^2+a^2)(y^2+b^2)} + \sqrt{(x^2+b^2)(y^2+a^2)} = (a+b)(x+y)$. Now applying the artifice of Art. 4, Chapter VII, we have

$$2\sqrt{(x^2+a^2)(y^2+b^2)} = (a+b)(x+y) + (a-b)(y-x) = 2(ay+bx),$$

whence by squaring, &c., $xy = ab$.]

29. $$\left.\begin{aligned}x-y-z &= 2\\ x^2+y^2-z^2 &= 22\\ xy &= 5\end{aligned}\right\}$$

[Putting u for $x - y$ we have $u - z = 2$, and $u^2 - z^2 = 12$.]

30. $$\left.\begin{aligned}(y-z)(z+x) &= 22\\ (z+x)(x-y) &= 33\\ (x-y)(y-z) &= 6\end{aligned}\right\}$$

31. $$\left.\begin{aligned}x+y+z &= 3\\ yz+zx+xy &= -1\\ \frac{1}{x}+\frac{1}{y}+\frac{1}{z} &= \frac{1}{3}\end{aligned}\right\}$$

(Calcutta University F. A. Paper, 1878.)

32. $$\left.\begin{aligned} \frac{1}{29}\left(x+\frac{y}{z}\right) = \frac{1}{34}\left(y+\frac{x}{z}\right) &= \frac{1}{6} \\ x+y+z &= 15 \end{aligned}\right\}$$

[We have $x+\frac{y}{z} = \frac{29}{6}$ and $y+\frac{x}{z} = \frac{34}{6}$; add and form a quadratic in z by the help of the 2nd equation.]

33. $$\left.\begin{aligned} xz+y &= 7z \\ yz+x &= 8z \\ x+y+z &= 12 \end{aligned}\right\}$$

34. $$\left.\begin{aligned} xy &= (x+y) \\ xz &= 2(x+z) \\ yz &= 3(y+z) \end{aligned}\right\}.$$

35. $$\left.\begin{aligned} x+y+z &= 13 \\ x^2+y^2+z^2 &= 91 \\ y^2 &= xz \end{aligned}\right\}$$

36. $$\left.\begin{aligned} xyz &= 231 \\ xyw &= 420 \\ xzw &= 660 \\ yzw &= 1540 \end{aligned}\right\}$$

37. $$\left.\begin{aligned} x^2+xy+xz &= 45 \\ y^2+yz+yx &= 75 \\ z^2+zx+zy &= 105 \end{aligned}\right\}$$

$$\left.\begin{aligned} x+y+z &= 13 \\ x^2+y^2+z^2 &= 65 \\ yz &= 10 \end{aligned}\right\}$$ [$(y+z)^2-(y^2+z^2) = 2yz$, hence find a quadratic in x.]

$$\left.\begin{aligned} x+y+z &= 13 \\ x^2+y^2+z^2 &= 61 \\ x(y+z) &= 2yz \end{aligned}\right\}$$ [Subtracting the 2nd equation from the square of the 1st, we get by the help of the 3rd equation, $yz = 18$.]

40. $$\left.\begin{aligned} x^2-y^2-z^2+2yz &= 3 \\ -x^2+y^2-z^2+2xz &= -9 \\ -x^2-y^2+z^2+2xy &= -3 \end{aligned}\right\}$$ (Cal. University F. A. Paper, 1866.)

[Putting u for $x+y-z$, v for $x-y+z$, and w for $y+z-x$, we have $uv = 3$, $uw = -9$ and $vw = -3$. Finding u, v, w it must be observed that $x+y+z = u+v+w$ and is ∴ known.]

41. $$\left.\begin{aligned} x^2+3xy+5xz &= 62 \\ 3y^2+5yz+yx &= 93 \\ 5z^2+zx+3zy &= 124 \end{aligned}\right\}$$ [It can be easily seen that $\frac{x}{2} = \frac{y}{3} = \frac{z}{4}$.]

42. $$\left.\begin{aligned} y+z : z+x : x+y &= a : b : c \\ (y+z)^2+(z+x)^2+(x+y)^2 &= 1. \end{aligned}\right\}$$ (C. U. F. A. P. 1885.)

43. $$\left.\begin{aligned} x^2+xy+y^2 &= 13 \\ y^2+yz+z^2 &= 49 \\ z^2+zx+x^2 &= 31 \end{aligned}\right\}$$

44. $$\left.\begin{aligned} x+y+z &= 12 \\ x^2+y^2+z^2 &= 50 \\ x^3+y^3+z^3 &= 216 \end{aligned}\right\}$$

45. $$\left.\begin{aligned} x^4+2x^3y+x^2y^2+2xy^3+y^4 &= 41 \\ \frac{x}{y}+\frac{y}{x} &= \frac{5}{2} \end{aligned}\right\}$$

(Calcutta University F. A. Paper, 1889.)

[From the 2nd equation we have $\frac{y}{x} = 2$, or $\frac{1}{2}$.]

CHAPTER X.

EQUATIONAL PROBLEMS.

1. What are eggs a dozen when two more in a shilling's worth lowers the price one penny per dozen?

Let x = the number of eggs we get for a shilling.

Then the price of each egg $= \frac{12}{x}$ pence,

and $\therefore$ the price of a dozen $= \frac{144}{x}$ pence .... (1)

If two more were obtained for a shilling, *i.e.*, if $(x+2)$ eggs were worth a shilling, the price of a dozen would, for a similar reason, be $\frac{144}{x+2}$ pence.

But by the condition of the problem, the latter price is one penny less than the former price, hence

$$\begin{aligned} \frac{144}{x+2} &= \frac{144}{x} - 1; \\ \therefore\quad x^2+2x &= 288, \\ \therefore\quad x^2+2x+1 &= 289, \\ \therefore\quad x+1 &= 17. \\ \therefore\quad x &= 16. \end{aligned}$$

Hence, from (1), the price per dozen = 9*d*.

2. Find two numbers, whose difference, multiplied by the difference of their squares $= 160$; and whose sum, multiplied by the sum of their squares gives the number 580.

Let $x+y$ and $x-y$ be the numbers.

Then, by the 1st condition of the problem,

$$2y.(4xy) = 160$$

$$\text{or,} \qquad xy^2 = 20 \quad . \quad . \quad . \quad . \quad . \quad (1)$$

By the 2nd condition of the problem,

$$2x\{2(x^2+y^2)\} = 580$$

$$\text{or,} \qquad x(x^2+y^2) = 145 \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Dividing (1) by (2),

$$\frac{y^2}{x^2+y^2} = \frac{20}{145} = \frac{4}{29};$$

$$\therefore \quad 25y^2 = 4x^2;$$

$$\therefore \quad \frac{x}{5} = \frac{y}{2} = k \text{ (suppose).}$$

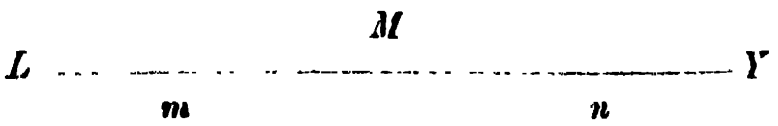
Hence from (1),

$$20k^3 = 20, \quad \therefore \quad k = 1;$$

$$\therefore \quad x = 5, \text{ and } v = 2.$$

Hence the required numbers are 7 and 3.

3. *A* sets off from London to York and *B* at the same time from York to London, and they travel uniformly; *A* reaches York 16 hours and *B* reaches London 36 hours, after they have met on the road. Find in what time each has performed the journey.



Let *L*, *Y* represent London and York respectively, and *M* the place where the travellers meet. Let *m*, *n* be the measures of *LM*, *MY* respectively in miles.

Now, since *A* travels *n* miles (*i e*, from *M* to *Y*) in 16 hours, he travels 1 mile in $\frac{16}{n}$ hours and $\therefore$ *m* miles in $\frac{16}{n}.m$ hours; hence the time in which *A* travelled from *L* to *M*

$$= \frac{16}{}.m \text{ hours.}$$

Similarly the time in which B travelled from Y to M

$$= \frac{36}{m}.n \text{ hours.}$$

Now, since they started at the same instant, the time in which A travelled from L to M is evidently equal to the time in which B travelled from Y to M;

$$\therefore \quad \frac{16}{n}m = \frac{36}{m}.n,$$

$$\text{whence} \quad \frac{m}{n} = \frac{3}{2}.$$

Hence the time in which A performed the journey

$$= \left(\frac{16}{n}.m + 16\right) \text{ hours} = 40 \text{ hours};$$

and the time in which B performed the journey

$$= \left(\frac{36}{m}.n + 36\right) \text{ hours} = 60 \text{ hours.}$$

4. A fraudulent tradesman contrives to employ his *false* balance both in buying and selling a certain article, thereby gaining 11 per cent. more on his outlay than he would gain, were the balance *true*. If however the scale pans, in which the article is weighed when bought and sold respectively, were interchanged, he would neither gain nor lose by the transaction. Determine the legitimate gain per cent on the article.

[In a *false* balance if any weight be placed on one of the scale pans the weight to be put on the other pan in order to make the beam horizontal will be *different*. For instance, if in buying rice a five-seer counterpoise be put on the pan, the quantity of rice put on the other will be either more or less than 5 seers. Suppose when the five seer counterpoise is put on the scale pan A, we are required to put on the pan B, a quantity of rice whose real weight is greater than 5 seers; but whatever may be its real weight, as its weight now is supposed to be equal to the weight of the counterpoise, we take it to be 5 seers. Thus we take for 5 seers what is really more than 5 seers. Hence if the merchant contrives to put the counterpoise on A and the article bought on B, he will evidently take away more of the article than he is sopposed to do; let the supposed weight of the article so bought be w lbs.; if then W lbs. be the *real* weight of the article, w is less than W. Again in selling the article if he puts the counterpoises on B and the article on A and if W' be the weight of the counterpoises, then W' is greater than W. By this contrivance then the merchant buys W lbs. of the article at the price of w lbs. and sells away these W lbs. again at the price of W' lbs. Hence in such a transaction the merchant's gain is two-fold, he buys more of the article than he pays for and the whole quantity thus bought he sells away at the price of a still greater quantity.]

Let w and W' be the *apparent* weights of the article when bought and sold respectively.

Then evidently w is less, and W' greater, than the true weight.

Let p = prime cost of a unit of weight,
x = the legitimate gain per cent.

Then the selling price of a unit of weight

$$= p + x \textit{ hundredths} \text{ of } p = p\left(1+\frac{x}{100}\right).$$

Hence the price paid by the merchant in buying the article, *i. e.*, his outlay $= w.p$, and the price realised by selling it $= W'.p\left(1+\frac{x}{100}\right)$.

$\therefore$ by the condition of the problem,

$$W'.p\left(1+\frac{x}{100}\right) = w.p + (x+11) \textit{ hundredths} \text{ of } w.p$$
$$= w.p\left(1+\frac{x+11}{100}\right) \quad \ldots \quad \ldots \quad (1)$$

If the scale pans were interchanged, the cost of buying the article would be $W'.p$, and the price realised by sale, $w.p\left(1+\frac{x}{100}\right)$; hence by the 2nd condition of the problem,

$$w.p\left(1+\frac{x}{100}\right) = W'.p \quad \ldots \quad \ldots \quad (2)$$

From (1) and (2),

$$\frac{1+\frac{x+11}{100}}{1+\frac{x}{100}} = 1+\frac{x}{100}$$

$$\text{or, } \left(1+\frac{x}{100}\right)^2 = 1+\frac{x+11}{100},$$

$$\text{or, } \left(\frac{x}{100}\right)^2 + \frac{x}{100} + \frac{1}{4} = \frac{11}{100} + \frac{1}{4} = \frac{36}{100},$$

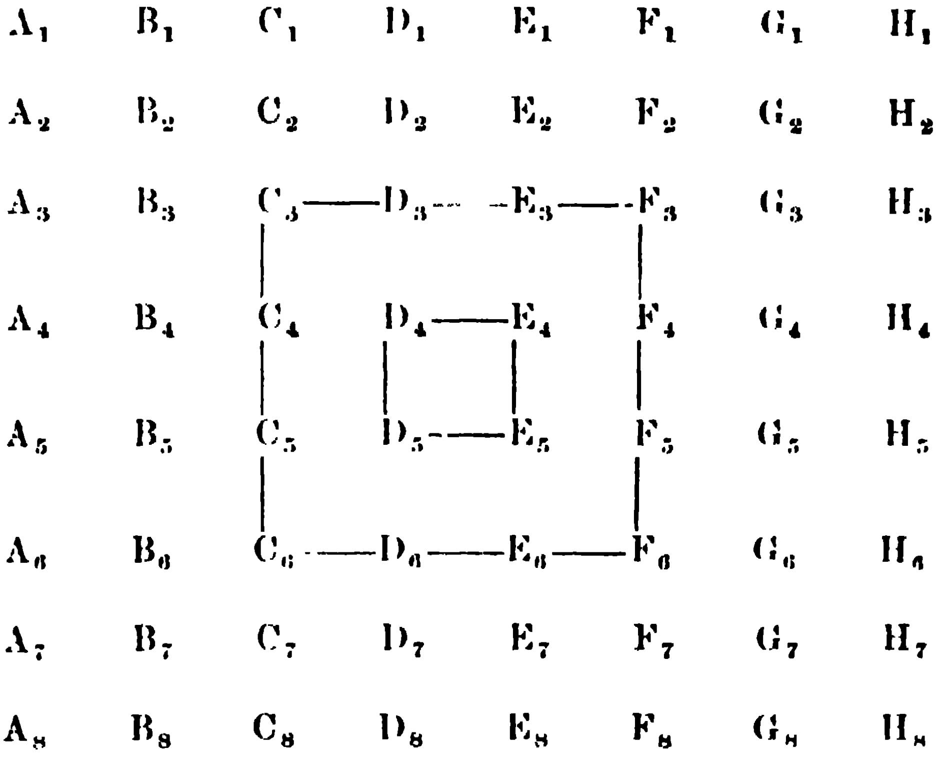
$$\therefore \frac{x}{100} = \frac{6}{10} - \frac{1}{2} = \frac{1}{10},$$

$$\therefore x = 10.$$

i.e., the legitimate gain is 10 per cent.

5. A body of men were formed into a hollow square, three deep, when it was observed that with the addition of 25 to their number, a solid square might be formed, of which the number of men in each side would be greater by 22 than the square root of the number of men in each side of the hollow square. Required the number of men in the hollow square.

[A number of men are said to be arranged in a *solid* square when they are arranged in parallel rows and the number of rows is equal to the number of men in each row. The following diagram, in which A_1, B_1, &c., represent men, will clearly illustrate the matter.


A_1 B_1 C_1 D_1 E_1 F_1 G_1 H_1
A_2 B_2 C_2 D_2 E_2 F_2 G_2 H_2
A_3 B_3 C_3 D_3 E_3 F_3 G_3 H_3
A_4 B_4 C_4 D_4 E_4 F_4 G_4 H_4
A_5 B_5 C_5 D_5 E_5 F_5 G_5 H_5
A_6 B_6 C_6 D_6 E_6 F_6 G_6 H_6
A_7 B_7 C_7 D_7 E_7 F_7 G_7 H_7
A_8 B_8 C_8 D_8 E_8 F_8 G_8 H_8

The above diagram represents an arrangement in which there are 8 rows, each containing 8 men. This is a *solid* square. If the square C_3 F_3 F_6 C_6 be removed from inside, the remainder will be a *hollow* square *two deep* having 8 men in each side; if, however, the square D_4 E_4 E_5 D_5 be removed the remainder will be a *hollow* square *three deep*.

Hence, the number of men in a *hollow* square *two deep* having x men in each side $= x^2-(x-4)^2$; in one *three deep* $= x^2-(x-6)^2$; and so on; thus the number of men in a hollow square n deep having x men in each side $=x^2-(x-2n)^2$.]

Let x = the number of men in a side of the hollow square; then the whole number of men

$$= x^2-(x-6)^2 \quad \ldots \quad (1)$$

Hence, by the 2nd condition of the problem,

$$x^2 - (x-6)^2 + 25 = (x^{\frac{1}{2}} + 22)^2$$

or, $$12x - 11 = x + 44x^{\frac{1}{2}} + 484\ ;$$

$$\therefore \quad 11x - 44x^{\frac{1}{2}} = 495,$$

or, $$x - 4x^{\frac{1}{2}} = 45\ ;$$

$$\therefore \quad x - 4x^{\frac{1}{2}} + 4 = 49,$$

$$\therefore \quad x^{\frac{1}{2}} - 2 = 7\ ;$$

$$\text{whence } x = 81.$$

Hence, from (1), the whole number of men

$$= 81^2 - 75^2$$

$$= 156 \times 6 = 936.$$

6. *K* engages to play a game of chess with *B* on the following conditions :– that *B* should name a certain number and put into *K*'s possession twenty four rupees together with as many rupees as equal the square of this number and that at the conclusion of the game *K* should return to *B* only a number of rupees equal to eight times the number named. What number could *B* name with the greatest advantage possible to himself ?

Let x = the number which *B* should name ; then he has to deposit with *K*, $(24 + x^2)$ rupees and get back at the end of the game only $8x$ rupees ;

hence *B* has altogether to lose $(x^2 + 24 - 8x)$ rupees ;

∴ x must be such that this loss may be as small as possible.

Now, since $x^2 - 8x + 24 = (x-4)^2 + 8$, which is always greater than 8 except when $x = 4$, the loss will for all values of x be greater than Rs. 8 except when x has this value.

Hence in order that the loss may be a minimum *B* should name the number 4.

7. With the object of examining a student of the 1st year as regards his progress in Algebra, I undertake to engage in a certain contract with him, which is as follows :—he is to give me a certain number of books, each worth as many rupees as the number of books, and to get from me in return six times as many rupees as any of those books is worth and also 21 rupees more. How many books should he bring me, with the greatest possible advantage to himself?

Let x = the number of books that the student brings me;

then, since the price of each book is x rupees, evidently I get x^2 rupees from him; and in return I give him $(6x+21)$ rupees.

Hence his gain (or loss as the case may be)

$$= (21 + 6x - x^2) \text{ rupees}$$

$$\begin{aligned} \text{Now,} \quad 21 + 6x - x^2 &= 21 - (x^2 - 6x) \\ &= 30 - (x^2 - 6x + 9) \\ &= 30 - (x - 3)^2. \end{aligned}$$

Evidently therefore the student is a loser if $(x - 3)$ be greater than 5, *i.e.*, if x be greater than 8, and he is a gainer if x be 8 or less than 8.

But not only should the student be a gainer but his gain must be the greatest possible, which evidently is the case when $(x - 3)^2$ is the least possible, *i.e.*, when $x = 3$.

Hence the student should bring me only three books.

8. Ram, Lakshman and Bharat went to visit a Rishi and brought their wives with them. The Rishi knew the wives' names to be Urmila, Mandavi and Sita, but forgot which was the wife of each hero. They told the Rishi that they had given presents to Pandits, and that each of the six had rewarded as many Pandits, as he or she had given gold mudras to each Pandit. Ram had rewarded 23 more Pandits than Urmila, and Lakshman had rewarded 11 Pandits more than Mandavi, likewise each hero had given away 63 gold mudras more than his wife. The Rishi having thought on what they said, dismissed them with his blessing, naming correctly the wife of each hero. From the conditions given, do you also find out the names of the wives.

Let x = the number of Pandits rewarded by any hero,

and y = the number of Pandits rewarded by his wife;

then the number of gold mudras given away by the hero = x^2,

and the number of gold mudras given away by his wife = y^2.

Hence, by the last condition of the problem, we have

$$x^2 - y^2 = 63,$$

$$\text{or,} \quad (x + y)(x - y) = 63$$

But $63 = 63 \times 1$, or 21×3, or 9×7;

hence, since $x+y$ and $x-y$ are positive integers, and $x+y$ is necessarily greater than $x-y$, we get the following three pairs of values for $x+y$ and $x-y$ and *no other* :—

$$(1)\ \left.\begin{array}{l} x+y = 63 \\ x-y = 1 \end{array}\right\}, \quad (2)\ \left.\begin{array}{l} x+y = 21 \\ x-y = 3 \end{array}\right\}, \quad (3)\ \left.\begin{array}{l} x+y = 9 \\ x-y = 7 \end{array}\right\}.$$

Hence we have the following three pairs of values for x and y :—

$$(1)\ \left.\begin{array}{l} x = 32 \\ y = 31 \end{array}\right\}, \quad (2)\ \left.\begin{array}{l} x = 12 \\ y = 9 \end{array}\right\}, \quad (3)\ \left.\begin{array}{l} x = 8 \\ y = 1 \end{array}\right\} \ldots\ldots (A)$$

i.e., the wife of the hero who rewarded 32 Pandits, rewarded 31 Pandits;

the wife of the hero who rewarded 12 Pandits, rewarded 9 Pandits; ... ... (α)

and the wife of the hero who rewarded 8 Pandits, rewarded only one Pandit. ... ... (β)

Now let us find out the names of the wives from the other conditions of the problem.

The number of Pandits rewarded by Ram may be 32, 12 or 8; but since he is known to have rewarded 23 *more* Pandits than somebody else, the number of Pandits rewarded by him *must be* 32.

The number of Pandits rewarded by Lakshman may then be either 12 or 8, but as he is known to have rewarded 11 *more* Pandits than somebody else, the number of Pandits rewarded by him *must be* 12. ... ... (a)

Hence the number of Pandits rewarded by Bharat *must be* 8. ... ... (b)

Again, since the number of Pandits rewarded by Urmila is 23 less than the number rewarded by Ram, it *must be* 9; hence by (α) and (a) Urmila is the wife of Lakshman;

also, since the number of Pandits rewarded by Mandavi is 11 less than the number rewarded by Lakshman, it *must be* 1; and ∴ by (β) and (b) Mandavi is the wife of Bharat;

evidently ∴ Sita is the wife of Ram.

Thus we have—

$$\left.\begin{array}{l} \textbf{Ram} \\ \textbf{Sita} \end{array}\right\} \qquad \left.\begin{array}{l} \textbf{Lakshman} \\ \textbf{Urmila} \end{array}\right\} \qquad \left.\begin{array}{l} \textbf{Bharat} \\ \textbf{Mandavi} \end{array}\right\}.$$

Exercise (50).

1. A person bought a certain number of oxen for £80; if he had bought 4 more for the same sum, each ox would have cost £1 less; find the number of oxen and the price of each.

2. A gentleman sends a lad into the market to buy a shilling's worth of oranges. The lad having eaten a couple, the gentleman pays at the rate of a penny for fifteen more than the market price; how many did the gentleman get for his shilling?

3. The plate of a looking glass is 18 inches by 12, and is to be framed with a frame of equal width, whose area is to be equal to that of the glass. Required the width of the frame.

4. *A* and *B* lay out some money on speculation. *A* disposes of his bargain for £11, and gains as much *per cent.* as *B* lays out; *B*'s gain is £36, and it appears that *A* gains four times as much *per cent.* as B. Required the capital of each.

5. A boat's crew row 3½ miles down a river and back again in 1 hour and 40 minutes. Supposing the river to have a current of 2 miles per hour, find the rate at which the crew would row in still water.

6. What two numbers are those whose sum multiplied by the greater is 204; and whose difference multiplied by the less is 35?

7. What two numbers are those whose sum added to the sum of their squares is 42 and whose product is 15?

8. *A* and *B* distribute £60 each among a certain number of persons. *A* relieves 40 persons more than *B* does, and *B* gives to each 5*s*. more than *A*. How many persons did *A* and *B* respectively relieve?

9. The product of two numbers added to their sum is 23; and five times their sum taken from the sum of their squares leaves 8; required the numbers.

10. A horse dealer buys a horse, and pays a certain sum for it; he afterwards sells it again for Rs. 171, and gains exactly as much per cent. as the horse had cost him. How much did he pay for the horse?

11. The small wheel of a bicycle makes 135 revolutions more than the large wheel in a distance of 260 yards; if the circumference of each were one foot more, the small wheel would make 27 revolutions more than the large wheel in a distance of 70 yards ; find the circumference of each wheel.

12. By lowering the price of apples and selling them one penny a dozen cheaper, an apple-woman finds that she can sell 60 more than she used to do for 5*s*. At what price per dozen did she sell them at first ?

13. There is a number between 10 and 100 ; when multiplied by the digit on the left the product is 280 ; if the sum of the digits be multipled by the same digit the product is 55 ; required the number.

14. *A* and *B* are two stations 300 miles apart. Two trains start simultaneously from *A* and *B*, each to the opposite station. The train from *A* reaches *B* nine hours, the train from *B* reaches *A* four hours, after they meet : find the rate at which each train travels.

15. By selling a horse for £24, I lose as much per cent. as it cost me. What was the prime cost of it ?

16. Find three numbers, such that if the first be multiplied by the sum of the second and third, the second by the sum of the first and the third and the third by the sum of the first and the second, the products shall be 408, 480 and 504.

17. There are two square buildings that are paved with stones, a foot square each. The side of one building exceeds that of the other by 12 feet, and both their pavements taken together contain 2120 stones. What are the lengths of them separately ?

18. There are three numbers, the difference of whose differences is 5 ; their sum is 44, and continued product 1950 ; find the numbers.

19. A train *A* starts to go from *P* to *Q*, two stations 240 miles apart, and travels uniformly. An hour later, another train *B* starts from *P*, and after travelling for 2 hours, comes to a point that *A* had passed 45 minutes previously. The pace of *B* s now increased by 5 miles an hour, and it overtakes *A* just on entering *Q*. Find the rates at which they started.

20. A square court-yard has a rectangular gravel walk round it. The side of the court wants 2 yards of being 6 times the breadth of the gravel walk ; and the number of square-yards in the walk exceeds the number of yards in the periphery of the court by 92. Required the area of the court.

21. Divide the number 26 into three such parts that their squares may have equal differences, and that the sum of those squares may be 300.

22. The number of soldiers present at a review is such that they could all be formed into a solid square and also could be formed into four hollow squares each 4 deep and each containing 24 more men in the front rank than when formed into a solid square : find the whole number.

23. A and B run a race round a two-mile course. In the first heat B reaches the winning post 2 minutes before A. In the second heat A increases his speed 2 miles an hour, and B diminishes his by the same quantity ; and A then reaches the winning post 2 minutes before B. Find at what rate each ran in the first heat.

24. From a vessel of wine containing a gallons, b gallons are drawn off, and the vessel is filled up with water. Find the quantity of wine remaining in the vessel when this has been repeated 4 times.

25. A wall was built round a rectangular court to a certain height. Now the length of one side of the court was two yards less, whilst three times the length of the other was 25 yards greater, than 8 times the height of the wall ; and the number of square yards in the court was greater than the number in the wall by 178. Required the dimensions of the court, and the height of the wall.

26. A person bought a number of £20 railway shares when they were at a certain rate per cent. discount for £1,500 ; and afterwards when they were at the same rate per cent. premium sold them all but 60 for £1,000. How many did he buy and what did he give for each of them ?

27. The sum of 4 numbers is 44 ; the sum of the product of the 1st and 2nd, and 3rd and 4th is 250 ; of the 1st and 3rd, and 2nd and 4th is 234 ; and of the 1st and 4th, and 2nd and 3rd is 225. Find them.

28. To complete a certain work A requires m times as long a time as B and C together ; B requires n times as long as A and C together ; and C requires p times as long as A and B together. Compare the times in which each would do it and prove that

$$\frac{1}{m+1}+\frac{1}{n+1}+\frac{1}{p+1}=1.$$

29. In a certain village there lived in the year 1872 a number of families each consisting of as many members as there were families. Ten years afterwards it was found that during this interval there were 670 births in the village and that on the average 50 lives were lost per family. Prove that the number of persons, living in the village at the time of this calculation, could not be less than 45, and if this number be actually 45, find out the number of souls that lived in the village in the year 1872.

30. Suppose you agree to give me out of your landed property a square plot of ground and receive in exchange a circular plot of land whose area is 76 square feet and also a rectangular plot, one of whose sides is 36 feet and the other is equal to a side of the piece of land you give me. What must be the area of the plot you give me, so that you can profit most by the exchange ?

CHAPTER XI.

IMAGINARY QUANTITIES.

1. Definition and Conventions. Just as $\sqrt{a}$ is regarded as a symbol such that $\sqrt{a} \times \sqrt{a} = a$, so $\sqrt{-a}$ is defined to be such that $\sqrt{-a} \times \sqrt{-a} = -a$.

The symbol $\sqrt{-a}$ is always regarded as equivalent to $\sqrt{a} \times \sqrt{-1}$: thus we have to deal with only one imaginary expression, namely, $\sqrt{-1}$, in all investigations.

$$\text{Hence } \sqrt{-a} \times \sqrt{-b} = \sqrt{a} \times \sqrt{-1} \times \sqrt{b} \times \sqrt{-1}$$
$$= \sqrt{ab} \times (\sqrt{-1})^2 = -\sqrt{ab}.$$

The imaginary expression $a\sqrt{-1}$ is considered to vanish when $a = 0$.

Any algebraical expression of the form $a + b\sqrt{-1}$ where a and b are real quantities is regarded as an imaginary expression ; it is said to consist of a real part a and an imaginary part $b\sqrt{-1}$.

N. B. The letter i is generally used to denote the symbol $\sqrt{-1}$.

2. Powers of *i*.

$$\left.\begin{array}{l}(i)^1 = i. \\ (i)^2 = -1.\end{array}\right\}, \quad \left.\begin{array}{l}(i)^3 = (i)^2.(i) = -i. \\ (i)^4 = (i)^2.(i)^2 = 1.\end{array}\right\}.$$

By the help of these 4 results we can very easily determine any other power of the symbol.

For instance,

$$\begin{aligned}(i)^{45} &= \{(i)^4\}^{11} \times i = i; \\ (i)^{51} &= \{(i)^4\}^{12}.(i)^3 = (i)^3 = -i: \\ (i)^{66} &= \{(i)^4\}^{16}.(i)^2 = (i)^2 = -1: \\ (i)^{72} &= \{(i)^4\}^{18} = 1.\end{aligned}$$

Exercise (51).

Simplify the following expressions :—

1. i^{25}. 2. i^{38}. 3. i^{77}. 4. i^{88}. 5 i^{58}
6. i^{15}. 7. i^{105}. 8. i^{203}. 9. i^{27}. 10. i^{34}

3. If $a+bi = 0$, then $a = 0$ and $b = 0$.

For if $a+bi = 0$,

then $bi = -a$,

$\therefore -b^2 = a^2$,

$\therefore a^2+b^2 = 0$.

Now since a^2 and b^2 are neither of them negative, their sum cannot be zero unless each of them is separately zero ; thus we have $a = 0$, and $b = 0$.

4. If $a+bi = c+di$, then $a = c$ and $b = d$.

Since $a+bi = c+di$,

$\therefore$ by transposition $(a-c)+(b-d)i = 0$.

Hence, by the last article, $a-c = 0$ and $b-d = 0$;

that is $a = c$, and $b = d$.

Thus *in order that two imaginary expressions may be equal it is necessary and sufficient that the real parts should be equal, and the imaginary parts should be equal.*

5. Conjugate imaginary expressions.—

Two imaginary expressions are said to be *conjugate* when they differ only in the sign of the coefficient of i. Thus $a+bi$ and $a-bi$ are conjugte.

Now, since $(a+bi)+(a-bi) = 2a$,

and $(a+bi)(a-bi)$

$$= a^2-b^2i^2 = a^2+b^2,$$

we conclude that *the sum and the product of two conjugate imaginary expressions are both real.*

6. Definition. The positive value of the square root of a^2+b^2 is called the *modulus* of each of the expressions $a+bi$ and $a-bi$.

(i) To prove that *the modulus of the product of two imaginary expressions is equal to the product of their moduli.*

Let $a+bi$ and $c+di$ be any two imginary expressions.

Then their product $= (ac-bd)+(ad+bc)i$,

$$\begin{aligned}\text{whose modulus} &= \sqrt{(ac-bd)^2+(ad+bc)^2}\\ &= \sqrt{a^2c^2+b^2d^2+a^2d^2+b^2c^2}\\ &= \sqrt{(a^2+b^2)(c^2+d^2)}\\ &= \sqrt{a^2+b^2}\times\sqrt{c^2+d^2},\end{aligned}$$

which proves the proposition.

(ii) To prove that *the modulus of the quotient of two imaginary expressions is the quotient of their moduli.*

The quotient obtained by dividing the first of the above expressions by the second

$$\begin{aligned}&= \frac{a+bi}{c+di}\\ &= \frac{(a+bi)(c-di)}{(c+di)(c-di)}\\ &= \frac{(ac+bd)+(bc-ad)i}{c^2+d^2}\\ &= \frac{ac+bd}{c^2+d^2}+\frac{bc-ad}{c^2+d^2}i,\end{aligned}$$

$$\text{whose modulus} = \sqrt{\left(\frac{ac+bd}{c^2+d^2}\right)^2+\left(\frac{bc-ad}{c^2+d^2}\right)}$$

$$= \frac{\sqrt{a^2c^2+b^2d^2+b^2c^2+a^2d^2}}{c^2+d^2}$$

$$= \frac{\sqrt{(a^2+b^2)(c^2+d^2)}}{c^2+d^2}$$

$$= \frac{\sqrt{a^2+b^2}}{\sqrt{c^2+d^2}},$$ which proves the proposition.

7. To find the square root of $a+bi$.

Assume $\sqrt{a+bi} = x+yi$, where x and y are real quantities.

Then $a+bi = x^2-y^2+2xyi$;

$\therefore$ by equating real and imaginary parts,

$$\left.\begin{aligned} x^2-y^2 &= a \quad \ldots \quad \ldots \quad \ldots \quad (1)\\ \text{and} \qquad 2xy &= b \quad \ldots \quad \ldots \quad \ldots \quad (2)\end{aligned}\right\}.$$

Hence,

$$\begin{aligned}(x^2+y^2)^2 &= (x^2-y^2)^2+4x^2y^2\\ &= a^2+b^2,\\ \therefore \quad x^2+y^2 &= \sqrt{a^2+b^2}. \qquad \ldots \quad (3)\end{aligned}$$

From (1) and (3), we get

$$x^2 = \frac{\sqrt{a^2+b^2}+a}{2}, \quad y^2 = \frac{\sqrt{a^2+b^2}-a}{2};$$

$$\therefore \quad x = \pm\left\{\frac{\sqrt{a^2+b^2}+a}{2}\right\}^{\frac{1}{2}}, \quad y = \pm\left\{\frac{\sqrt{a^2+b^2}-a}{2}\right\}^{\frac{1}{2}}.$$

N. B. Since x and y are real quantities, x^2+y^2 is positive and $\therefore$ in (3) the positive sign must be put before the quantity $\sqrt{a^2+b^2}$.

Also from (2) we see that the product xy must have the same sign as b; hence x and y must have the *same* sign if b be *positive*, and *different* signs if b be *negative*.

Cor. Supposing $a = 0$ and $b = \pm 1$, we have

$$\sqrt{+i} = \pm\frac{1+i}{\sqrt{2}} \text{ and } \sqrt{-i} = \pm\frac{1-i}{\sqrt{2}}.$$

8. The cube roots of unity.

Let $x = \sqrt[3]{1}$, then $x^3 = 1$, or $x^3-1 = 0$,

i.e., $(x-1)(x^2+x+1) = 0$,

$$\therefore \text{ either } x - 1 = 0 \quad \ldots \quad (1)$$
$$\text{or, } x^2 + x + 1 = 0 \quad \ldots \quad (2)$$
$$\text{From (1), } x = 1,$$
$$\text{,, (2), } x = \frac{-1 \pm \sqrt{-3}}{2}$$

Thus the cube roots of 1 are

$$1, \quad \frac{-1 + \sqrt{-3}}{2}, \text{ and } \frac{-1 - \sqrt{-3}}{2}.$$

Now with regard to the latter two roots, both of which are imaginary, we observe that

$$\left(\frac{-1 + \sqrt{-3}}{2}\right)^2 = \frac{1 + (-3) - 2\sqrt{-3}}{4}$$
$$= \frac{-2 - 2\sqrt{-3}}{4}$$
$$= \frac{-1 - \sqrt{-3}}{2},$$

$$\text{and } \left(\frac{-1 - \sqrt{-3}}{2}\right)^2 = \frac{1 + (-3) + 2\sqrt{-3}}{4}$$
$$= \frac{-2 + 2\sqrt{-3}}{4}$$
$$= \frac{-1 + \sqrt{-3}}{2}$$

That is, if ω denote *either* of the imaginary roots, the other is ω^2.

Hence it is usual to denote the three cube roots of unity by 1, ω, ω^2.

Cor. 1. Since ω satisfies the equation $x^2 + x + 1 = 0$, we have $\omega^2 + \omega + 1 = 0$; that is, the sum of the three cube roots of unity is zero.

Cor. 2. Any positive integral power of ω is equal to 1, or ω, or ω^2; thus $\omega^7 = (\omega^3)^2.\omega = 1^2.\omega = \omega$.

$$\omega^{29} = (\omega^3)^9.\omega^2 = \omega^2 \text{ ; and so on.}$$

9. A few miscellaneous examples.

Example 1. Show that $(\omega m + \omega^2 n)(\omega^2 m + \omega n) = m^2 - mn + n^2$.

$$\begin{aligned}(\omega m + \omega^2 n)(\omega^2 m + \omega n) &= \omega^3 m^2 + mn(\omega^4 + \omega^2) + \omega^3 n^2\\ &= m^2 - mn + n^2,\end{aligned}$$

$$(\text{since } \omega^4 + \omega^2 = \omega^3 . \omega + \omega^2 = \omega + \omega^2 = -1).$$

Example. 2. Show that $(x + \omega y + \omega^2 z)(x + \omega^2 y + \omega z)$

$$= x^2 + y^2 + z^2 - yz - zx - xy.$$

$$\begin{aligned}(x + \omega y + \omega^2 z)(x + \omega^2 y + \omega z) &= x^2 + \omega^3 y^2 + \omega^3 z^2 + xy(\omega + \omega^2)\\ &\quad + yz(\omega^2 + \omega^4) + zx(\omega^2 + \omega)\\ &= x^2 + y^2 + z^2 - xy - yz - zx,\end{aligned}$$

$$(\text{since } \omega + \omega^2 = -1 \text{ and } \omega^2 + \omega^4 = \omega^2 + \omega = -1).$$

Example 3. Show that

$$\frac{4-5i}{2+3i} \times \frac{3+2i}{7+i} = \frac{81-583i}{650}.$$

$$\begin{aligned}\text{The given expression} &= \frac{12 - 10i^2 - 7i}{14 + 3i^2 + 23i}\\ &= \frac{22 - 7i}{11 + 23i}\\ &= \frac{(22-7i)(11-23i)}{(11+23i)(11-23i)}\\ &= \frac{242 + 161i^2 - 583i}{121 - 529i^2}\\ &= \frac{242 - 161 - 583i}{121 + 529}\\ &= \frac{81 - 583i}{650}.\end{aligned}$$

Exercise (52).

1. Multiply $3 + 7\sqrt{-3}$ by $5 - 4\sqrt{-3}$
2. Multiply $3\sqrt{-5}$ by $2\sqrt{-3}$
3. Multiply $3\sqrt{-7} + 5\sqrt{-3}$ by $2\sqrt{-7} - 7\sqrt{-3}$
4. Reduce $\dfrac{(2+i)^3}{3+2i}$ to the form $A + Bi$.

5 Express $\dfrac{a+bi}{c+di}$ in the form $A+Bi$.

6. Multiply $x-\dfrac{1+\sqrt{-3}}{2}$ by $x-\dfrac{1-\sqrt{-3}}{2}$.

7. Show that $\dfrac{3+2i}{2-5i}+\dfrac{3-2i}{2+5i}=-\dfrac{8}{29}$.

Extract the square roots of :—

8. $1+4\sqrt{-3}$. **9.** $7-30\sqrt{-2}$. **10.** $4\sqrt{-5}-1$.

11. $-3-4i$. **12.** $2i$.

13. Find the square of

$$\sqrt{9+10i}+\sqrt{9-40i}.$$

14. Find the value of

$$(4+3\sqrt{-20})^{\frac{1}{2}}+(4-3\sqrt{-20})^{\frac{1}{2}}.$$

15. Extract the square root of

$$x+i\sqrt{x^4+x^2+1}.$$

(Madras University F. A. Paper, 1888.)

16. Find the value of $\sqrt[3]{-1}$.

17. Prove that $\left(\dfrac{-1+\sqrt{-3}}{2}\right)^{2n}+\left(\dfrac{-1-\sqrt{-3}}{2}\right)^{2n}=-1$.

18. Prove that $\left(\dfrac{-1+\sqrt{-3}}{2}\right)^{21}+\left(\dfrac{-1-\sqrt{-3}}{2}\right)^{21}=2$.

19. Determine the simplest value of

$$\left(1-m\right)^2.\left(m-m^2\right)^2.\left(1-m^2\right)^2,\text{ when } m=\frac{-1+\sqrt{-3}}{2}.$$

If 1, ω, ω^2 are the three cube roots of unity, show that—

20. $(1+\omega)^3-(1+\omega^2)^3=0$.

21. $\omega^4+2\omega^6+\omega^8=1$.

22. $(1-\omega+\omega^2)^3=(1+\omega-\omega^2)^3=-8$.

23. $x^3+y^3=(x+y)(\omega x+\omega^2y)(\omega^2x+\omega y)$.

24. $x^3-y^3=(x-y)(\omega x-\omega^2y)(\omega^2x-\omega y)$.

25. $(a+b+c)(a+b\omega+c\omega^2)(a+b\omega^2+c\omega)=a^3+b^3+c^3-3abc$.

26. $(x+y)^2+(x\omega+y\omega^2)^2+(x\omega^2+y\omega)^2=6xy$.

27. Prove that $(x^2+a^2)^3 = (x^3-3xa^2)^2+(3x^2a-a^3)^2$.

28 Prove that $(x^2+a^2)^4 = (x^4-6x^2a^2+a^4)^2$
$+(4x^3a-4xa^3)^2$.

29. Prove that $(x^2+a^2)^5 = (x^5-10x^3a^2+5xa^4)^2$
$+(5x^4a-10x^2a^3+a^5)^2$.

30. Prove that $(x^2+a^2)^7 = (x^7-21x^5a^2+35x^3a^4-7xa^6)^2$
$+(7x^6a-35x^4a^3+21x^2a^5-a^7)^2$.

31. If $X = ax+cy+bz$, $Y = cx+by+az$,
$Z = bx+ay+cz$,

show that $X^2+Y^2+Z^2-YZ-ZX-XY$
$= (a^2+b^2+c^2-bc-ca-ab)(x^2+y^2+z^2-yz-zx-xy)$.

32. If $X = ax+cy+bz$, $Y = cx+by+az$,
$Z = bx+ay+cz$,

show that $X^3+Y^3+Z^3-3XYZ$
$= (a^3+b^3+c^3-3abc)(x^3+y^3+z^3-3xyz)$.

CHAPTER XII.

ARITHMETICAL PROGRESSION.

1. Definition.—Quantities are said to be in Arithmetical Progression when they increase or decrease regularly by a *common difference*.

Thus each of the following series of quantities is in Arithmetical progression:—

2,	5,	8,	11,	14,	&c.
9,	5,	1,	-3,	-7,	&c.
a,	$a+b$,		$a+2b$,	$a+3b$,	&c.
a,	$a-b$,		$a-2b$,	$a-3b$,	&c.

In the first of the above examples the quantities increase by 3, whereas in the second the quantities decrease by 4; so the common differences in these two cases are said to be 3 and -4 respectively. Similarly in the third example the common difference is b and in the fourth it is $-b$.

Note. If a be the first term and b the common difference of a series of numbers in Arithmetical Progression, we have the second term $= a+b$, the 3rd term $= a+2b$, the 4th term $= a+3b$,........., the 10th term $= a+9b$,........., the 21st term $= a+20b$, and so on. Hence the nth term $= a+(n-1)b$.

Example 1. Find the 19th term of the series 10, 8, 6, 4, &c.

The first term $= 10$, and the common difference $= -2$.

Hence the 19th term $= 10 + 18.(-2) = 10 - 36 = -26$.

Example 2. What term of the series 5, 7, 9, 11, &c. is 25 ?

Let the rth term of the given series be the required term ; then we must have

$$25 = 5 + (r-1).2$$
$$= 3 + 2r ;$$
$$\text{whence } r = 11.$$

Thus the 11th term of the given series $= 25$.

Exercise (53).

1. Find the 8th, 20th and $(n-3)$th terms of the series 2, 4, 6, 8, &c.

2. What terms of the series 9, 11, 13, 15, &c. are 65, 99, and $6n - 13$?

3. The first term of a given series is 3 and the 7th term 39 ; find the common difference.

4. If there be 60 terms in *A. P.* of which the first term is 8 and the last term 185, find the 31st term.

5. If a be the first term and l the last term of a series of numbers in *A. P.*, shew that the 5th term from the beginning + the 5th term from the end $= a + l$.

6. In the preceding example, shew that the rth term from the beginning + the rth term from the end $= a + l$.

2. To find the sum of n terms of an Arithmetic series of which the first term is a and the common difference, b.

Let S denote the required sum, and l, the last term (*i.e.* the nth term).

Then $S = a + (a+b) + (a+2b) + (a+3b) + \&c. + \{a+(n-1)b\}$.

And, by writing the series in the reverse order, we have also $S = l + (l-b) + (l-2b) + (l-3b) + \&c. + \{l-(n-1)b\}$.

Therefore, by addition,

$$2S = (a+l)+(a+l)+(a+l)+\&c.\dots\dots\text{to } n \text{ terms}$$
$$= n(a+l)\ ;$$
$$\therefore\ S = \frac{n}{2}(a+l) \quad \dots \quad \dots \quad \dots \quad \dots \quad (1)$$

Thus the sum of n terms in *A. P.* is n times the semi-sum of the first and last terms, or, in other words, n times the average of the first and last terms.

Also, since $l = a+(n-1)b$,

$$\therefore\ S = \frac{n}{2}\Big[\,a+\Big\{a+(n-1)b\Big\}\Big]$$
$$= \frac{n}{2}\Big\{2a+(n-1)b\Big\} \quad \dots \quad \dots \quad \dots \quad (2)$$

N. B.—The formulæ (1) and (2) should be carefully remembered so that they might readily be applied in any suitable case.

Example 1. Find the sum to 20 terms of the series 5, $4\frac{1}{3}$ $3\frac{2}{3}$, &c.

The first term = 5, and the common difference

$$= \tfrac{13}{3} - 5 = -\tfrac{2}{3}.$$

Hence, the required sum $= \frac{20}{2}\{2 \times 5 + (20-1) \times (-\frac{2}{3})\}$

$$= 10(10 - \tfrac{38}{3})$$
$$= 10(-\tfrac{8}{3}) = -26\tfrac{2}{3}.$$

Example 2. Find the value of $1+2+3+4+\&c.$ to 100 terms.

The last term of the series evidently = 100.

Hence, the required sum $= \frac{100}{2}(1+100)$

$$= 50 \times 101$$
$$= 5050.$$

Exercise (54).

1. Sum 1, [illegible] 3, 4, &c. to 25 terms.
2. Sum 1, 3, 5, 7, &c. to 30 terms.
3. Sum 5, $4\frac{3}{4}$, $4\frac{1}{2}$, &c. to 21 terms.
4. Sum 13, $12\frac{1}{3}$, $11\frac{2}{3}$, &c. to 40 terms.

5. Sum 2, 7, 12, &c. to 101 terms.

6. Sum $\frac{n-1}{n}$, $\frac{n-2}{n}$, $\frac{n-3}{n}$, &c. to n terms.

7. Sum $\frac{a-b}{a+b}$, $\frac{3a-2b}{a+b}$, $\frac{5a-3b}{a+b}$, &c. to n terms.

3. Applications of the formulæ (1) and (2) of the preceding article.—The following examples will illustrate some important applications of those formulæ.

Example 1. The first term of a series in *A. P.* is 17, the last term $-12\frac{3}{8}$, and the sum $25\frac{7}{16}$; find the common difference.

Let n = the number of terms; then we must have

$$\begin{aligned} 25\tfrac{7}{16} &= \tfrac{n}{2}\{17+(-12\tfrac{3}{8})\} \\ &= \tfrac{n}{2}(17-12\tfrac{3}{8}) \\ &= \tfrac{n}{2}\times 4\tfrac{5}{8}, \end{aligned}$$

$$\text{or,}\quad \tfrac{407}{16} = \tfrac{37n}{16}, \quad \therefore \quad n = \tfrac{407}{37} = 11.$$

If then b be the required common difference, we must have

$$-12\tfrac{3}{8}\,(=\text{the 11th term}) = 17+10b,$$

$$\begin{aligned} \therefore \quad 10b &= -12\tfrac{3}{8}-17 \\ &= -29\tfrac{3}{8} = -\tfrac{235}{8}, \end{aligned}$$

$$\therefore \quad b = -\frac{235}{8\times 10} = -\frac{5\times 47}{5\times 2\times 8} = -\frac{47}{16}.$$

Example 2. The sum of a series in *A. P.* is 72, the first term 17, and the common difference -2; find the number of terms, and explain the double answer.

Let n = the number of terms.

Then we must have

$$\begin{aligned} 72 &= \tfrac{n}{2}\{2\times 17+(n-1)\times(-2)\} \\ &= \tfrac{n}{2}\{34-2(n-1)\} \\ &= \tfrac{n}{2}(36-2n) \\ &= 18n-n^2; \end{aligned}$$

$$\therefore \quad n^2-18n+72 = 0;$$

$$\text{or} \quad (n-6)(n-12) = 0;$$

$$\therefore \quad n = 6 \text{ or } 12.$$

The double answer shows that there are two sets of numbers satisfying the conditions of the problem, and this can be easily verfied. For, the series to 6 terms is 17, 15, 13, 11, 9, 7; and to 12 terms it is 17, 15, 13, 11, 9, 7, 5, 3, 1, -1, -3, -5; now since the sum of the last 6 terms of the latter set of numbers $= 0$, evidently therefore the sum of 6 terms of the series is exactly the same as that of 12 terms

Example 3. How many terms of the series, $-8, -6, -4$, &c., amount to 52 ?

Let n = the required number.

Then we must have

$$\begin{aligned} 52 &= \tfrac{n}{2}\{2\times(-8)+(n-1)\times 2\} \\ &= \tfrac{n}{2}(2n-18) \\ &= n^2-9n. \\ \therefore\ n^2-9n-52 &= 0; \\ \text{or } (n-13)(n+4) &= 0; \\ \therefore\qquad n &= 13, \text{ or } -4. \end{aligned}$$

Hence, since the number of terms can only be a positive integer, we must reject the negative value and take 13 to be the answer to the question.

Example 4. The sum of p terms of an $A.\ P.$ is q, and the sum of q terms is p; find the sum of $p+q$ terms

Let a be the first term, and b the common difference; then since the sum of p terms $= q$, we must have

$$q = \tfrac{p}{2}\{2a+(p-1)b\},$$

$$\left.\begin{aligned} \text{or,}\quad 2q &= p\cdot 2a+p(p-1)b \quad \ldots\ \ldots\ \ldots\ (1) \\ \text{Similarly}\quad 2p &= q\cdot 2a+q(q-1)b \quad \ldots\ \ldots\ \ldots\ (2) \end{aligned}\right\}$$

Subtracting (2) from (1), we have

$$\begin{aligned} 2(q-p) &= (p-q).2a+\{(p^2-q^2)-(p-q)\}b \\ &= (p-q).2a+(p-q)(p+q-1)b; \\ \therefore\quad -2 &= 2a+(p+q-1)b. \end{aligned}$$

Hence, the sum of $p+q$ terms

$$\begin{aligned} &= \frac{p+q}{2}\left\{2a+(p+q-1)b\right\} \\ &= \frac{p+q}{2}\times(-2) \\ &= -(p+q). \end{aligned}$$

Exercise (55).

1. The first term of an A. P. is 5, the number of terms 30, and their sum 1455; find the common difference.

2. The first term of a series being 2, and the 5th term being 7, find how many terms must be taken so that the sum may be 63.

3. What is the common difference when the first term is 1, the last 50, and the sum 204?

4. How many terms of the series, 19, 17, 15, &c., amount to 91?

5. The sum of a certain number of terms of the series, 21, 19, 17, &c., is 120. Find the last term and the number of terms.

6. How many terms of the series, 54, 51, 48, &c., must be taken to make 513? Explain the double answer.

7. If the sum of 8 terms of an A. P. is 64, and the sum of 19 terms is 361, find the sum of n terms.

8. Find the series of which the nth term is $\frac{3+n}{4}$; and also find the sum of the series to 105 terms.

9. Find the series whose rth term is $2r-1$; find the sum of the series to n terms.

10. The sum of n terms of an A. P. is $3n^2-n$ and the common difference 6; find the first term.

11. The sum of n terms of an A. P. is 40, the common difference 2, and the last term 13; find n.

12. Prove that the latter half of $2n$ terms of any Arithmetical series = $\frac{1}{3}$rd. of the sum of $3n$ terms of the same series.

13. If $2n+1$ terms of the series, 1, 3, 5, 7, 9, &c., be taken, then the sum of the alternate terms, 1, 5, 9, &c., will be to the sum of the remaining terms, 3, 7, 11, &c., as $n+1$ is to n.

14. Prove that (i) $$b = \frac{l^2-a^2}{2s-(l+a)},$$

and (ii) $$s = \frac{l+a}{2b}(l-a+b).$$

4. Arithmetic means.—

Definition 1. When three quantities are in Arithmetical Progression, the middle one is said to be the **Arithmetic mean** between the other two.

Thus 5 is the Arithmetic mean between 3 and 7.

Definition 2. If A and B be any two quantities and x_1, x_2, x_3, x_4, &c., x_{n-1}, x_n a number of others such that A, x_1, x_2, x_3, &c., x_{n-1}, x_n, B are in Arithmetical Progression, then, x_1, x_2, x_3, &c., are called the **Arithmetic means** between A and B.

Thus 3, 4, 5, 6, 7 are Arithmetic means between 2 and 8, and so are the numbers $3\frac{1}{2}$, 5 and $6\frac{1}{2}$; for both the series 2, 3, 4, 5, 6, 7, 8, and 2, $3\frac{1}{2}$, 5, $6\frac{1}{2}$, 8 are in A. P.

Note. It is evident from the above example that between any two quantities the number of different sets of Arithmetic means is unlimited.

5. To insert a given number of Arithmetic means between two given quantities.

Let a and c be the two given quantities, and n the number of means to be inserted.

Then we have to find out n quantities x_1, x_2, x_3, &c., x_{n-2}, x_{n-1}, x_n such that a, x_1, x_2, x_3, &c., x_{n-1}, x_n, c may be in A. P. Evidently the series [a, x_1, x_2, x_3, &c., x_{n-1}, x_n, c] consists of $n+2$ terms of which a is the first term and c the last.

Hence, if b be the common difference, we must have

$$c = a+(n+1)b,$$

$$\text{whence } b = \frac{c-a}{n+1}.$$

$$\text{Hence, } x_1 = a+b = a+\frac{c-a}{n+1};$$

$$x_2 = a+2b = a+\frac{2(c-a)}{n+1};$$

&c. &c. &c.

$$x_n = a+nb = a+\frac{n(c-a)}{n+1}.$$

Example 1. Find the Arithmetic mean between any two quantities a and b.

Let x = the quantity sought.

Then a, x, b are in A. P., and ∴ we must have $x-a=b-x$,

$$\text{whence } x = \frac{a+b}{2}.$$

Example 2. Insert 4 Arithmetic means between 3 and 18.

Let x_1, x_2, x_3, x_4, be the means.

Then $3, x_1, x_2, x_3, x_4, 18$ are in A. P.

Hence, if b = the common difference,

we must have $18 = 3 + 5b$, $\therefore\ b = 3$.

$$\text{Hence,}\quad \left.\begin{aligned} x_1 &= 3 + b = 6 \\ x_2 &= x_1 + b = 9 \\ x_3 &= x_2 + b = 12 \\ x_4 &= x_3 + b = 15 \end{aligned}\right\}.$$

Thus the required means are 6, 9, 12 and 15. ✓

Exercise (56).

1. Find the Arithmetic mean between (i) 5 and 8 ; (ii) -5, and 21 ; (iii) $m - n$ and $m + n$; (iv) $(a + x)^2$ and $(a - x)^2$.

2 Insert 2 Arithmetic means between (i) 8 and 12 ; (ii) -6 and 14.

3. Insert 3 Arithmetic means between 117 and 477.

4. Insert 4 Arithmetic means between 2 and -18.

5. Insert 17 Arithmetic means between $3\frac{1}{2}$ and $-41\frac{1}{2}$.

6. There are n Arithmetic means between 1 and 31, such that the 7th mean : $(n - 1)$th mean = 5 : 9 ; required n.

6. The Natural Numbers.—The numbers 1, 2, 3, &c. are called the *natural numbers.*

(i) To find the sum of the first n natural numbers.

Let S denote the sum ; then,

$$S = 1 + 2 + 3 + \ldots\ldots + n$$

$$= \frac{n}{2}(1 + n) = \frac{n(n + 1)}{2} \quad \ldots \quad \ldots \quad \ldots \quad (A)$$

(ii) To find the sum of the first n odd natural numbers.

Let S denote the sum ; then,

$$S = 1 + 3 + 5 + 7 \ldots \quad \ldots \quad \ldots \quad \ldots \text{ to } n \text{ terms}$$

$$= \frac{n}{2}\left\{2 + (n-1) \times 2\right\}$$

$$= \frac{n}{2} \times 2n = n^2 \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \text{(B)}$$

(iii) To find the sum of the squares of the first n natural numbers.

Let S denote the sum ; then,

$$S = 1^2 + 2^2 + 3^2 + 4^2 + \dots\dots\dots + n^2.$$

We have $n^3 - (n-1)^3 = 3n^2 - 3n + 1.$

Hence, putting 1, 2, 3, &c., for n, we have

$$\begin{aligned} 1^3 - 0^3 &= 3.1^2 - 3.1 + 1, \\ 2^3 - 1^3 &= 3.2^2 - 3.2 + 1, \\ 3^3 - 2^3 &= 3.3^2 - 3.3 + 1, \\ 4^3 - 3^3 &= 3.4^2 - 3.4 + 1, \\ \dots\dots & \quad \dots\dots\dots \\ (n-1)^3 - (n-2)^3 &= 3(n-1)^2 - 3.(n-1) + 1, \\ n^3 - (n-1)^3 &= 3\,n^2 \quad - 3.n \quad + 1. \end{aligned}$$

Hence, by addition,

$$n^3 = 3(1^2 + 2^2 + 3^2 + \dots + n^2) - 3(1 + 2 + 3 + \dots + n) + n$$

$$= 3S - 3.\frac{n(n+1)}{2} + n,$$

$$\therefore \quad 3S = n^3 - n + \frac{3n(n+1)}{2}$$

$$= n(n+1)\left\{(n-1) + \frac{3}{2}\right\},$$

$$\therefore \quad S = \frac{n(n+1)(2n+1)}{6}. \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \text{(C)}$$

(iv) To find the sum of the cubes of the first n natural numbers.

Let S denote the sum ; then,

$$S = 1^3 + 2^3 + 3^3 + \dots\dots\dots\dots + n^3.$$

We have $n^4 - (n-1)^4 = 4n^3 - 6n^2 + 4n - 1.$

Hence, putting 1, 2, 3, &c., for n, we have

$$1^4 - 0^4 = 4.1^3 - 6.1^2 + 4.1 - 1,$$

$$2^4-1^4 = 4.2^3-6.2^2+4.2-1,$$
$$3^4-2^4 = 4.3^3-6.3^2+4.3-1,$$

$$(n-1)^4-(n-2)^4 = 4.(n-1)^3-6.(n-1)^2+4.(n-1)-1,$$
$$n^4-(n-1)^4 = 4.n^3 \quad -6.n^2 \quad +4.n \quad -1.$$

Hence, by addition,

$$n^4 = 4(1^3+2^3+3^3+\&c.+n^3)-6(1^2+2^2+3^2+\&c.+n^2) +4(1+2+3+\&c.+n)-n$$
$$= 4S-6\cdot\frac{n(n+1)(2n+1)}{6}+4\cdot\frac{n(n+1)}{2}-n;$$
$$\therefore\ 4S = n^4+n+n(n+1)(2n+1)-2n(n+1)$$
$$= n(n+1)\{(n^2-n+1)+(2n+1)-2\}$$
$$= n(n+1)(n^2+n),$$
$$\therefore\ S = \frac{n^2(n+1)^2}{4} = \left\{\frac{n(n+1)}{2}\right\}^2 \dots\ \dots\ \dots\ \dots\ \dots\ \text{(D)}$$

Thus *the sum of the cubes of the first* n *natural numbers is equal to the square of the sum of those numbers.*

Example 1. Sum the series $1.2+2.3+3.4+\&c.$ to n terms.

The nth term of the series evidently $= n(n+1) = n^2+n$.

Hence, putting $n = 1$, the 1st term $= 1^2+1$,

" " $n = 2$, " 2nd term $= 2^2+2$,

" " $n = 3$, " 3rd term $= 3^2+3$,

and so on.

Hence, if S denote the sum of the given series,

we have $S = (1^2+1)+(2^2+2)+(3^2+3)+\&c.$ to n terms

$$= (1^2+2^2+3^2+\&c.+n^2)+(1+2+3+\&c.+n)$$
$$= \frac{n(n+1)(2n+1)}{6}+\frac{n(n+1)}{2}$$
$$= \frac{n(n+1)}{2}\left\{\frac{2n+1}{3}+1\right\}$$
$$= \frac{n(n+1)(n+2)}{3}.$$

Example 2. Sum the series—

$$1^2 + 3^2 + 5^2 + 7^2 + \&c. \text{ to } n \text{ terms.}$$

Since evidently each term of the given series is equal to the square of the *corresponding* term of the series 1, 3, 5, 7, &c., $\therefore$ the nth term of the given series = the square of the nth term of the series 1, 3, 5, 7, &c.,

$$\begin{aligned} \text{and } \therefore \text{ the } n\text{th term} &= \{1 + (n-1) \times 2\}^2 \\ &= (2n-1)^2 \\ &= 4n^2 - 4n + 1. \end{aligned}$$

$$\begin{aligned} \text{Hence, putting } n &= 1, 2, 3, \&c., \text{ we have} \\ \text{the 1st term} &= 4.1^2 - 4.1 + 1, \\ \text{,, 2nd ,,} &= 4.2^2 - 4.2 + 1, \\ \text{,, 3rd ,,} &= 4.3^2 - 4.3 + 1, \\ &\ldots\ldots\ldots \\ &\text{and so on.} \end{aligned}$$

Hence, if S denote the sum of the given series, we must have

$$\begin{aligned} S &= 4(1^2 + 2^2 + 3^2 + \&c. + n^2) - 4(1 + 2 + 3 + \&c. + n) + n \\ &= 4 \cdot \frac{n(n+1)(2n+1)}{6} - 4 \cdot \frac{n(n+1)}{2} + n \\ &= 2n(n+1)\left\{\frac{(2n+1)}{3} - 1\right\} + n \\ &= \frac{2n(n+1) \times 2(n-1)}{3} + n \\ &= \frac{n}{3}\left\{4\left(n^2 - 1\right) + 3\right\} \\ &= \frac{n}{3}\left(4n^2 - 1\right). \end{aligned}$$

Example 3. Sum the series—

$$1^2 + (1^2 + 2^2) + (1^2 + 2^2 + 3^2) + \&c. \text{ to } n \text{ terms.}$$

The nth term of the series

$$\begin{aligned} &= 1^2 + 2^2 + 3^2 + \&c. + n^2 \\ &= \frac{n(n+1)(2n+1)}{6} \\ &= \frac{n(2n^2 + 3n + 1)}{6} \\ &= \tfrac{1}{3}n^3 + \tfrac{1}{2}n^2 + \tfrac{1}{6}n. \end{aligned}$$

Hence, the 1st term $= \frac{1}{3}.\,1^3 + \frac{1}{2}.1^2 + \frac{1}{6}.1,$

" 2nd " $= \frac{1}{3}\,2^3 + \frac{1}{2}.2^2 + \frac{1}{6}.2,$

" 3rd " $= \frac{1}{3}.\,3^3 + \frac{1}{2}.3^2 + \frac{1}{6}.3,$

. . . . ,

and so on

Hence, if S denote the required sum, we must have

$$S = \frac{1}{3}(1^3 + 2^3 + 3^3 + \&c. + n^3) + \frac{1}{2}(1^2 + 2^2 + 3^2 + \&c. + n^2) + \frac{1}{6}(1 + 2 + 3 + \&c + n)$$

$$= \frac{1}{3}.\frac{n^2(n+1)^2}{4} + \frac{1}{2}.\frac{n(n+1)(2n+1)}{6} + \frac{1}{6}.\frac{n(n+1)}{2}$$

$$\frac{n(n+1)}{12}\left\{n(n+1) + (2n+1) + 1\right\}$$

$$= \frac{n(n+1)}{12}\left(n^2 + 3n + 2\right)$$

$$= \frac{n(n+1)^2(n+2)}{12}.$$

Exercise (57).

Sum the series :—

1. $2^2 + 5^2 + 8^2 +$ &c. to n terms.
2. $1.2^2 + 2.3^2 + 3.4^2 +$ &c. to n terms.
3. $1.3 + 3.5 + 5.7 + 7.9 +$ &c to n terms.
4 $1^3 + 3^3 + 5^3 +$ &c. to n terms.
5. $1 + (1+2) + (1+2+3) +$ &c. to n terms.

7. Miscellaneous Examples.

Example 1. Prove that if the number of terms of an $A.P.$ be *odd*, twice the middle term is equal to the sum of the first and last terms.

Since the number of terms is odd, let it be denoted by $2n+1$.

Evidently the middle term is one which has n terms on either side of it ; hence it is the $(n+1)$th term from the beginning and also the $(n+1)$th term from the end.

Hence, putting M for the middle term, we must have

$$M = a + (n + 1 - 1)b$$

$$= a + nb \; \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

and also $M = l - \overline{(n+1-1)}b$

$$= l - nb \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

Hence, by addition,

$$2M = a + l.$$

Example 2 Prove that the sum of an odd number of terms in *A. P.* is equal to the middle term multiplied by the number of terms.

Let $2n+1$ = the number of terms.

Then the sum of the terms

$$= \frac{2n+1}{2}\left(a+l\right) = \frac{2n+1}{2} \times 2M \qquad \text{[last example]}$$

$$= (2n+1) \times M.$$

Example 3. Find the first five terms of the series of which the sum to n terms $= 5n^2 + 3n$

Let t_1, t_2, t_3, &c., t_n denote respectively the 1st., 2nd., 3rd., &c., nth terms of the series;

and let s_1, s_2, s_3, &c., s_n denote respectively the sums of 1, 2, 3, &c., n terms of the series.

Evidently then $s_1 = t_1$; $s_2 = t_1 + t_2$;

$s_3 = t_1 + t_2 + t_3$; and so on.

Now, by the question, we have

$$s_n = 5n^2 + 3n.$$

(*i.e.*, the sum to *any number* of terms = 5 times the square of *that number* + 3 times *that number*).

Hence, putting $n = 1$, we have $s_1 = 5 + 3 = 8$,

„ $n = 2$, „ $s_2 = 20 + 6 = 26$,

„ $n = 3$, „ $s_3 = 45 + 9 = 54$,

„ $n = 4$, „ $s_4 = 80 + 12 = 92$,

„ $n = 5$, „ $s_5 = 125 + 15 = 140$,

and so on.

Hence, $t_1 = s_1 = 8$,

$t_2 = s_2 - s_1 = 26 - 8 = 18$,

$t_3 = s_3 - s_2 = 54 - 26 = 28$,

$t_4 = s_4 - s_3 = 92 - 54 = 38$,

$t_5 = s_5 - s_4 = 140 - 92 = 48$,

and so on.

Thus the first five terms of the series are 8, 18, 28, 38 and 48.

Example 4. Sum the series—

$$1+5+12+22+35+\&c. \text{ to } n \text{ terms.}$$

[The peculiarity of the series is that the successive differences of the terms are in A. P.]

Let S denote the required sum and let t_n denote the nth term of the series. Then we have

$$S = 1+5+12+22+ \; .\;.\;.\;. \; + t_n\,;$$

also

$$S = 0+1+\;5+12+ \; .\;.\;.\;. \; + t_{n-1}+t_n.$$

Hence, by subtraction,

$$\begin{aligned} 0 &= 1\;+4+7+10+\&c.+(t_n-t_{n-1})-t_n \\ &= \{1+4+7+10+\&c. \text{ to } n \text{ terms}\}-t_n\,; \end{aligned}$$

$$\begin{aligned} \therefore\quad t_n &= \frac{n}{2}\left\{\,2+(n-1)3\right\} \\ &= \frac{n(3n-1)}{2}, \end{aligned}$$

i.e., the nth term of the given series $= \frac{3}{2}\cdot n^2 - \frac{1}{2}n$.

Hence, the 1st. term $= \frac{3}{2}\cdot 1^2 - \frac{1}{2}\cdot 1$,

2nd. " $= \frac{3}{2}\cdot 2^2 - \frac{1}{2}\cdot 2$,

3rd. " $= \frac{3}{2}\cdot 3^2 - \frac{1}{2}\cdot 3$,

and so on.

$$\begin{aligned} \text{Hence, } S &= \tfrac{3}{2}(1^2+2^2+3^2+\&c.+n^2)-\tfrac{1}{2}(1+2+3+\&c.+n) \\ &= \frac{3}{2}\cdot\frac{n(n+1)(2n+1)}{6}-\frac{1}{2}\cdot\frac{n(n+1)}{2} \\ &= \frac{n(n+1)}{4}\cdot 2n = \frac{n^2(n+1)}{2}. \end{aligned}$$

Example 5. Sum the series—

$$\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\&c. \text{ to } n \text{ terms.}$$

Let S denote the sum to n terms.

Now, we have

$$t_1 = \frac{1}{1.2} = 1-\frac{1}{2},$$

$$t_2 = \frac{1}{2.3} = \frac{1}{2} - \frac{1}{3},$$

$$t_3 = \frac{1}{3.4} = \frac{1}{3} - \frac{1}{4},$$

$$\&c., \quad \&c., \quad \&c.,$$

$$t_n = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

$$\text{Hence, } S = 1 - \frac{1}{n+1} = \frac{n}{n+1}.$$

Example 6. Find three numbers in A. P. whose product $= 120$ and whose sum $= 15$.

Let $\alpha - \beta$, α and $\alpha + \beta$ be the numbers;

then we have

$$\left.\begin{aligned} (\alpha+\beta).\alpha.(\alpha+\beta) &= 120\ldots\ldots\ldots(1) \\ \text{and} \quad (\alpha-\beta)+\alpha+(\alpha+\beta) &= 15\ldots\ldots\ldots(2) \end{aligned}\right\}.$$

$$\text{From } (2),\ 3\alpha = 15, \quad \therefore \alpha = 5.$$

$$\text{From } (1),\ \alpha(\alpha^2 - \beta^2) = 120,$$

$$\therefore \quad 5.(25 - \beta^2) = 120,$$

$$\therefore \quad 25 - \beta^2 = 24,$$

$$\therefore \quad \beta^2 = 1, \therefore \beta = \pm 1.$$

Hence, the numbers are 4, 5, 6.

Example 7. If a^2, b^2, c^2 be in A. P., then

$$\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b} \text{ are in A. P.}$$

$$\text{Evidently } \frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b} \text{ are in A. P.,}$$

$$\text{if } \frac{1}{c+a} - \frac{1}{b+c} = \frac{1}{a+b} - \frac{1}{c+a},$$

$$\text{i.e., if } \frac{b-a}{(c+a)(b+c)} = \frac{c-b}{(a+b)(c+a)},$$

$$\text{i.e., if } (b-a)(b+a) = (c-b)(c+b),$$

$$\text{i.e., if } b^2 - a^2 = c^2 - b^2;$$

but this is true by hypothesis.

$$\therefore \frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b} \text{ are in A. P.}$$

Example 8. Determine the relation which must exist between a, b and c, in order that they may be respectively the pth, qth and rth terms of an A. P.

Let α denote the first term and β the common defference of the A. P. of which a, b and c are the pth, qth and rth terms; then we must have

$$\left.\begin{aligned} a &= \alpha + (p-1)\beta \quad \ldots \quad (1)\\ b &= \alpha + (q-1)\beta \quad \ldots \quad (2)\\ c &= \alpha + (r-1)\beta \quad \ldots \quad (3)\end{aligned}\right\}.$$

Now we have to eliminate α and β from these three equations. Subtracting (2) from (1), and (3) from (2), we have

$$a - b = (p - q)\beta,$$

$$b - c = (q - r)\beta.$$

Hence, $\quad (a - b)(q - r) = (b - c)(p - q),$

or, $\quad a(q - r) + b(r - p) + c(p - q) = 0;$

which is the relation required.

Exercise (58).

1. The $(n + 1)$th term of a series in A. P. is $\dfrac{ma - nb}{a - b}$, required the sum of the series to $(2n + 1)$ terms.

2. Find the first five terms of the series of which the sum to n terms is $2n^2 + 7n$.

3. The sum to n terms of an A. P. is $3n^2 + 10n$; find the first term and the common difference.

4. Find the 35th term of the series of which the sum to n terms is $n^2 + n$.

5. Sum the series—

$1 + 3 + 6 + 10 + 15 +$ &c., to n terms.

6. Sum the series—

$2 + 5 + 10 + 17 +$ &c., to n terms.

7. Sum the series—

$2 + 7 + 14 + 23 + 34 +$ &c., to n terms.

8. Sum the series—

$\dfrac{1}{3.5} + \dfrac{1}{5.7} + \dfrac{1}{7.9} +$ &c., to n terms.

9. Find 4 numbers in A. P., such that their sum shall be 56, and the sum of their squares 864.

[Let $a - 3\beta$, $a - \beta$, $a + \beta$ and $a + 3\beta$ be the numbers.]

10. The sum of three numbers in A. P. is 15, and the sum of the squares of the two extremes is 58. What are the numbers ?

11. There are four numbers in A. P., the sum of the two extremes is 8, and the product of the means is 15. What are the numbers ?

12. Find six numbers in A. P. such that the sum of the two extremes may be 16, and the product of the two middle terms 63.

[Let $a - 5\beta$, $a - 3\beta$, $a - \beta$, $a + \beta$, $a + 3\beta$, $a + 5\beta$ be the numbers.]

13. If $(b - c)^2$, $(c - a)^2$, $(a - b)^2$ are in A. P., shew that

$$\frac{1}{b - c}, \quad \frac{1}{c - a}, \quad \frac{1}{a - b} \text{ are in A. P.}$$

14. Determine the relation which must exist between a, b and c, in order that they may be respectively the sums of p, q and r terms of an A. P.

15. Given P and Q the mth and nth terms of an Arithmetic series, find the rth term.

16. There are n Arithmetic means between 3 and 54, such that the 8th mean : $(n - 2)$th mean $= 3 : 5$; find n.

17. If S_1, S_2, S_3 be the sums of n terms of three Arithmetic series, the first term of each being 1 and the respective common differences 1, 2, 3 ; prove that

$$S_1 + S_3 = 2S_2 .$$

18. If there be r Arithmetical Progressions, each beginning from unity, whose common differences are 1, 2, 3, &c., r ; shew that the sum of their nth terms is $= \frac{1}{2}.\{(n - 1) r^2 + (n + 1) r\}$.

19. Sum the series—

$$n.1 + (n - 1).2 + (n - 2).3 + (n - 3).4 + \&c. + 1.n.$$

[The rth term of the series $= \{n - (r - 1)\}.r = (n + 1) r - r^2$. Hence the required sum $= (n + 1)\{1 + 2 + 3 + \ldots + n\} - \{1^2 + 2^2 + 3^2 + \ldots + n^2\} = $ &c.]

20. On the ground are placed n stones ; the distance between the first and second is one yard, between the 2nd and 3rd three yards, between the 3rd and 4th five yards, and so on. How far will a person have to travel who shall bring them, one by one, to a basket placed at the first stone ?

CHAPTER XIII.

GEOMETRICAL PROGRESSION.

1. **Definition.**—Quantities are said to be in Geometrical Progression when each is equal to the product of the preceding and some constant factor.

The constant factor is called the *common ratio* of the series, and it is found by dividing *any* term by that which immediately *precedes* it.

Thus each of the following series forms a Geometrical Progression :—

$$1,\quad 2,\quad 4,\quad 8,\quad 16,\quad \&c.\ ;$$
$$1,\quad \tfrac{1}{2},\quad \tfrac{1}{4},\quad \tfrac{1}{8},\quad \tfrac{1}{16},\quad \&c.\ ;$$
$$1,\quad -\tfrac{1}{3},\quad \tfrac{1}{9},\quad -\tfrac{1}{27},\quad \tfrac{1}{81},\quad \&c.\ ;$$
$$a,\quad ar,\quad ar^2,\quad ar^3,\quad ar^4,\quad \&c.$$

In the first example the common ratio is 2, in the second $\tfrac{1}{2}$, in the third $-\tfrac{1}{3}$, and in the fourth r.

Note. If a be the first term and r the common ratio of a Geometric series, we have the 2nd. term $= a.r$, the 3rd. term $= a.r^2$, the fourth term $= a.r^3$, ... the 10th term $= a.r^9$, ... the 21st term $= a.r^{20}$, and so on. Hence the nth term $= a.r^{n-1}$.

Exercise (59).

1. Find the 8th term of the series 4, 12, 36, &c.

[The common ratio $= \tfrac{12}{4} = 3$; hence the 8th term $= 4.3^7 =$ &c.]

2. Find the 6th term of the series $3\tfrac{3}{8}$, $2\tfrac{1}{4}$, $1\tfrac{1}{2}$, &c.

3. Find the 9th term of the series 1, 4, 16, 64, &c.

4. Find the 6th term of the series 1, -3, 9, -27, &c.

5. Find the 5th term and the $(n-1)$th term of the series $\tfrac{3}{2}$, -1, $\tfrac{2}{3}$, &c.

6. Find the 7th term of the series -21, 14, $-9\tfrac{1}{3}$, &c.

7. The first two terms of a series in G. P. are 125 and 25 ; what are the 6th and 7th terms ?

2. To find the sum of a number of terms in Geometrical Progression.

Let a be the first term, r the common ratio, n the number of terms, and S the sum required.

Then

$$S = a + ar + ar^2 + ar^3 + \&c. + ar^{n-1},$$

$$\therefore \quad Sr = \quad ar + ar^2 + ar^3 + \&c. + ar^{n-1} + ar^n.$$

Hence, by subtraction,

$$Sr - S = ar^n - a,$$

$$\therefore \quad S(r-1) = a(r^n - 1),$$

$$\therefore \quad S = \frac{a(r^n - 1)}{r - 1} \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

$$\text{or,} \quad = \frac{a(1 - r^n)}{1 - r} \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

Cor. If l denote the last (or the nth) term of the series we have $l = ar^{n-1}$; hence from (1),

$$S = \frac{rl - a}{r - 1} \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

Note. The formula (2) may conveniently be used in all cases *except when r is positive and greater than* 1.

Example 1. Find the sum of $\frac{16}{27} - \frac{8}{9} + \frac{4}{3} - \&c.$ to 7 terms.

The common ratio $= -\frac{8}{9} \div \frac{16}{27} = -\frac{8}{9} \times \frac{27}{16} = -\frac{3}{2}$.

Hence, by formula (2),

$$\text{the sum} = \frac{\frac{16}{27}\left\{1 - \left(-\frac{3}{2}\right)^7\right\}}{1 + \frac{3}{2}}$$

$$= \frac{\frac{16}{27}\left\{1 + \frac{2187}{128}\right\}}{\frac{5}{2}}$$

$$= \frac{16}{27} \times \frac{2315}{128}$$

$$= \frac{463}{108}$$

$$= 4\tfrac{31}{108}.$$

Example 2. Find the sum of

$$3 + 4\tfrac{1}{2} + 6\tfrac{3}{4} + \&c. \text{ to 5 terms.}$$

The common ratio $= 4\frac{1}{2} \div 3 = \frac{9}{2} \times \frac{1}{3} = \frac{3}{2}$.

Hence, if S denote the required sum, we have by formula (1),

$$S = \frac{3.\left\{\left(\frac{3}{2}\right)^5 - 1\right\}}{\frac{3}{2} - 1} = \frac{3.\left\{\frac{243}{32} - 1\right\}}{\frac{1}{2}} = 3 \times \frac{211}{32} \times 2$$

$$= \frac{633}{16}$$

$$= 39\tfrac{9}{16}.$$

Exercise (60).

1. Sum $1 + 3 + 9 + 27 + \&c.$, to 12 terms.

2. Sum $81 - 27 + 9 - \&c.$, to 8 terms.

3. Sum $2 - 4 + 8 - \&c.$, to 10 terms.

4. Sum $\frac{4}{9} - \frac{1}{3} + \frac{1}{4} - \&c.$, to 5 terms.

5. Sum $2 - 4 + 8 - \&c.$, to $2r$ terms.

6. Sum $2\frac{1}{2} - 1 + \frac{2}{5} - \&c.$, to n terms.

7. Shew that the sum of n terms of a G. P. beginning with the pth term is r^{p-q} times the sum of an equal number of terms of the same series beginning with the qth term.

3. If n be an integer and r a given proper fraction, to prove that r^n diminishes as n increases.

Let $r = \frac{3}{7}$. Now, since $\frac{3}{7}$ of any number is undoubtedly less than that number,

$(\frac{3}{7})^2$ is less than $\frac{3}{7}$, because $(\frac{3}{7})^2 = \frac{3}{7}$ of $\frac{3}{7}$;
$(\frac{3}{7})^3$ is less than $(\frac{3}{7})^2$, because $(\frac{3}{7})^3 = \frac{3}{7}$ of $(\frac{3}{7})^2$;
$(\frac{3}{7})^4$ is less than $(\frac{3}{7})^3$, because $(\frac{3}{7})^4 = \frac{3}{7}$ of $(\frac{3}{7})^3$;
and so on.

Hence, it is clear that in the series $\frac{3}{7}$, $(\frac{3}{7})^2$, $(\frac{3}{7})^3$, $(\frac{3}{7})^4$,...each term is less than the preceding ; which is briefly expressed by saying that $(\frac{3}{7})^n$ diminishes as n increases.

Similarly the proposition may be proved for any other value of r which is less than 1.

Hence, generally speaking, if r has any given value less than 1, r^n diminishes as n increases.

Note. From the above it is quite clear that if r be a proper fraction, r^n is very small when n is infinitely large.

✓ 4. Geometrical series continued to infinity.

Let us consider the series a, ar, ar^2, ar^3, &c.

If S denote the sum to n terms, we have

$$S = \frac{a(1 - r^n)}{1 - r}$$

$$= \frac{a}{1 - r} - \frac{ar^n}{1 - r}.$$

If then r be a *proper fraction*, the larger n is, the smaller will (r^n and $\therefore$) $\frac{ar^n}{1-r}$ be ; hence by sufficiently increasing the value of n we can make $\frac{ar^n}{1-r}$ less than any assigned quantity, however small ; and therefore *by sufficiently increasing the value of* n, *the sum of* n *terms of the series can be made to differ from* $\frac{a}{1-r}$ *by as small a quantity as we please.*

This statement is usually put thus :—*The sum of an infinite number of terms of the Geometrical Progression is* $\frac{a}{1-r}$; or, more briefly, *the sum to infinity is* $\frac{a}{1-r}$.

Let us apply all these remarks to a particular example.

Consider the series 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, &c.

Here $a = 1$, $r = \frac{1}{2}$; hence the sum to n terms

$$= \frac{1}{1-\frac{1}{2}}\left(1 - \frac{1}{2^n}\right) = 2\left(1 - \frac{1}{2^n}\right) = 2 - \frac{1}{2^{n-1}}.$$

Now, by taking n large enough, 2^{n-1} can be made as large as we please, and therefore $\frac{1}{2^{n-1}}$ as small as we please. Hence we may say that *by taking* n *large enough, the sum of* n *terms of the series can be made to differ from* 2 *by as small a quantity as we please* : or, briefly, *the sum of an infinite number of terms of this series is* 2.

N. B. It must be borne in mind that the sum of n terms of a Geometrical Progression *approaches a fixed limit* as n increases indefinitely *only when* r *is less than unity. If* r *be greater than unity there is no such fixed limit.*

Example 1. Prove that in a decreasing Geometrical Progression continued to infinity each term bears a constant ratio to the sum of all which follow it.

Let the series be a, ar, ar^2, ar^3, &c., where r is less than unity.

Then the n^{th} term $= ar^{n-1}$ and the sum of all the terms which follow this

$$= ar^n(1 + r + r^2 + r^3 + \&c., \text{ to infinity})$$

$$= ar^n \cdot \frac{1}{1-r}.$$

Hence the ratio of the n^{th} term to the sum of all which follow it

$$= \left(ar^{n-1} \div \frac{ar^n}{1-r}\right) = \frac{1-r}{r}.$$

Now this is constant *whatever value n may have*, which proves the proposition.

Example 2. Sum to infinity $\frac{3}{2} - \frac{2}{3} + \frac{8}{27} - \&c.$

Here $a = \frac{3}{2}$, and $r = -\frac{2}{3} \div \frac{3}{2} = -\frac{4}{9}$.

Hence, the required sum $= \dfrac{\frac{3}{2}}{1 + \frac{4}{9}}$

$= \frac{3}{2} \times \frac{9}{13}$

$= \frac{27}{26}$.

Exercise (61).

Sum to infinity each of the following series :—

1. $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \&c$ **2.** $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \&c.$

3. $\frac{5}{8} + \frac{1}{2} + \frac{2}{5} + \frac{8}{25} + \&c.$ **4.** $1 - \frac{2}{3} + \frac{4}{9} - \&c.$

5. $3\frac{3}{8} + 2\frac{1}{4} + 1\frac{1}{2} + \&c.$

6. $\frac{2}{5} + \frac{3}{5^2} + \frac{2}{5^3} + \frac{3}{5^4} + \&c.$ [Split this up into two series.]

7. $\frac{4}{7} + \frac{5}{7^2} + \frac{4}{7^3} + \frac{5}{7^4} + \&c$ **8.** $\sqrt{3} + \frac{1}{\sqrt{3}} + \frac{1}{3\sqrt{3}} + \&c.$

9. $(\sqrt{2} + 1) + 1 + (\sqrt{2} - 1) + \&c.$

10. Find the common ratio of a G. P. continued to infinity in which each term is ten times the sum of all the terms which follow it.

5. Recurring Decimals.—Recurring decimals furnish a good illustration of infinite Geometrical Progressions. Thus, for example, $\cdot 2\dot{3}\dot{4} = \cdot 234343434\ldots\ldots\ldots$

$$= \left.\begin{array}{l} \cdot 2 \\ + \cdot 034 \\ + \cdot 00034 \\ + \cdot 0000034 \\ + \text{\&c., \&c.,} \end{array}\right\} = \frac{2}{10} + \frac{34}{10^3} + \frac{34}{10^5} + \frac{34}{10^7} + \text{\&c.}$$

Here the terms after $\frac{2}{10}$ constitute a Geometrical Progression, of which the first term is $\frac{34}{10^3}$, and the common ratio $\frac{1}{10^2}$.

Hence, we may take $\cdot 2\dot{3}\dot{4} = \frac{2}{10} + \frac{34}{10^3} \div \left\{1 - \frac{1}{10^2}\right\}$

$$= \frac{2}{10} + \frac{34}{990} = \frac{232}{990},$$ which agrees with the value found by the usual Arithmetical rule.

6 Geometric means. *Definition 1* When three quantities are in Geometrical Progression the middle one is called **the Geometric mean** between the other two.

Definition 2. When any number of quantities x_1, x_2, x_3, &c. are such that a, x_1, x_2, x_3, &c., b are in G. P., then x_1, x_2, x_3, &c. are called **Geometric means** between a and b.

(i) To find the Geometric mean between two given quantities.

Let a and b be the two given quantities ; G the Geometric mean.

Then since a, G, b are in G. P., we must have $\frac{G}{a} = \frac{b}{G}$, each being equal to the common ratio.

$$\therefore\ G^2 = ab, \text{ and } \therefore\ G = \sqrt{ab}.$$

(ii) To insert a given number of Geometric means between two given quantities.

Let a and b be the two given quantities ; and x_1, x_2, x_3, x_4, &c., x_n, the n means to be inserted.

Then a, x_1, x_2, x_3, &c., x_n, b are in G. P.

Let r denote the common ratio of the series ;
then b = the $(n+2)$th term $= a.r^{n+1}$,

$$\therefore\ r^{n+1} = \frac{b}{a},$$

$$\text{and} \quad \therefore \quad r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}.$$

$$\text{Hence } x_1 = a\cdot\left(\frac{b}{a}\right)^{\frac{1}{n+1}};\ x_2 = a.\left(\frac{b}{a}\right)^{\frac{2}{n+1}};\ x_3 = a.\left(\frac{b}{a}\right)^{\frac{3}{n+1}};$$

and so on.

Example. Insert 3 Geometric means between $\frac{1}{2}$ and 128.

Let x_1, x_2, x_3 be the means.

Then $\frac{1}{2}, x_1, x_2, x_3$, 128 are in G. P.

Hence if r be the common ratio of the series,

we must have 128 = the 5th term $= \frac{1}{2}\cdot r^4$,

$$\therefore \quad r^4 = 256, \text{ whence } r = 4.$$

$$\text{Hence,} \quad \left.\begin{aligned} x_1 &= \tfrac{1}{2}.4 = 2 \\ x_2 &= \tfrac{1}{2}\cdot 4^2 = 8 \\ x_3 &= \tfrac{1}{2}\cdot 4^3 = 32 \end{aligned}\right\}.$$

Exercise (62).

1. Insert 2 Geometric means between 3 and 24.
2. Insert 3 Geometric means between $2\frac{1}{4}$ and $\frac{4}{9}$.
3. Insert 4 Geometric means between $\frac{2}{3}$ and $-5\frac{1}{16}$.
4. Insert 5 Geometric means between $3\frac{5}{9}$ and $40\frac{1}{2}$.

7. Miscellaneous Examples.

Example 1. If $x < 1$, sum the series

$$1 + 2x + 3x^2 + 4x^3 + \&c. \text{ to infinity.}$$

Let S denote the required sum, then,

$$S = 1 + 2x + 3x^2 + 4x^3 + \&c.$$

$$\text{and} \quad \therefore \quad Sx = \quad x + 2x^2 + 3x^3 + \&c.$$

Hence, by subtraction,

$$S(1 - x) = 1 + x + x^2 + x^3 + \&c. \text{ to infinity}$$

$$= \frac{1}{1 - x},$$

$$\therefore \quad S = \frac{1}{(1 - x)^2}.$$

Example 2. Sum to n terms $5 + 55 + 555 + \&c.$

Let S denote the required sum ; then,

$$S = 5 + 55 + 555 + \&c. \text{ to } n \text{ terms}$$

$$= 5\{1 + 11 + 111 + \&c. \text{ to } n \text{ terms}\}$$

$$= \tfrac{5}{9} \times 9\{1 + 11 + 111 + \&c. \text{ to } n \text{ terms}\}$$

$$= \tfrac{5}{9}\{9 + 99 + 999 + \&c. \text{ to } n \text{ terms}\}$$

$$= \tfrac{5}{9}\{(10 - 1) + (10^2 - 1) + (10^3 - 1) + \&c. \text{ to } n \text{ terms}\}$$

$$= \tfrac{5}{9}\{(10 + 10^2 + 10^3 + \&c. \text{ to } n \text{ terms}) - n\}$$

$$= \frac{5}{9}\left\{\frac{10(10^n - 1)}{10 - 1} - n\right\}$$

$$= \frac{50}{81}\left(10^n - 1\right) - \frac{5n}{9}.$$

Example 3. Sum to n terms $1 + 5 + 13 + 29 + \&c.$

Let t_n denote the nth term of the series, and S the required sum ; then

$$S = 1 + 5 + 13 + 29 + \&c. + t_n ;$$

also $$S = 0 + 1 + 5 + 13 + \&c. + t_{n-1} + t_n.$$

Therefore, by subtraction,

$$0 = (1 + 4 + 8 + 16 + \&c. \text{ to } n \text{ terms}) - t_n ;$$

$$\therefore \quad t_n = 1 + \{4 + 8 + 16 + \&c. \text{ to } (n - 1) \text{ terms}\}$$

$$= 1 + \frac{4(2^{n-1} - 1)}{2 - 1}$$

$$= 1 + 2^2.\left(2^{n-1} - 1\right) = 2^{n+1} - 3.$$

Hence, the 1st. term $= 2^2 - 3$,

,, 2nd. ,, $= 2^3 - 3$,

,, 3rd. ,, $= 2^4 - 3$,

and so on.

Hence, $$S = (2^2 - 3) + (2^3 - 3) + (2^4 - 3) + \&c. + (2^{n+1} - 3)$$

$$= (2^2 + 2^3 + 2^4 + \&c. \text{ to } n \text{ terms}) - 3n$$

$$= \frac{2^2.(2^n - 1)}{2 - 1} - 3n$$

$$= 4(2^n - 1) - 3n.$$

Example 4. If a, b, c, d be in G. P., shew that

$$(a-d)^2 = (b-c)^2+(c-a)^2+(d-b)^2.$$

We have $\dfrac{b}{a} = \dfrac{c}{b} = \dfrac{d}{c}$, each of them being equal to the common ratio.

$$\therefore\ b^2 = ac\ ;\ c^2 = bd\ ;\ \text{and } bc = ad. \quad \ldots \quad \ldots \quad (\alpha)$$

Hence, $(b-c)^2+(c-a)^2+(d-b)^2$

$$= (b^2+c^2-2bc)+(c^2+a^2-2ca)+(d^2+b^2-2db)$$

$$= 2(b^2-ac)+2(c^2-bd)+a^2+d^2-2bc$$

$$= 2\times 0+2\times 0+a^2+d^2-2ad \qquad [\text{by } \alpha\ ]$$

$$= (a-d)^2.$$

Example 5. If a, b, c, d be in G. P., shew that

a^2-b^2, b^2-c^2, c^2-d^2 are in G. P.

Evidently a^2-b^2, b^2-c^2, c^2-d^2 are in G. P,

if $(a^2-b^2)(c^2-d^2) = (b^2-c^2)^2$.

Now, since a, b, c, d are in G. P., we have

$$\frac{b}{a} = \frac{c}{b} = \frac{d}{c}\ ;$$

$$\therefore \qquad ac = b^2,\ bd = c^2 \text{ and } ad = bc.$$

Hence, $(a^2-b^2)(c^2-d^2) = a^2c^2-b^2c^2-a^2d^2+b^2d^2$

$$= b^4-b^2c^2-b^2c^2+c^4$$

$$= b^4-2b^2c^2+c^4$$

$$= (b^2-c^2)^2.$$

$\therefore$ a^2-b^2, b^2-c^2, c^2-d^2 are in G. P.

Example 6. The continued product of three numbers in G. P. is 216, and the sum of the products of them in pairs is 156 ; find the numbers.

Let $\dfrac{a}{r}$, a, ar be the numbers ;

Then, by the conditions given, we must have

$$\left.\begin{array}{lll} & \dfrac{a}{r}.a.ar = 216 & \ldots\ (1) \\ \text{and} & \dfrac{a}{r}\cdot a+\dfrac{a}{r}\cdot ar+a.ar = 156 & \ldots\ (2) \end{array}\right\}$$

From (1), $\quad a^3 = 216, \quad \therefore \quad a = 6.$

Hence, from (2), $\quad \frac{1}{r} + 1 + r = \frac{156}{36} = \frac{13}{3},$

$$\therefore \quad 3(1 + r + r^2) = 13r,$$

$$\text{or,} \quad 3r^2 - 10r + 3 = 0,$$

$$\text{or,} \quad (r - 3)(3r - 1) = 0,$$

$$\therefore \quad r = 3, \text{ or } \tfrac{1}{3}.$$

Hence the numbers are 2, 6, 18.

Exercise (63).

1. Find by the method of summation of infinite Geometric series the values of :—

(i) $\cdot 02\dot{7}$; (ii) $1\cdot 1\dot{4}\dot{5}$; (iii) $\cdot 21\dot{5}0\dot{1}$; (iv) $\cdot\dot{1}4285\dot{7}$.

2. Sum $1 + 3x + 5x^2 + 7x^3 +$ &c. to infinity.

3. Sum $1.2x + 2.4x^2 + 3.8x^3 +$ &c. to infinity.

4. Sum $1.3x + 4.9x^2 + 7.27x^3 +$ &c. to infinity.

• **5.** Sum $a + 2a^2 + 3a^3 + 4a^4 +$ &c. to n terms

6. Sum $1 - 3x + 5x^2 - 7x^3 +$ &c. to infinity.

7. Sum $\frac{1}{3} + \frac{3}{9} + \frac{5}{27} +$ &c. to infinity.

8. Sum $1 + \frac{2}{2} + \frac{3}{2^2} + \frac{4}{2^3} +$ &c. to n terms.

9. Find the nth term, and the sum to n terms, of the series—

$$1.1, \quad 2.3, \quad 4.5, \quad 8.7, \quad \text{\&c.}$$

10. Sum $1 + \frac{2}{5} + \frac{3}{5^2} + \frac{4}{5^3} +$ &c., to n terms.

11. Sum to n terms $4 + 44 + 444 +$ &c.

12. Sum the series $\cdot 9 + \cdot 99 + \cdot 999 +$ &c., to n terms.

13. Sum the series $1 + 3 + 7 + 15 +$ &c., to n terms.

14. Sum to n terms $-6 - 4 + 0 + 8 + 24 +$ &c.

15. Find the sum of $6 + 9 + 21 + 69 + 261 +$ &c., to n terms.

16. If a, b, c, d be in G. P., shew that

$$(a^2 + b^2 + c^2)(b^2 + c^2 + d^2) = (ab + bc + cd)^2.$$

[We have $\frac{a}{b} = \frac{b}{c} = \frac{c}{d} = k$ (say) ; thus $a = bk$, $b = ck$, $c = dk$; hence $a^2+b^2+c^2 = k^2(b^2+c^2+d^2)$ and also $a^2+b^2+c^2 = k(ab+bc+cd)$.]

17. If a, b, c, d are in G. P., prove that

(i) $(b+c)(b+d) = (c+a)(c+d)$;

(ii) $(a+d)(b+c) - (a+c)(b+d) = (b-c)^2$.

18. Three numbers whose sum is 15 are in A. P. ; if 1, 4 and 19 be added to them respectively, the results are in G. P. Determine the numbers.

19. Three numbers whose product is 512 are in G. P. ; if 8 be added to the first and 6 to the second, the numbers are in A. P. Find the numbers.

20. The sum of three quantities in G. P. is $24\frac{4}{9}$, and their product is 64 ; find them.

21. Find the relation between a, b, c that they may be the pth, qth and rth terms of a Geometric series.

22. If P and Q be the pth and qth terms of a Geometric series, find the nth term.

23. If S be the sum, P the product and R the sum of the reciprocals of n terms in G. P., prove that $P^2 = \left(\frac{S}{R}\right)^n$.

24. Find the sum of n terms of the series, the rth term of which is $(2r+1)2^r$.

25. In a G. P. shew that the product of any two terms equidistant from a given term is always the same.

[If the nth term be taken as the given term, it can be easily shown that the product of the $(n+p)$th and $(n-p)$th terms is independent of p.]

26. If there be n terms in G. P., prove that the n^{th} root of their product is equal to the square root of the product of the first and last terms.

27. If n Geometrical means be found between two quantities a and c, shew that their product will be $(ac)^{\frac{n}{2}}$.

28. If a, b, c, d are in G. P., shew that the reciprocals of a^2-b^2, b^2-c^2, c^2-d^2 are also in G. P.

29. If S_1, S_2, S_3, &c., S_n are the sums of infinite Geometric series, whose first terms are 1, 2, 3, &c., n, and whose common ratios are $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, &c., $\frac{1}{n+1}$ respectively, prove that

$$S_1 + S_2 + S_3 + \&c. + S_n = \tfrac{n}{2}(n+3).$$

30. Find the sum of the infinite series—

$$1 + (1+a)r + (1+a+a^2)r^2 + (1+a+a^2+a^3)r^3 + \&c.,$$

r and a being proper fractions.

CHAPTER XIV.

HARMONIC PROGRESSION.

1. **Definition.** A series of quantities are said to be in *Harmonical Progression* when their reciprocals are in Arithmetical Progression.

Thus the series 1, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, &c., and $\frac{1}{2}$, $-\frac{1}{3}$, $-\frac{1}{8}$, $-\frac{1}{13}$, $-\frac{1}{18}$, &c., are in Harmonical Progression, their reciprocals, 1, 2, 3, 4, 5, &c., and 2, -3, -8, -13, -18, &c., being in Arithmetical Progression.

From the above definition we can deduce the following which is sometimes given as the defining property :—

If a, b, c be three consecutive terms of an Harmonical Progression, then $a : c = a - b : b - c$.

For, by definition, $\frac{1}{a}$, $\frac{1}{b}$, $\frac{1}{c}$ are in *A. P.*,

$$\therefore \frac{1}{b} - \frac{1}{a} = \frac{1}{c} - \frac{1}{b},$$

$$\text{or,}\quad \frac{a-b}{ab} = \frac{b-c}{bc}.$$

Hence, $\frac{a-b}{b-c} = \frac{ab}{bc} = \frac{a}{c}$, which proves the property in question.

2. Harmonic means.—*Definition 1.* If three quantities a, H, b are such that $\frac{1}{a}, \frac{1}{H}, \frac{1}{b}$ are in *A. P.*, then H is said to be the *Harmonic mean* between a and b.

Definition 2. If any number of quantities x_1, x_2, x_3, &c., x_n be such that $\frac{1}{a}, \frac{1}{x_1}, \frac{1}{x_2}, \frac{1}{x_3}$, &c., $\frac{1}{x_n}, \frac{1}{b}$ are in *A. P.*, then x_1, x_2, x_3, &c., x_n are said to be n *Harmonic means* between a and b.

(i) To find the Harmonic mean between two given quantities.

Let a, b be the two quantities and H their harmonic mean;

then $\frac{1}{a}, \frac{1}{H}, \frac{1}{b}$ are in *A. P.*, and therefore

$$\frac{1}{H} - \frac{1}{a} = \frac{1}{b} - \frac{1}{H},$$

$$\therefore \quad \frac{2}{H} = \frac{1}{a} + \frac{1}{b} = \frac{a+b}{ab}.$$

$$\text{Hence,} \quad H = \frac{2ab}{a+b}.$$

Example 1. Insert 18 harmonic means between 1 and $\frac{1}{20}$.

Let x_1, x_2, x_3, &c., x_{18} be the required means.

Then 1, $\frac{1}{x_1}, \frac{1}{x_2}, \frac{1}{x_3}$, &c., $\frac{1}{x_{18}}$, 20 are in *A. P.*

∴ If β denote the common difference of this series, we must have

$$20 = \text{the 20th term}$$
$$= 1 + 19\beta,$$

whence $\beta = 1.$

Hence,
$$\frac{1}{x_1} = 1 + \beta = 2;$$
$$\frac{1}{x_2} = 1 + 2\beta = 3;$$
$$\frac{1}{x_3} = 1 + 3\beta = 4;$$

&c. &c. &c.

$$\frac{1}{x_{17}} = 20 - 2\beta = 18;$$
$$\frac{1}{x_{18}} = 20 - \beta = 19.$$

Hence, the required means are

$$\tfrac{1}{2},\ \tfrac{1}{3},\ \tfrac{1}{4},\ \&c.,\ \tfrac{1}{18},\ \tfrac{1}{19}.$$

Example 2. Find the 7th term of the series $\frac{2}{9} + \frac{4}{17} + \frac{1}{4} + \&c.$

The reciprocals of the terms, namely, $\frac{18}{4}$, $\frac{17}{4}$, $\frac{16}{4}$, &c., are in *A. P.*, hence the given series is Harmonic, and $\therefore$ its 7th term must be the reciprocal of the 7th term of the Arithmetic series, $\frac{18}{4}$, $\frac{17}{4}$, $\frac{16}{4}$, &c.

Now since the common difference of this series $= -\frac{1}{4}$ and the first term $= \frac{18}{4}$,

$$\therefore \text{ its 7th term } = \tfrac{18}{4} - 6\cdot\tfrac{1}{4} = \tfrac{12}{4} = 3.$$

Hence the 7th term of the given series $= \frac{1}{3}$.

Example 3 Find the first three terms of an *H. P.* whose 5th term is $\frac{2}{5}$ and 9th term $\frac{1}{2}$.

Let a be the 1st term and b the common difference of the corresponding *A. P.*;

$$\begin{aligned} \text{then}\quad \tfrac{5}{2} &= \text{the 5th term of the } A.\,P \\ &= a + 4b \quad \ldots \quad \ldots \quad (1) \\ \text{and}\quad 2 &= \text{the 9th term of the } A.\,P \\ &= a + 8b \quad \ldots \quad \ldots \quad (2) \end{aligned}$$

From (1) and (2), by subtraction,

$$4b = -\tfrac{1}{2},\ \therefore\ b = -\tfrac{1}{8};$$

hence, from (2), $a = 2 - 8b = 3$.

Hence, the first three terms of the corresponding *A. P.* are 3, $3 - \frac{1}{8}$, $3 - \frac{2}{8}$, *i.e.*, 3, $\frac{23}{8}$ and $\frac{11}{4}$

Hence the first three terms of the *H. P.* are $\frac{1}{3}$, $\frac{8}{23}$ and $\frac{4}{11}$.

Exercise (64)

1. Find the harmonic mean between 4 and 8.
2. Insert 2 harmonic means between $\frac{1}{2}$ and $\frac{4}{17}$.
3. Insert 4 harmonic means between 4 and $\frac{1}{4}$.

4. Find the 5th term of the series—

(i) $4\frac{1}{2}$, 3, 2, &c. ;

(ii) $4\frac{1}{2}$, 3, $2\frac{1}{4}$, &c.

5. Find the 8th term of the series $\frac{2}{11}$, $\frac{1}{5}$, $\frac{2}{9}$, &c.

6. Continue to 3 terms each way the series $\frac{3}{4}$, $\frac{3}{5}$, $\frac{1}{2}$.

7. Find the first three terms of an *H. P.*, whose 4th term is $\frac{3}{5}$ and 8th term $\frac{1}{3}$.

8. Find the *n*th term of an *H. P.*, of which a, b are respectively the first and second terms.

9. Insert n harmonic means between x and y.

10. Find the *n*th term of the series

$$4 + 4\tfrac{2}{7} + 4\tfrac{8}{13} + 5 + \&c.$$

(Calcutta University F. A. Paper, 1886)

3. Relation between the Arithmetic, Geometric and Harmonic means between two real positive quantities.

Let A, G, H be the Arithmetic, Geometric and Harmonic means between any two *real positive* quantities a and b.

Then we have

$$A = \frac{a+b}{2} \quad \ldots \quad \ldots \quad (1)$$

$$G = +\sqrt{ab} \quad \ldots \quad \ldots \quad (2)$$

$$H = \frac{2ab}{a+b} \quad \ldots \quad \ldots \quad (3)$$

Hence, $AH = \frac{a+b}{2} \cdot \frac{2ab}{a+b} = ab = G^2$;

i.e., G is the Geometric mean between A and H.

Again, $A - G = \frac{a+b}{2} - \sqrt{ab} = \frac{1}{2}(\sqrt{a} - \sqrt{b})^2$;

and $G - H = \sqrt{ab} - \frac{2ab}{a+b} = \frac{\sqrt{ab}}{a+b}\Big(\sqrt{a} - \sqrt{\ }$

Now, since a and b are both positive, $\sqrt{a}$ and $\sqrt{b}$ are both *real*, therefore $(\sqrt{a} - \sqrt{b})^2$ is a positive quantity ;

also $\sqrt{ab}$ and $a+b$ are both positive.

Hence both $A - G$ and $G - H$ are positive. Therefore $A > G$ and $G > H$.

Thus it is proved that *the Arithmetic, Geometric and Harmonic means between any two real positive quantities are themselves in Geometric Progression; and that they are in descending order of magnitude.*

Miscellaneous Examples.

Example 1. If a, b, c be in Harmonic Progression, show that $a^2 + c^2 > 2b^2$.

We have the Harmonic mean between a and $c = b$.

Hence, since the geometric mean between any two quantities is greater than the Harmonic mean, we must have

$$\sqrt{ac} > b \text{ and } \therefore\ ac > b^2.$$

Now, since $(a - c)^2$ is a positive quantity,

$$\therefore\ a^2 + c^2 > 2ac$$

a fortiori $\therefore\ a^2 + c^2 > 2b^2$.

Example 2. If four positive quantities a, b, c, d be in $H.\ P.$, show that $a + d > b + c$.

$$\left.\begin{array}{l}\text{The Harmonic mean between } a \text{ and } c = b \\ \text{and } \quad ,, \quad ,, \quad ,, \quad ,, \quad b \text{ and } d = c\end{array}\right\}.$$

Hence, since the Arithmetic mean between two positive quantities is greater than the Harmonic mean, we must have

$$\frac{a+c}{2} > b$$

$$\text{and } \quad \frac{b+d}{2} > c$$

Hence, $\frac{1}{2}(a + d) + \frac{1}{2}(b + c) > b + c$,

$$\therefore\ \tfrac{1}{2}(a + d) > \tfrac{1}{2}(b + c),$$

$$\therefore\ a + d > b + c.$$

Example 3. If $a^x = b^y = c^z =$ &c., and a, b, c, &c. be in $G.\ P.$, then will x, y, z, &c. be in $H.\ P.$

Let $a^x = b^y = c^z = d = \&c. = K$.

Then,
$$\left.\begin{aligned} a &= K^{\frac{1}{x}} \\ b &= K^{\frac{1}{y}} \\ c &= K^{\frac{1}{z}} \\ d &= K^{\frac{1}{v}} \\ \&c., &\quad \&c. \end{aligned}\right\} \quad \therefore \quad \begin{aligned} \frac{b}{a} &= K^{\frac{1}{y}-\frac{1}{x}} \\ \frac{c}{b} &= K^{\frac{1}{z}-\frac{1}{y}} \\ \frac{d}{c} &= K^{\frac{1}{v}-\frac{1}{z}} \\ \&c., &\quad \&c. \end{aligned}$$

But a, b, c, d, &c. being in *G. P.*, we have

$$\frac{b}{a} = \frac{c}{b} = \frac{d}{c} = \&c\,;$$

$$\therefore \quad K^{\frac{1}{y}-\frac{1}{x}} = K^{\frac{1}{z}-\frac{1}{y}} = K^{\frac{1}{v}-\frac{1}{z}} = \&c.$$

Hence, $$\frac{1}{y} - \frac{1}{x} = \frac{1}{z} - \frac{1}{y} = \frac{1}{v} - \frac{1}{z} \quad \&c.$$

i.e., $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}, \frac{1}{v}$, &c are in *A. P.*

$\therefore$ x, y, z, v, &c., are in *H. P.*

Example 4. If a, b, c are in *H. P.*, shew that

$\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$ are also in *H. P.*

$\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$ are in *H. P.*,

if $\frac{b+c}{a}, \frac{c+a}{b}, \frac{a+b}{c}$ are in *A. P.*,

i.e., if $\frac{b+c}{a} + 1, \frac{c+a}{b} + 1, \frac{a+b}{c} + 1$ are in *A. P.*,

i.e., if $\frac{a+b+c}{a}, \frac{a+b+c}{b}, \frac{a+b+c}{c}$ are in *A. P.*

i.e., if $(a+b+c)\left(\frac{1}{b} - \frac{1}{a}\right) = (a+b+c)\left(\frac{1}{c} - \frac{1}{b}\right)$

i.e., if $\frac{1}{b} - \frac{1}{a} = \frac{1}{c} - \frac{1}{b}$,

i.e., if $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in *A. P.*,

i.e., if a, b, c are in *H. P.*

Thus the required result is established.

Example 5. An Arithmetical progression and an Harmonical progression have the same first term, the same last term, and the same number of terms: prove that the product of the r^{th} term from the beginning in one series and the r^{th} term from the end in the other is independent of r.

(Bombay University P. E. Paper, 1890.)

Let p be the first term, q the last term, and n the number of terms of each progression.

Let b be the common difference of the arithmetic series, and c that of the arithmetic series corresponding to the given harmonic series. Then we must have

$$\left.\begin{aligned} q &= p+(n-1)b \\ \text{and}\quad \frac{1}{q} &= \frac{1}{p}+(n-1)c \end{aligned}\right\}$$

$$\text{Hence,}\quad b = \frac{q-p}{n-1},\quad \text{and } c = \frac{p-q}{pq(n-1)},$$

$$\text{and}\ \therefore\ \text{we have } c = -\frac{b}{pq} \quad \ldots\quad \ldots\quad \ldots\quad (1)$$

Let t, t' be the r^{th} term of the $A.\ P.$, from the beginning and the r^{th} term of the $H.\ P.$ from the end, respectively.

Then we have

$$t = p+(r-1)b,$$

$$\begin{aligned} \text{and}\qquad \frac{1}{t'} &= \frac{1}{q}-(r-1)c \\ &= \frac{1}{q}+\frac{(r-1)b}{pq} \qquad [\text{ by } (1)\ ] \\ &= \frac{1}{pq}\left\{p+(r-1)b\right\}. \end{aligned}$$

Thus we have

$$\frac{1}{t'} = \frac{1}{pq}\times t,$$

$$\text{and}\qquad \therefore\ tt' = pq;$$

i.e., tt' is independent of r.

Exercise (65).

1. Prove that the three quantities a, b, c are in $A.\ P.$, $G.\ P.$, or $H.\ P.$, according as

$$\frac{a-b}{b-c} = \frac{a}{a} \text{ or } = \frac{a}{b} \text{ or } = \frac{a}{c}, \text{ respectively.}$$

2. If a, b, c be in arithmetical progression, b, c, d in geometrical progression, and c, d, e in harmonical progression, prove that a, c, e are in geometrical progression.

(Bombay University P. E. Paper, 1886.)

3. If a, b, c be the p^{th}, q^{th} and r^{th} terms respectively of an Harmonic series, shew that $(p-q)ab+(q-r)bc+(r-p)ca = 0$.

4. If the Harmonic mean between two quantities is to their Geometric mean as 12 to 13, prove that the quantities are in the ratio of 4 to 9

5. The Harmonic mean between two numbers is $14\frac{2}{5}$, and the Geometric mean 24 ; find the numbers.

6. If the mth term of an $H. P.$ be equal to n and the nth term be equal to m, find the $(m+n)$th term.

7. If a, b, c be in $H. P.$, show that $a^3+c^3>2b^3$.

8. Show that b^2 is greater than, equal to, or less than ac, according as a, b, c are in $A. P.$, $G. P.$, or $H. P.$

9. If a, b, c be in $H. P.$, shew that

$$\frac{1}{b-a}+\frac{1}{b-c}=\frac{1}{a}+\frac{1}{c}.$$

10. If $\frac{a+b}{2}$, b, $\frac{b+c}{2}$ be in $H. P.$, then a, b, c are in $G. P.$

11. If a, b, c be in $A. P.$; α, β, γ in $H. P.$; $a\alpha$, $b\beta$, $c\gamma$ in $G. P.$, then will $\frac{\alpha}{\gamma}+\frac{\gamma}{\alpha}=\frac{a}{c}+\frac{c}{a}$.

12. α, β, γ are the Geometric means between ca, ab ; ab, bc ; bc, ca respectively. Prove that if a, b, c are in $A. P.$, α^2, β^2, γ^2 are also in $A. P.$, and $\beta+\gamma$, $\gamma+\alpha$, $\alpha+\beta$ are in $H. P.$

(Madras University F. A. Paper, 1890.)

13. If s_1, s_2, s_3 denote the sums of n terms of three arithmetic series whose first terms are unity and their common differences in harmonic progression ; prove that

$$n=\frac{2s_3s_1-s_1s_2-s_2s_3}{s_1-2s_2+s_3}.$$ (Bombay University P. E. Paper, 1889.)

[If b_1, b_2, b_3 respectively be the common differences, it is easy to see that $\frac{s_1-n}{b_1}=\frac{s_2-n}{b_2}=\frac{s_3-n}{b_3}$.]

14. If $2(y-a)$ is an $H. M.$ between $y-x$ and $y-z$, then $x-a$, $y-a$, $z-a$ form a $G. P.$

(Allahabad University I. E. Paper, 1890.)

[It can be shown that $\frac{(y-a)+(x-a)}{(y-a)-(x-a)}=\frac{-(y-a)-(z-a)}{(y-a)-(z-a)}$ whence the result follows by componendo and dividendo.]

CHAPTER XV.

PERMUTATIONS AND COMBINATIONS.

1. **Definitions.** (1) When we have n things at our disposal the *different orders* in which r positions can be filled up, placing one thing in each position, are called the *permutations* or *arrangements* of the n things taken r at a time.

Thus the permutations of the three letters a, b, c taken *two* at a time are ab, ba, ac, ca, bc, cb;
and the permutations of them taken *all* at a time are

$$abc,\ acb,\ bac,\ bca,\ cab,\ cba.$$

(2) The *different groups* or *sets* of r things that can be formed out of n things, *without regard to the order in which the things are placed in each group*, are called the *combinations* or *selections* of the n things taken r at a time.

Thus the combinations of the three letters a, b, c taken *two* at a time are ab, ac, bc;
and if all the three letters be taken, we have the single combination abc.

N. B. It should be clearly understood that ab and ba are regarded as *two different permutations* because a and b respectively occupy *different positions* in them; but leaving *position* out of account, both of them have to be regarded as the *same combination*; similarly abc, acb, bca, bac, cab, cba, though *different permutations*, have to be regarded as the *same combination*.

It should also be borne in mind that two combinations are considered different whenever they do *not* consist of *exactly the same* things; thus $abcde$ and $abfde$ are *two different* combinations, because the former contains a, b, d, e and c, whereas the latter a, b, d, e and f; *i. e.*, for c in the one we have got f in the other, the other things remaining unchanged. The student should particularly attend to this point as he not unfrequently feels disinclined to consider such combinations as different.

2. Before taking up the question of *finding the number of permutations of n things taken r at a time*, it is deemed advisable to give the student a clear view of the question itself and thus prepare him for a ready comprehension of its solution. The following examples have been devised with this object in view, and it is desired that the student should carefully attend to the reasoning by which each result is arrived at.

Example 1. In a two storied building one room in the 2nd. story and one in the lower have to be let, each room being capable of accommodating only one person; if there be 6 candidates for the rooms, in how many ways can the rooms be disposed of?

Call the candidates A, B, C, D, E, F.

Now suppose the upper room is given to A; then the lower room can be given to *any one* of the *remaining five*; thus we have the following 5 arrangements:—

$$\begin{array}{ccccc} A & A & A & A & A \\ B & C & D & E & F \end{array} \quad \ldots \quad \ldots \quad (1)$$

Similarly, we have the following 5 sets of arrangements in which the upper room is given respectively to B, C, D, E and F:—

$$\begin{array}{ccccc} B & B & B & B & B \\ A & C & D & E & F \end{array} \quad \ldots \quad \ldots \quad (2)$$

$$\begin{array}{ccccc} C & C & C & C & C \\ A & B & D & E & F \end{array} \quad \ldots \quad \ldots \quad (3)$$

$$\begin{array}{ccccc} D & D & D & D & D \\ A & B & C & E & F \end{array} \quad . \quad \ldots \quad (4)$$

$$\begin{array}{ccccc} E & E & E & E & E \\ A & B & C & D & F \end{array} \quad \ldots \quad \ldots \quad (5)$$

$$\begin{array}{ccccc} F & F & F & F & F \\ A & B & C & D & E \end{array} \quad \ldots \quad \ldots \quad (6)$$

Hence, the total number of ways of disposing of the rooms $= 6 \times 5 = 30$.

Example 2. Shew that the number of permutations of 4 things taken 3 at a time is 24.

Let the things be denoted by a, b, c, d; then we have to find the number of ways in which we can fill up three positions, *placing one in each position.*

Putting a in the 1st. position, *any one of the rest* can be put in the 2nd.; thus we can fill up the first two positions in 3 ways, namely, $ab, \quad ac, \quad ad. \quad \ldots \quad (1)$

Similarly, the first two positions can also be filled up in the following three sets of ways in which b, c, d respectively occupy the 1st. position:—

$$ba, \quad bc, \quad bd, \quad \ldots \quad \ldots \quad (2)$$
$$ca, \quad cb, \quad cd, \quad \ldots \quad \ldots \quad (3)$$
$$da, \quad db, \quad dc. \quad \ldots \quad \ldots \quad (4)$$

Thus altogether the number of ways of filling up the first two positions $= 4 \times 3 = 12$, and they are—

$$\left.\begin{array}{cccccc} ab, & ac, & ad, & ba, & bc, & bd, \\ ca, & cb, & cd, & da, & db, & dc. \end{array}\right\}$$

Now, evidently for *each* of the above ways of filling up the first two positions there are *only two* ways of filling up the third (for instance, a and b occupying the 1st. and 2nd. positions respectively, the third position can be occupied either by c or by d) ; hence we have altogether the following 12 *pairs* of ways of filling up the three positions :—

$$\left.\begin{matrix} abc \\ abd \end{matrix}\right\},\ \left.\begin{matrix} acb \\ acd \end{matrix}\right\},\ \left.\begin{matrix} adb \\ adc \end{matrix}\right\},\ \left.\begin{matrix} bac \\ bad \end{matrix}\right\},\ \left.\begin{matrix} bca \\ bcd \end{matrix}\right\},\ \left.\begin{matrix} bda \\ bdc \end{matrix}\right\},$$

$$\left.\begin{matrix} cab \\ cad \end{matrix}\right\},\ \left.\begin{matrix} cba \\ cbd \end{matrix}\right\},\ \left.\begin{matrix} cda \\ cdb \end{matrix}\right\},\ \left.\begin{matrix} dab \\ dac \end{matrix}\right\},\ \left.\begin{matrix} dba \\ dbc \end{matrix}\right\},\ \left.\begin{matrix} dca \\ dcb \end{matrix}\right\}.$$

Thus the number of permutations of 4 things taken 3 at a time $= 12 \times 2 = 24$.

Example 3. Suppose that the 1st., 2nd and 3rd. teacherships in a certain school are vacant, and that 4 candidates apply each for *any one* of the appointments ; in how many ways can the teacherships be disposed of ?

Call the candidates A, B, C, D.

Now, since the 1st. teachership can be given to any one of the candidates, suppose it is given to A : then the 2nd. can be given either to B, or to C, or to D, and thus the first two teacherships can be disposed of in the following three ways :—

AB, AC, AD. ... (1)

Similarly, giving the 1st. teachership to B, C, D respectively, we have the following three sets of arrangements for the first two appointments :—

BA, BC, BD, ... (2)
CA, CB, CD, ... (3)
DA, DB, DC. ... (4)

Hence the total number of arrangements for the first two teacherships $= 4 \times 3 = 12$.

Now, evidently, after having disposed of the first two teacherships in any one way there are *only two* ways of disposing of the 3rd. (for instance, A and B getting the 1st. and 2nd. appointments respectively, the 3rd. can be given either to C or to D) ; hence we have altogether the following 12 *pairs* of arrangements for the three appointments in question :—

$$\left.\begin{matrix} \mathrm{ABC} \\ \mathrm{ABD} \end{matrix}\right\},\ \left.\begin{matrix} \mathrm{ACB} \\ \mathrm{ACD} \end{matrix}\right\},\ \left.\begin{matrix} \mathrm{ADB} \\ \mathrm{ADC} \end{matrix}\right\},\ \left.\begin{matrix} \mathrm{BAC} \\ \mathrm{BAD} \end{matrix}\right\},\ \left.\begin{matrix} \mathrm{BCA} \\ \mathrm{BCD} \end{matrix}\right\},\ \left.\begin{matrix} \mathrm{BDA} \\ \mathrm{BDC} \end{matrix}\right\},$$

$$\left.\begin{matrix} \mathrm{CAB} \\ \mathrm{CAD} \end{matrix}\right\},\ \left.\begin{matrix} \mathrm{CBA} \\ \mathrm{CBD} \end{matrix}\right\},\ \left.\begin{matrix} \mathrm{CDA} \\ \mathrm{CDB} \end{matrix}\right\},\ \left.\begin{matrix} \mathrm{DAB} \\ \mathrm{DAC} \end{matrix}\right\},\ \left.\begin{matrix} \mathrm{DBA} \\ \mathrm{DBC} \end{matrix}\right\},\ \left.\begin{matrix} \mathrm{DCA} \\ \mathrm{DCB} \end{matrix}\right\}.$$

Thus the total number of ways in which the appointments can be disposed of = $12 \times 2 = 24$.

Example 4. Suppose there are 5 appointments vacant in a Magistrate's office ; in how many ways it is possible for the Magistrate to provide for three select candidates A, B, C ?

If one of the appointments be given to A, *any one* of the *remaining four* can be given to B ; thus, corresponding to *each* way of providing for A there are *four* ways of providing for A and B ; hence there are altogether 5×4 ways of providing for A and B.

Again, giving one of the appointments to A and another to B, *any one* of the *remaining three* can be given to C ; thus corresponding to *each* way of providing for A and B, there are *hree* ways of providing for A, B and C.

Hence, the total number of ways of providing for A, B and $C = (5 \times 4) \times 3 = 60$.

Exercise (66).

N. B. The following examples are intended as an exercise for the student and it is strongly recommended that he should work out the results exactly in the way pointed out in the above examples :—

1. Show that the number of permutations of 4 things taken 2 at a time is 12.

2. Suppose I have 7 flags of seven different colours (red, orange, yellow, green, blue, indigo, violet), and 2 flagstaffs of unequal heights. How many different signals can I make using both the flagstaffs for a signal ?

3. A student has got 6 subjects to read (English, Mathematics, Sanskrit, Physics, History and Logic), but having got holidays he wants to read only 2 of them, one in the morning and one in the evening : in how many ways can he draw up a routine ?

4. Shew that the number of permutations of *three* things taken *all* together is 6.

5. A gentleman has got 6 rooms to spare ; in how many ways can he accommodate 3 guests (A, B, C), each in a separate room ?

6. A gentleman has got 3 rooms to spare, but he has got 6 guests to accommodate ; if after accommodating three of the guests one in each room he takes the other three elsewhere, shew that the number of ways in which the three rooms can be disposed of is 120.

7. If there are 32 stations on the Eastern Bengal Railway from Calcutta to Goalundo, find the number of tickets required in order that a person may travel from any one station to any other.

3. To find the number of permutations of n different things taken r at a time.

Let a, b, c, d, e, &c., l, k denote the n things. Then we have to find the number of ways in which we can fill up r positions *placing one in each position.*

Now, if we put a in the 1st. position, we can put *any one* of the *remaining* $(n-1)$ things in the 2nd.; thus we can fill up the first two positions in $(n-1)$ ways, namely,

$$ab,\quad ac,\quad ad,\quad ae,\quad \&c.,\quad al,\quad ak.\quad \ldots\quad (1)$$

Similarly, putting b, c, d, &c., k respectively in the first position we can also fill up the first two positions as follows:—

$$\begin{array}{llllllll}
ba, & bc, & bd, & be, & \&c., & bl, & bk, & \ldots\quad (2)\\
ca, & cb, & cd, & ce, & \&c., & cl, & ck, & \ldots\quad (3)\\
da, & db, & dc, & de, & \&c., & dl, & dk, & \ldots\quad (4)\\
\&c & & \&c. & & & \&c. & & \\
ka, & kb, & kc, & kd, & ke, & \&c., & kl. & \ldots\quad (n)
\end{array}$$

Hence the total number of ways in which the first two positions can be filled up $= n(n-1)$.

Now it is evident that for *each* of the above ways of filling up the first two positions there are only $(n-2)$ ways of filling up the 3rd. (for instance, a and b occupying the 1st. and 2nd. positions respectively, the 3rd. position can be occupied by any one of the *remaining* $n-2$ things); *i.e.*, for *each* way of filling up the first two positions there are $(n-2)$ ways of filling up the first three Hence the total number of ways of filling up the first three positions $= n(n-1)(n-2)$.

Evidently again for *each* way of filling up the first three positions there are only $(n-3)$ ways of filling up the 4th (for instance, a, b and c respectively occupying the 1st., 2nd. and 3rd. positions, the 4th position can be occupied by any one of the *remaining* $n-3$ things); *i.e.*, for *each* way of filling up the first three positions there are $(n-3)$ ways of filling up the first four. Hence the total number of ways of filling up the first four positions $= n(n-1)(n-2)(n-3)$.

Proceeding thus, we find that the total number of ways of filling up any number of positions always = the product of an equal number of factors and that these factors commencing from n successively diminish by 1.

Hence, the total number of ways of filling up r positions

$$= n(n-1)(n-2)(n-3) \text{ \&c. to } r \text{ factors}$$

$$= n(n-1)(n-2)(n-3)\ldots(n-r-1).$$

Thus the number of permutations of n things taken r at a time $= n(n-1)(n-2)(n-3)\ \ldots\ (n-r+1)$.

Cor. The number of permutations of n things taken *all* together $= n(n-1)(n-2)$ &c. to n factors

$$= n(n-1)(n-2)\ldots3.2.1.$$

This product is usually denoted by $\underline{|n}$, which is read *factorial n*; thus $\underline{|n}$ denotes the product of the natural numbers from 1 to n inclusive.

NOTE. For the sake of convenience the symbol ${}^nP_r$ is usually employed to denote the number of permutations of n things taken r at a time; thus ${}^5P_4$ denote the number of permutations of 5 things taken 4 at a time. It should also be observed that $\underline{|n} = n\underline{|n-1} = n(n-1)\underline{|n-2}$, and so on.

Example 1. Find the value of ${}^6P_4$.

The required value = the number of permutations of 6 things taken 4 at a time

$$= 6 \times 5 \times 4 \times 3 = 360.$$

Example 2. How many numbers can be formed taking only 3 out of the 5 digits 1, 2, 3, 4, 5?

Since each arrangement of 3 digits gives us a new number, the number of different numbers that can be formed is evidently the same as the number of permutations of 5 things taken 3 at a time.

Hence, the required number $= 5 \times 4 \times 3$

$$= 60.$$

Example 3. Of the different words that can be formed from the letters of the word *courtesy* how many will begin with c and end with y?

We have altogether 8 letters, namely, c, o, u, r, t, e, s, y.

Now if c and y always occupy respectively the first and last positions, then we can form only as many words as there are ways of arranging among themselves the six intermediate etters.

Hence, the required number of words

$$= {}^6P_6 = 6.5.4.3.2.1$$
$$= 720.$$

Example 4. In how many ways can the 7 letters A, B, C, D, E, F, G be arranged that B and C may always occupy contiguous positions?

Since B and C will always be together, let us put them within brackets as (BC), and find the number of ways in which the *six* things, namely :—A, (BC), D, E, F, G can be arranged

The number of such arrangements evidently $= {}^6P_6$

$$= 6.5.4.3.2.1$$
$$= 720,$$

and in each of these the letters B, C stand in the order BC;

similarly there are 720 arrangements in each of which they stand in the order CB.

Hence, there are altogether 2×720 or 1440 arrangements in each of which the letters B, C occupy *contiguous* positions.

Example 5. In how many ways can the letters of the word *valedictory* be arranged so that the vowels may never be separated?

Let us put the vowels within brackets, as $(aeio)$; then we have got only 8 things, namely, v, l, d, c, t, r, y, $(aeio)$ to arrange.

Hence, the number of arrangements of the letters in which the vowels stand together in the order *aeio*

$$= {}^8P_8 = 8.7.6.5.4.3.2.1. = 40320.$$

Similarly, corresponding to *each* order in which the vowels may stand together there are 40320 arrangements of the letters.

But the vowels, being 4 in number, may stand together in 4.3.2.1 or 24 different orders.

∴ the total number of arrangements of the given letters under the given condition

$$= 4 \times 40320$$
$$= 967680.$$

N. B. If the condition were that the vowels should not only occupy contiguous positions but stand, with respect to each other, in one and the same invariable order, then the number of arrangements would only be = 40320.

Example 6. Find the number of ways in which the letters of the word *machine* can be arranged such that the vowels may occupy only odd positions.

We have altogether 7 letters to deal with, 4 consonants, and 3 vowels.

Let us mark out the positions to be filled up as follows :—

$$\overset{1}{(a)} \quad \overset{2}{(\ )} \quad \overset{3}{(i)} \quad \overset{4}{(\ )} \quad \overset{5}{(e)} \quad \overset{6}{(\ )} \quad \overset{7}{(\ )}$$

Now, since the vowels can be placed only in three out of the four positions marked 1, 3, 5, 7, the total number of ways in which they can be made to stand in odd positions

$$= {}^4P_3 = 4.3.2 = 24.$$

Suppose one arrangement of the vowels is as shewn in the diagram ; then for this particular arrangement of the vowels the number of ways in which the 4 consonants can be made to occupy the remaining 4 positions (marked 2, 4, 6, 7)

$$= {}^4P_4 = 4.3.2.1.$$

Hence, for *each* particular way of putting the vowels in odd positions there are 24 arrangements of the whole set ;

∴ the total number of arrangements of the given letters under the given condition

$$= 24 \times 24 = 576.$$

Example 7. You are given 8 balls of different colours (black, white, red, yellow, green, blue, indigo, violet) ; in how many ways can you arrange them in a row so that the black and white balls may never come together ?

Let x = the required number of permutations, *i.e.*, those in which the black and white balls are *not* together,
and y = the number of permutations in which they *are* together.

Then evidently $x + y = {}^8P_8$.

Now, as in Example 4,

$$y = 2 \times {}^7P_7 = 2 \times \underline{|7}.$$

Hence,

$$\begin{aligned} x &= {}^8P_8 - y \\ &= \underline{|8} - 2 \times \underline{|7} \\ &= (8 - 2) \times \underline{|7} \\ &= 6 \times 7.6.5.4.3.2.1 \\ &= 30240. \end{aligned}$$

Exercise (67).

1. In the permutations formed out of a, b, c, d, e, f, g, taken all together, how many begin with ab? How many with abc? How many with $abcd$?

2. Find the values of ${}^{10}P_6$ and ${}^{12}P_8$.

3. If ${}^{n}P_4 = 12 \times {}^{n}P_2$, find n.

4. If ${}^{2n+1}P_{n-1} : {}^{2n-1}P_n = 3 : 5$, find n.

5. If ${}^{20}P_r = 13 \times {}^{20}P_{r-1}$, find r.

6. How many numbers each lying between 100 and 1000 can be formed with the digits 1, 2, 3, 4, 5, 6, 7?

7. How many numbers each lying between 10 and 100 can be formed with the digits 3, 4, 0, 5, 6?

8. How many numbers each lying between 10 and 1000 can be formed with the digits 2, 3, 4, 0, 8, 9?

9. How many numbers each greater than 1000 can be formed with the digits 5, 6, 7, 8, 9?

10. How many changes may be rung with 6 bells out of 8; and how many with the whole peal?

11. How many different signals may be formed by means of 12 different flags which can be hoisted 4 at a time above each other?

12. There are m men and n monkeys, n being greater than m; find the number of ways in which each man may become the owner of one monkey.

(Bombay University P. E. Paper, 1891.)

13. In how many ways can the letters of the word *dogmatic* be rearranged?

14. The first year class in a certain college consists of 10 students of whom one is a Mahomedan, one a Christian and the rest are Hindus; in how many ways can the students be arranged in a row if the Mahomedan and Christian students have always to occupy extreme positions?

15. You are given a copy of the following books :—Todhunter's Algebra, Hamblin Smith's Trigonometry, Milton's Paradise Lost, Bacon's Advancement of Learning, Wrigley's Collections of Problems, Shakspeare's Hamlet, Fowler's Logic, and Raghuvansam; in how many ways can you arrange these books on a shelf so that the Mathematical books may be always put together ?

16. In the preceding example, in how many ways can the books be arranged so that the books on English Literature *also* may be put together ?

17. You are given a gold mohar, a rupee, an eight-anna piece, a four-anna piece, a two-anna piece, a pice and a half-pice ; in how many ways can you arrange them in a line so that the copper coins may always stand in odd positions ?

18. In the preceding example find the number of ways of arranging the coins if the Mohar *also* has to be put in an odd position.

19. On the E. B. Railway there are 12 stations from Barackpur to Ranmagar ; if a booking clerk is to be appointed in each of these stations out of 12 candidates of whom one is a Uriya, one a Marhatta and the rest are Bengalis, find the number of ways of appointing the men so that the Uriya and the Marhatta may never be appointed in two consecutive stations.

20. Find the number of arrangements that can be made of the letters of the word *youngster* so that the vowels may not *all* be in consecutive positions in any of them.

21. There are three works each of 2 volumes and two works each of 3 volumes ; in how many ways can the 12 books be arranged on a shelf so that volumes of the same work are not separated ?

22. A shelf contains 20 books, of which 4 are single volumes, and the others form sets of 8, 5 and 3 volumes respectively : find in how many ways the books may be arranged on the shelf, the volumes of each set being in their due order.

[Evidently the volumes of each set may be in due order two ways, either from left to right or from right to left.]

4. To find the number of combinations of n different things taken r at a time.

Let x = the required number, *i.e*, the number of ways in which r things can be selected out of n things.

Now if with the n things at our disposal we are asked to fill up r positions in all possible ways, putting one thing in each position, our work will clearly resolve itself into the following two operations :—

(i) To select a group of r things out of the n things;

(ii) To arrange the r things thus selected in the r positions in as many ways as possible.

Thus for each way of selecting r things the proposed work can be done in $\underline{|r}$ ways.

Hence the total number of ways of doing the work $= x \times \underline{|r}$.

But, by article 3, the same number

$$= n(n-1)(n-2) \; \ldots \; \ldots \; (n-r+1).$$

Hence, $x \times \underline{|r} = n(n-1)(n-2) \; \ldots \; \ldots \; (n-r+1)$;

$$\therefore \quad x = \frac{n(n-1)(n-2) \; \ldots \; \ldots \; (n-r+1)}{\underline{|r}}.$$

Thus denoting the number of combinations of n things taken r at a time by the symbol ${}^nC_r$, we have

$${}^nC_r = \frac{n(n-1)(n-2)\ldots(n-r+1)}{\underline{|r}} \quad \ldots \quad \ldots \quad (1)$$

Cor. We can also express this result in a different form; for if we multiply both numerator and denominator of the right hand expression by $\underline{|n-r}$ it becomes

$$= \frac{n(n-1)(n-2)\ldots(n-r+1) \times \underline{|n-r}}{\underline{|r} \times \underline{|n-r}},$$

and the new numerator is evidently the product of all the natural numbers from n to 1; [See note, Art 3.]

$$\therefore \quad {}^nC_r = \frac{\underline{|n}}{\underline{|r}\;\underline{|n-r}}. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

The formula (1) may conveniently be used whenever numerical calculations have to be made, whereas (2) will do very well as an *algebraical expression.*

Note. If in (2) we put $r = n$ we have

$${}^nC_n = \frac{\underline{|n}}{\underline{|n}\;\underline{|0}} = \frac{1}{\underline{|0}}.$$

But ${}^nC_n$ evidently $= 1$; hence the symbol $\underline{|0}$ must be considered as equivalent to 1.

4. (a) To prove that the number of combinations of n things taken r at a time is equal to the number of combinations of n things taken $n-r$ at a time.

Since $\quad {}^nC_r = \dfrac{\underline{|n}}{\underline{|r}\,\underline{|n-r}}$,

putting $n-r$ for r, we have

$$ {}^nC_{n-r} = \frac{\underline{|n}}{\underline{|n-r}\,\underline{|n-(n-r)}} $$

$$ = \frac{\underline{|n}}{\underline{|n-r}\,\underline{|r}}. $$

Hence, ${}^nC_{n-r} = {}^nC_r$.

Alternative method :—

Evidently for every group of r things that we can take out of n things, a group of $n-r$ things is left out ; thus *corresponding to each group* of r things there is one group of the remaining $n-r$ things. Hence the number of different groups of $n-r$ things is exactly equal to the number of different groups of r things ;

$$ i.\,e.,\ {}^nC_{n-r} = {}^nC_r\,. $$

Such combinations are called *complementary.*

Cor. If ${}^nC_x = {}^nC_y$, then $x = y$ or $x+y = n$; for y can have only either of two values, namely, x or $n-x$.

Note. We have ${}^nC_0 = {}^nC_{n-0} = {}^nC_n = 1$.

Example 1. Find the values of ${}^{12}C_8$, ${}^{15}C_9$, and ${}^{25}C_{22}$.

$$ {}^{12}C_8 = {}^{12}C_{12-8} = {}^{12}C_4 = \frac{12\times11\times10\times9}{1\times2\times3\times4} $$

$$ = 11\times5\times9 $$

$$ = 495. $$

$$ {}^{15}C_9 = {}^{15}C_{15-9} = {}^{15}C_6 = \frac{15\times14\times13\times12\times11\times10}{1\times2\times3\times4\times5\times6} $$

$$ = 7\times13\times11\times5 $$

$$ = 5005. $$

$$ {}^{25}C_{22} = {}^{25}C_{25-22} = {}^{25}C_3 = \frac{25\times24\times23}{1\times2\times3} $$

$$ = 2300. $$

Example 2. If ${}^nC_{14} = {}^nC_{16}$, find ${}^nC_{28}$ and ${}^{32}C_n$.

Since ${}^nC_{14} = {}^nC_{16}$, $\therefore\ n = 14+16 = 30.$

Hence, $$^{n}C_{28} = {}^{30}C_{28} = {}^{30}C_2 = \frac{30 \times 29}{1 \times 2} = 435;$$

and $$^{32}C_n = {}^{32}C_{30} = {}^{32}C_2 = \frac{32 \times 31}{1 \times 2} = 496.$$

Example 3. Prove that $^{4n}C_{2n} : {}^{2n}C_n$
$$= \{1.3.5 \ldots (4n-1)\} : \{1.3.5 \ldots (2n-1)\}^2.$$

Since $$^{4n}C_{2n} = \frac{\underline{|4n}}{\underline{|2n}\ \underline{|2n}}, \text{ and } {}^{2n}C_n = \frac{\underline{|2n}}{\underline{|n}\ \underline{|n}},$$

$\therefore$ the required ratio

$$= \frac{\underline{|4n}}{\underline{|2n}\ \underline{|2n}} \times \frac{\underline{|n}\ \underline{|n}}{\underline{|2n}}$$

Now, $$\underline{|4n} = 1.2.3.4.5 \ldots 4n$$
$$= \{1.3.5 \ldots (4n-1)\} \times \{2.4.6 \ldots 4n\}$$
$$= \{1.3.5 \ldots (4n-1)\} \times 2^{2n}.\{1.2.3 \ldots 2n\},$$
$$\therefore \frac{\underline{|4n}}{\underline{|2n}} = \{1.3.5 \ldots (4n-1)\} \times 2^{2n}. \quad \ldots \quad (1)$$

Also, $$\underline{|2n} = \{1.3.5 \ldots (2n-1)\} \times \{2.4.6 \ldots$$
$$= \{1.3.5 \ldots (2n-1)\} \times 2^n\{1.2.3 \ldots$$
$$\frac{\underline{|2n}}{\underline{|n}} = \{1.3.5 \ldots (2n-1)\} \times 2^n,$$

and $$\left\{\frac{\underline{|n}}{\underline{|2n}}\right\}^2 = \frac{1}{\{1.3.5 \ldots (2n-1)\}^2 \times 2^{2n}}. \quad (2)$$

Hence from (1) and (2), the required ratio
$$= \frac{1.3.5 \ldots (4n-1)}{\{1.3.5 \ldots (2n-1)\}^2}.$$

Example 4. Sixteen clerkships are vacant in a merchant's office; how many different batches of men can be chosen out of twenty candidates? How often may any particular candidate be selected?

We have only to find out the number of different *groups* of 16 men that can be formed out of 20, *without any reference to the appointment to be given to each.*

Hence, the required number of ways

$$= {}^{20}C_{16} = {}^{20}C_4$$

$$= \frac{20 \times 19 \times 18 \times 17}{1 \times 2 \times 3 \times 4}$$

$$= 5 \times 19 \times 3 \times 17$$

$$= 4845.$$

Let us now find out how many times a particular candidate may be chosen.

Every time that a particular candidate is selected the *other* 15 candidates will have to be chosen from the remaining 19 candidates.

Hence a particular man may be selected as many times as we can select a group of 15 men out of the remaining 19.

Hence the required number of times

$$= {}^{19}C_{15} = {}^{19}C_4$$

$$= \frac{19 \times 18 \times 17 \times 16}{1 \times 2 \times 3 \times 4}$$

$$= 19 \times 3 \times 17 \times 4$$

$$= 3876.$$

Example 5. A father with eight children takes three at a time to the Zoological gardens, as often as he can without taking the same three children together more than once. How often will he go, and how often will each child go?

(Calcutta University F. A. Paper, 1877.)

Since the father is to accompany *each new* group of children he will evidently have to go as many times as there are different groups of 3 children formed out of 8.

Hence, the number of times that the father will have to go

$$= {}^{8}C_3 = \frac{8 \times 7 \times 6}{1 \times 2 \times 3} = 56.$$

Again, each child will go only as many times as the *other two* can be taken out of the remaining 7; hence, the number of times that each child will go $= {}^{7}C_2 = \frac{7 \times 6}{1 \times 2} = 21.$

Example 6. Out of 10 consonants and 4 vowels, how many words can be formed, each containing 3 consonants and 2 vowels?

The number of different groups of 3 consonants that can be formed out of 10 = $^{10}C_3$;

also the number of different groups of 2 vowels that can be formed out of 4 = $^4C_2$.

Now combining each of the first set of groups with each of the second set, we have altogether $^{10}C_3 \times {}^4C_2$ groups each consisting of 3 consonants and 2 vowels.

Then, as each of these groups contains 5 different letters, which can be arranged among themselves in $\underline{|5}$ ways, we get from each such group $\underline{|5}$ words

Hence, the required number of words

$$= {}^{10}C_3 \times {}^4C_2 \times \underline{|5}$$

$$= \frac{10 \times 9 \times 8}{1 \times 2 \times 3} \times \frac{4 \times 3}{1 \times 2} \times \underline{|5}$$

$$= 120 \times 6 \times \underline{|5}$$

$$= 720 \times 120$$

$$= 86400.$$

Example 7. If words be formed taking at a time only 5 of the letters of the word *metaphysic*, in how many of them will the letter *t* occur ?

There are altogether 10 letters at our disposal.

We have to find the number of ways in which five positions can be filled up, one or other of them being occupied by *t*.

For every position of *t* the remaining four positions can evidently be filled up in $^9P_4$ ways.

Hence the required number $= 5 \times {}^9P_4$

$= 5 \times 9.8.7.6$

$= 15120.$

Example 8. In how many ways can 21 white balls and 19 black balls be arranged in a row so that no two black balls may be together ?

Let us first arrange the white balls as shewn below :—

× *W* × *W* × *W* . . . . *W* × *W* ×

Then evidently in order that no two black balls may occupy contiguous positions, the black balls can only be put in the positions marked ×.

Now the white balls being 21 in number, the number of *such* positions (*i.e.*, those marked ×) is evidently 22. [For instance, when there are 4 white balls there are 5 such positions, thus :—

$\times \quad W \quad \times \quad W \quad \times \quad W \quad \times \quad W \quad \times$].

Hence in order to place the black balls we must choose any 19 out of these 22 places.

Hence, the required number of ways

$$= {}^{22}C_{19} = {}^{22}C_3$$

$$= \frac{22 \times 21 \times 20}{1 \times 2 \times 3}$$

$$= 11 \times 7 \times 20$$

$$= 1540.$$

Example 9. A boat's crew consists of 8 men, 3 of whom can only row on one side and 2 only on the other. Find the number of ways in which the crew can be arranged.

Call the men A, B, C, D, E, F, G, H, and suppose A, B, C remain only on one side and D, E on the other as represented here,—

$$\begin{array}{c|c} A & D \\ B & E \\ C & \end{array}$$

Then, since 4 men must row on each side, of the remaining 3, *one* must be placed on the side of A, B, C and the *other two* on the side of D, E; and this can evidently be done in 3 ways, for we can place any one of the three on the side of A, B, C.

Of these 3 ways of distributing the crew let us first consider one, say that in which F is on the side of A, B, C.

$$\begin{array}{c|c} A & D \\ B & E \\ C & G \\ F & H \end{array}$$

Now A, B, C, F can be arranged among themselves in $\underline{|4}$ ways, and evidently for *each* of these arrangements there are again $\underline{|4}$ ways of arranging D, E, G, H among themselves; hence in this case the total number of ways of arranging the men $= \underline{|4} \times \underline{|4}$.

Similarly, in *each* of the other two cases also we have $\underline{|4} \times \underline{|4}$ ways of arranging the men.

Hence, on the whole, the number of ways of arranging the crew

$$= 3 \times \lfloor\underline{4} \times \lfloor\underline{4}$$
$$= 3 \times 24 \times 24$$
$$= 1728.$$

Example 10. In how many of the permutations of n things taken r together will three given things occur?

(Calcutta University F. A. Paper, 1880.)

Let us first find out the number of *combinations* of n things taken r together *in each of which three given things occur.*

Evidently *each* of the combinations containing the given things *also* contains a group of $r-3$ things taken out of the remaining $n-3$ things. Hence, there are as many combinations of r things each containing three given things as there are combinations of $n-3$ things taken $r-3$ together; *i.e.*, the required number of combinations

$$= {}^{n-3}C_{r-3} = \frac{\lfloor\underline{n-3}}{\lfloor\underline{r-3}\,\lfloor\underline{n-r}}.$$

But each of these combinations, consisting of r things, gives us $\lfloor\underline{r}$ permutations.

Hence the number of *permutations* in each of which three given things occur

$$= \frac{\lfloor\underline{n-3}}{\lfloor\underline{r-3}\,\lfloor\underline{n-r}} \times \lfloor\underline{r}$$
$$= \frac{\lfloor\underline{n-3}}{\lfloor\underline{n-r}} \times r(r-1)(r-2).$$

Exercise (68).

1. Find the values of ${}^{50}C_{47}$, ${}^{27}C_{23}$ and ${}^{59}C_{52}$.

2. If ${}^{n}C_{18} = {}^{n}C_{7}$, find the values of ${}^{n}C_{23}$ and ${}^{27}C_{n}$.

3. If ${}^{n}C_{r-5} = {}^{n}C_{r+9}$ and if $\frac{n}{r} = \frac{5}{2}$; find n and r.

4. If there be twenty pears at three a penny, how many different selections can be made in buying six penny worth? and in how many of these will a particular pear occur?

(Calcutta University F. A. Paper, 1861.)

5. How many triangles can be formed by joining the angular points of a decagon; how many diagonals has it?

[Any three angular points give a triangle : the straight line joining *any two* angular points *that are not adjacent* is a diagonal.]

6. How many diagonals can be drawn in a figure of n straight lines as sides?

(Calcutta University F. A. Paper, 1867.)

7. Four persons are chosen by lot out of ten; in how many ways can this be done and how often would any one person be chosen?

(Bombay University P. E. Paper, 1882.)

8. Suppose 20 clerks have to be appointed out of 23 candidates of whom 2 are Mahomedans and the rest Hindus. How many different selections can be made so that none of the Mahomedan candidates may be excluded?

9. Find the number of combinations which can be formed by taking the letters of the alphabet 6 at a time, each combination containing two vowels and no more.

10. Out of 17 consonants and 5 vowels how many words can be formed, each consisting of 2 vowels and 3 consonants?

11. A committee of 7 members is to be chosen out of 20 Municipal Commissioners of whom 15 are Hindus and 5 Mahomedans, in such a way that 5 Hindus and 2 Mahomedans shall be on the committee. In how many different ways can such a committee be constituted, and from how many of these will a particular Hindu Commissioner be excluded?

(Calcutta University F. A. Paper, 1883.)

[If a particular Hindu Commissioner be excluded, the 5 committee members will have to be chosen from the remaining 14 Commissioners.]

12. Suppose 8 Professors are wanted for a College newly started; if 3 Premchand Roychand Students and 12 M. A.'s apply for the Professorships, how many selections can be made taking in (i) all the Premchand Roychand Students, (ii) *at least* one of them.

13. A cricket team consisting of eleven players is to be selected from two sets consisting of six and eight players respectively. In how many ways can the selection be made, on the supposition that the set of six shall contribute not fewer than four players?

(Madras University F. A. Paper, 1890.)

14. Prove that the number of words which can be fromed of the letters a, b, c, d, e, f taken *three* together, each word containing one vowel *at least*, is 96.

(Calcutta University F. A. Paper, 1866.)

15. If words be formed with 6 only of the letters of the word *centrifugal*, in how many of them will *all the three* letters c, t, f, occur ?

16. If words be formed with 7 only of the letters of the word *polygamist*, in how many of them will both the letters p and t occur ?

17. In how many ways can 24 rupees and 23 pice be arranged in a row so that no two pice may be together ?

18. In how many ways can 37 literary and 35 Mathematical books be arranged in a column one above the other so that no two Mathematical books may be together, the books of each class being considered alike ?

19. A boat's crew consists of 10 men, 3 of whom can only row on one side and 2 only on the other. Find the number of ways in which the crew can be arranged.

20. An English School and a Vernacular School are both under *one* superintendent. Suppose that the superintendentship, the first four teacherships in the English School and also the first four teacherships in the Vernacular School are vacant; if there be altogether 11 candidates for the appointments, 3 of whom apply *exclusively* for the superintendentship, and 2 *exclusively* for appointments in the English School, in how many ways can the different appointments be disposed of ?

21. There are 12 points in a plane, 5 *only* of which are in the same straight line; find the number of triangles which can be formed by joining the points.

22. Find the number of straight lines which result from joining the points in the preceding example.

23. There are n points in a plane of which no three are in a straight line except m, which are all in a straight line. Find the number of triangles formed by joining the points.

(Bombay University P. E. Paper, 1889.)

24. In how many ways can 9 things be divided equally among 3 persons ?

25. There are 3 asses capable of carrying weights of 3, 4 and 5 maunds respectively; in how many ways can 12 packets each weighing a maund be distributed among the asses?

5. To find the number of combinations of n things taken r at a time without assuming the formula for the number of permutations.

Let ${}^nC_r$ denote the number of combinations of n things taken r at a time; and let the n things be denoted by the n letters a, b, c, d, &c.

Then it is evident that among the combinations of these letters, r at a time, each of those that contain any particular letter, say a, contains *also* a group of $r-1$ letters out of the remaining $n-1$ letters. Hence there are as many combinations of r letters each containing a as there are combinations of $n-1$ letters taken $r-1$ at a time.

Thus we see that of the combinations of n things taken r at a time, the number of those which contain a is ${}^{n-1}C_{r-1}$;

similarly, the number of those which contain b is ${}^{n-1}C_{r-1}$; and so for each of the n letters.

Hence if all the ${}^nC_r$ combinations be written down, every particular letter will have been written down ${}^{n-1}C_{r-1}$ times; because it will be found in ${}^{n-1}C_{r-1}$ combinations and only once in each of them. Hence the total number of letters written down will evidently be $n \times {}^{n-1}C_{r-1}$.

But the total number of letters written down is also evidently equal to $r \times {}^nC_r$, because each combination contains r letters.

Hence, $r \times {}^nC_r = n \times {}^{n-1}C_{r-1}$,

or, $${}^nC_r = \frac{n}{r} \times {}^{n-1}C_{r-1}.$$

In this formula change n and r first into $n-1$ and $r-1$ respectively, then into $n-2$ and $r-2$ respectively, and so on; thus we have

$${}^{n-1}C_{r-1} = \frac{n-1}{r-1} \times {}^{n-2}C_{r-2};$$

$${}^{n-2}C_{r-2} = \frac{n-2}{r-2} \times {}^{n-3}C_{r-3};$$

&c. &c. &c.

$$^{n-r+2}C_2 = \frac{n-r+2}{2} \times {}^{n-r+1}C_1\,;$$

$$^{n-r+1}C_1 = \frac{n-r+1}{1} \qquad \text{(as is evident).}$$

Now multiply together the vertical columns and cancel like factors from each side ; thus we have

$$^{n}C_r = \frac{n(n-1)(n-2)\ldots(n-r+1)}{r(r-1)(r-2)\ldots 2.1}$$

$$= \frac{n(n-1)(n-2)\ldots(n-r+1)}{\lfloor r}.$$

6. To find for what value of r ^nC_r is greatest.

$$\text{Since } {}^nC_r = \frac{n(n-1)(n-2)\ldots(n-r+2)(n-r+1)}{1.2.3.\ldots(r-1).r}$$

$$\text{and } {}^nC_{r-1} = \frac{n(n-1)(n-2)\ldots(n-r+2)}{1.2.3.\ldots(r-1)},$$

$$\therefore \quad {}^nC_r = {}^nC_{r-1} \times \frac{n-r+1}{r}$$

$$= {}^nC_{r-1} \times \left(\frac{n+1}{r} - 1\right).$$

Evidently then $^nC_r >= \text{ or } < {}^nC_{r-1}$, according as the multiplying factor $\left(\frac{n+1}{r} - 1\right)$ is $>= \text{ or } < 1$;

i.e., according as $\frac{n+1}{r} >= \text{ or } < 2$,

i.e., according as $\frac{\frac{1}{2}(n+1)}{r} >= \text{ or } < 1$,

i.e., according as $r <= \text{ or } > \frac{1}{2}(n+1)$. . . .(A)

Now, since n may be odd or even, let us consider these two cases separately.

(i) Let n be odd, say, equal to $2m+1$, where m is some positive integer.

Then, by (A), $^nC_r >= \text{ or } < {}^nC_{r-1}$, according as $r <= \text{ or } > m+1$.

Hence, of the terms $^nC_1$, $^nC_2$, $^nC_3$, . . . $^nC_{m-1}$, ^nC_m, each is *greater* than the preceding ; . . . . (1)

$^nC_{m+1} = {}^nC_m$; . . . . (2)

and of the terms ${}^nC_{m+1}$, ${}^nC_{m+2}$, ${}^nC_{m+3}$, ... ${}^nC_{2m}$, ${}^nC_{2m+1}$, each is *less* than the preceding. ... ... (3)

Hence, from (1), (2) and (3), it is clear that ${}^nC_m$ and ${}^nC_{m+1}$ are the greatest terms of the series ${}^nC_1$, ${}^nC_2$, ${}^nC_3$, ... ${}^nC_{2m}$, ${}^nC_{2m+1}$;

$$i.e.,\ {}^nC_r \text{ is greatest when } r = m = \frac{(2m+1)-1}{2} = \frac{n-1}{2},$$

$$\text{or when } r = m+1 = \frac{(2m+1)+1}{2} = \frac{n+1}{2}.$$

(ii) Let n be even, say, equal to $2m$, where m is some positive integer.

then, by (A), ${}^nC_r > = \text{ or } < {}^nC_{r-1}$ according as $r < = \text{ or } > m+\frac{1}{2}$.

Hence, of the terms ${}^nC_1$, ${}^nC_2$, ${}^nC_3$, . . . ${}^nC_{m-1}$, ${}^nC_m$, each is *greater* than the preceding ; . . . (α)

and of the terms ${}^nC_m$, ${}^nC_{m+1}$, ${}^nC_{m+2}$, ... ${}^nC_{2m-1}$, ${}^nC_{2m}$, each is *less* than the preceding. ... ... (β)

Hence, from (α) and (β), it is clear that ${}^nC_m$ is the greatest term of the series ${}^nC_1$, ${}^nC_2$, ${}^nC_3$, ... ${}^nC_{2m-1}$, ${}^nC_{2m}$;

$$i.e.,\ {}^nC_r \text{ is greatest when } r = m = \frac{n}{2}.$$

Thus it is proved that if n be odd, ${}^nC_r$ is greatest when $r = \frac{1}{2}(n \mp 1)$ and that if n be even, ${}^nC_r$ is greatest when $r = \frac{n}{2}$.

Exercise (69).

1. How many letters of the word *pantheism* should be taken to form a group so that the number of different groups may be the greatest? In how many of these groups will the letters p and m occur?

2. A person wishes to make up as many different parties as he can out of 20 friends, each party consisting of the same number : how many should he invite at a time? In how many of these would the same man be found?

3. Find the greatest number of different groups that can be formed from the letters of the word *Barouche*, each group consisting of the same number of letters. In how many of them will the letter c occur?

4. Find the ratio of ${}^{20}C_r$ to ${}^{25}C_r$ when each of them has the greatest value possible.

5. Find the difference between the greatest values of ${}^{15}C_r$ and ${}^{12}C_r$.

7. To find the number of permutations of n things taken all together when the things are not all different.

Let the n things be represented by n letters and suppose p of them to be a's, q of them to be b's, r of them to be c's, and the rest to be unlike.

To find the number of ways in which we can put these n letters side by side in n consecutive positions.

Evidently we can do this in the following way: Choose any p of the n positions and put a in each, *then* choose any q of the remaining $(n-p)$ positions and put b in each, *then* choose any r of the remaining $(n-p-q)$ positions and put c in each, and *then* put the $(n-p-q-r)$ different letters in the remaining $(n-p-q-r)$ positions.

Let P, Q, R, D respectively represent the number of ways in which the four operations pointed out above can be performed. Clearly then,

$$P = \frac{\underline{|n}}{\underline{|p}\,\underline{|n-p}},$$

$$Q = \frac{\underline{|n-p}}{\underline{|q}\,\underline{|n-p-q}},$$

$$R = \frac{\underline{|n-p-q}}{\underline{|r}\,\underline{|n-p-q-r}},$$

$$D = \underline{|n-p-q-r}.$$

Now, since for each way of performing the first operation there are Q ways of performing the 2nd, the total number of ways of performing the first two operations $= P \times Q$;

again, since for each way of performing the first two operations there are R ways of performing the third, the total number of ways of performing the first three operations $= P \times Q \times R$;

lastly, for each way of performing the first three operations there being D ways of performing the 4th, the total number of ways of performing the four operations $= P \times Q \times R \times D$.

Hence, the required number of ways

$$= P \times Q \times R \times D$$

$$= \frac{\underline{|n}}{\underline{|p}\,\underline{|q}\,\underline{|r}}.$$

And similarly any other case may be treated.

NOTE. It should be clearly noticed that if the p a's were p unlike letters different from any of the rest, then, for each way of *selecting* p positions for them there would be $\underline{|p}$ ways of *arranging* them ; hence, by the mode of reasoning followed in the present article it is easy to see that if the a's, b's, c's, were respectively p, q and r *unlike* letters, the total number of ways of arranging the n letters would be $(P\underline{|p}) \times (Q\underline{|q}) \times (R\underline{|r}) \times D$ or $\underline{|n}$, a result already obtained, in Art. 3, from a different point of view.

Otherwise :—

The result of this article can also be obtained as follows :—

Let N represent the required number of permutations. Then if in *any one* of these permutations the p a's were changed into p unlike letters different from any of the rest, then *without altering the position of the remaining letters*, from this single permutation alone we could from $\underline{|p}$ different permutations. Hence if this change were made in *each* of the N permutations, the whole number of permutations would be $N \times \underline{|p}$.

[For instance, if there be 3 a's then from the single permutation $b\,c\,a\,f\,g\,a\,a\,k$, when the a's are changed into three different letters a_1, a_2, a_3, we can form the following $\underline{|3}$ permutations *without altering the position of any of the remaining letters* :—(1) $bca_1fga_2a_3k$, (2) $bca_1fga_3a_2k$, (3) $bca_2fga_1a_3k$, (4) $bca_2fga_3a_1k$, (5) $bca_3fga_1a_2k$, (6) $bca_3fga_2a_1k$.]

Similarly, if in each of these new permutations the q letters b were changed into q unlike letters different from any of the rest, the whole number of permutations would be $N \times \underline{|p} \times \underline{|q}$;

and if in each of these permutations again the r letters c were changed into r unlike letters different from any of the rest, the whole number of permutations would be $N \times \underline{|p} \times \underline{|q} \times \underline{|r}$.

Thus when all the letters are different from one another the total number of permutations $= N \times \underline{|p} \times \underline{|q} \times \underline{|r}$.

But the same number also $= \underline{|n}$.

$$\therefore \quad N \times \underline{|p} \times \underline{|q} \times \underline{|r} = \underline{|n} ;$$

$$\therefore \quad N = \frac{\underline{|n}}{\underline{|p}\,\underline{|q}\,\underline{|r}}.$$

Example 1. How many different words can be formed out of the letters of the word *constantinople*? In how many of these will the 3 *n*'s be consecutive letters?

(Calcutta University F. A. Paper, 1876.)

We have altogether 14 letters of which 2 are *o*, 3 are *n*, and 2 are *t*.

Hence, the required number of words

$$= \frac{\underline{|14}}{\underline{|2}\,\underline{|3}\,\underline{|2}}$$

$$= 14.13.12.11.10.9.8.7.6.5$$

$$= 3632428800.$$

If the 3 *n*'s are to be consecutive letters in every word, then they may be regarded as one letter and thus we have only 12 letters of which 2 are *o* and 2 are *t*.

Hence, the number of such words

$$= \frac{\underline{|12}}{\underline{|2}\,\underline{|2}}$$

$$= 12.11.10.9.8.7.6.5.3.2$$

$$= 119750400.$$

Example 2. There are fifteen boat clubs; two of the clubs have each three boats on the river, five others have two, and the remaining eight have one: find an expression for the number of ways in which a list can be formed of the order of the 24 boats, observing that the second boat of a club cannot be above the first.

Since the 1st boat of any club must always occupy a higher position in the list than the 2nd, and the 2nd, a higher position than the 3rd, it is evident that after we have chosen any one set of positions in the list for the boats of any club there is only one way of putting them in these positions, and therefore the boats of any club must be *virtually* regarded as all *alike*.

Thus the number of ways of forming the list is quite the same as the number of permutations of 24 letters taken all together of which 3 are *a*, 3 are *b*, 2 are *c*, 2 are *d*, 2 are *e*, 2 are *f*, 2 are *g*, and the rest unlike.

Hence the required expression

$$= \frac{\underline{|24}}{\{\underline{|3}\}^2\{\underline{|2}\}^5}.$$

Example 3. In how many ways can the letters of the word *multiple* be re-arranged—

(1) without changing the order of the vowels ;

(2) without changing the place of any vowel ;

(3) without changing the relative order of vowels and consonants ?

We have altogether got 8 letters of which two are *l*, and the rest unlike.

(1) If the order of the 3 vowels in any arrangement is to remain the same the vowels must be regarded as like letters, for in that case choosing *any three* positions for them there is *only one* way of putting them in those 3 positions (namely *a* in the first of those positions, *i* in the 2nd and *e* in the 3rd, *i.e.*, in the order *u i e* in which they initially stand).

Hence, the total number of ways of arranging the letters

$$= \frac{\underline{|8}}{\underline{|3}\,\underline{|2}} = 8.7.6.5.2 = 3360.$$

Therefore, the number of ways of *re-arranging the letters*

$$= 3359.$$

(2) If the vowels have to remain *fixed* in their positions the total number of arrangements of the letters is evidently the same as the number of arrangements of the other 5 letters in the remaining 5 positions and is $\therefore\ \frac{\underline{|5}}{\underline{|2}} = 60.$

Hence, the number of ways of *re-arranging* the letters = 59.

(3) If the *relative order* of the vowels and consonants remain the same in every arrangement, then the 1st position will always be occupied by a consonant, the 2nd by a vowel, the 3rd and 4th by consonants, the 5th by a vowel, the 6th and 7th by consonants and the 8th by a vowel ; *i.e.*, the vowels will continue to occupy the 2nd, 5th and 8th positions as they do initially, *irrespective of the order in which they may occupy these positions.*

Now, the vowels can be placed in the specified places in $\underline{|3}$ ways, and the 5 consonants in the remaining 5 places in $\frac{\underline{|5}}{\underline{|2}}$ ways.

Hence, the total number of arrangements of the letters

$$= \lfloor\underline{3} \times \frac{\lfloor\underline{5}}{\lfloor\underline{2}} = 360\,;$$

and ∴ the number of ways of *re-arranging* the letters = 359.

Example 4. How many numbers greater than a million can be formed with the digits 2, 3, 0, 3, 4, 2, 3 ?

Since each number is to consist of not less than 7 digits we shall have to use all the 7 digits in forming the numbers.

Now among these 7 digits there are 2 two's and 3 three's; hence, the total number of ways of arranging the digits

$$= \frac{\lfloor\underline{7}}{\lfloor\underline{2}\,\lfloor\underline{3}} = 420.$$

But out of these arrangements we have to reject those that begin with zero, for they are no numbers.

Now evidently there are as many such arrangements as there are ways of arranging the remaining 6 digits among themselves,

and ∴ their number $= \dfrac{\lfloor\underline{6}}{\lfloor\underline{2}\,\lfloor\underline{3}} = 60.$

Hence, the required number

$$= 420 - 60$$

$$= 360.$$

Example 5. In how many ways can the letters of the word "arrange" be arranged? How many arrangements can be made, (*a*) if the two *r's* are not allowed to come together, (*b*) if neither the two *r's* nor the two *a's* are allowed to come together?

(Madras University F. A. Paper, 1888.)

We have altogether 7 letters of which 2 are *r*, 2 are *a*, and the rest unlike.

Hence, the total number of ways in which the letters can be arranged

$$= \frac{\lfloor\underline{7}}{\lfloor\underline{2}\,\lfloor\underline{2}}$$

$$= 7.6.5.2.3$$

$$= 1260.$$

(*a*) Let x = the required number of arrangements, *i.e.*, those in which the two *r's* are *not* consecutive letters;

and y = the number of those in which they *are* so.

Then we must have $x+y = 1260$.

To find y we must regard rr as one letter, and find the number of arrangements of the six letters a, (rr), a, n, g, e, taken all together.

$$\text{Hence, } y = \frac{\underline{|6}}{\underline{|2}}$$
$$= 6.5.4.3$$
$$= 360.$$
$$\text{Hence, } x = 1260 - 360$$
$$= 900.$$

(*b*) Let N be the required number of arrangements, *i.e.*, those in which *neither* the two r's *nor* the two a's are consecutive letters.

Let l = the number of arrangements in which the two r's *as well as* the two a's are consecutive letters,

m = the number of those in which the two r's are consecutive letters, but the two a's are not so,

n = the number of those in which the two a's are consecutive letters, but the two r's are not so.

Then we must have
$$\text{N} = 1260 - (l+m+n).$$

To find l we have to find the number of arrangements of the five letters (aa), (rr), n, g, e taken all together.

$$\text{Hence } l = \underline{|5} = 120 \quad \ldots \quad \ldots \quad \ldots \quad (\alpha)$$

Also from the solution of the previous part of the problem, we see that

$l+m$ = the number of arrangements in which the two r's are consecutive letters

$$= 360 \quad \ldots \quad \ldots \quad \ldots \quad (\beta)$$

and $l+n$ = the number of arrangements in which the two a's are consecutive letters

$$= 360 \quad \ldots \quad \ldots \quad \ldots \quad (\gamma)$$

Now subtracting (α) from the sum of (β) and (γ) we have

$$l+m+n = 720 - 120$$
$$= 600.$$
$$\text{Hence, N} = 1260 - 600$$
$$= 660.$$

Exercise (70).

1. How many different words can be made out of the letters which form the word *Allahabad*? In how many will the vowels occupy the even places?

(Allahabad University I. E. Paper, 1891.)

2. In how many different ways may the letters of the continued product $c^3 \, d \, e^4 \, f \, g^2$ be written?

3. Find how many words could be made from the letters of the word *Orion*, supposing (i) the letters may stand in any order and (ii) supposing that the two consonants may not stand together.

(Calcutta University F. A. Paper, 1875.)

4. In how many ways can the letters of the word *rationalisation* be re-arranged? In how many of the new arrangements will the 2 *t*'s be together, and in how many will the 3 *a*'s occupy the first three consecutive positions?

5. If as many more words as possible be formed out of the letters of the word *transcendentalism*, in how many of them will the first three positions be occupied by the 3 *n*'s and the last two by the 2 *t*'s?

6. * In how many ways can the letters of the word *tachinnation* be re-arranged without changing the order of the letters *h* and *t*?

7. In how many ways can the letters of the word *Utilitarianism* be re-arranged without changing the position of any of the vowels.

8. In how many ways can the letters of the word *Civilisation* be re-arranged without changing the relative order of vowels and consonants?

9. You are given 12 balls of which 4 are red, 3 black and 5 white; in how many ways can you arrange the balls so that no two white balls may occupy contiguous positions?

10. Suppose from a certain College at the B. A. Examination of the year 1888, 3 candidates obtain first class Honours in each of the following subjects: English, Mathematics and Physical Science, no candidate obtaining Honours in more than one subject; in how many ways may 9 scholarships of different values be awarded to the 9 candidates provided only that due regard be paid to the places obtained by candidates in any one subject.

11. How many different numbers each of six digits can be expressed by means of the digits of the number 121, 202?

(Madras University F. A. Paper, 1886.)

12. How many numbers can be formed with the digits 9, 8, 5, 2, 3, 4, 3, 2, 5, 8, 5, 2, 3, taken all together, so that the even digits may always occupy the even places?

13. A train consists of 12 carriages, of which 5 are first class, 4 second and the rest third class. In how many different ways may the carriages be arranged? In how many different ways may they be arranged so that all the first class may be together?

14. Of 15 balls of which some are white and the rest black, how many should be white so that the number of ways in which the balls can be arranged in a row may be the greatest possible?

8. To find the number of permutations of n things taken r at a time when each may occur once, twice, thrice, &c., up to r times in any permutation.

Let the n letters a, b, c, d, &c, represent the n things. Then it is implied that we have at least r of each of these letters, (*i.e.*, r a's, r b's, r c's, &c.) at our disposal so that in filling up r positions we can put any letter in one or more (as we like), up to all, of these r positions.

Suppose we put a in the 1st position; then in the 2nd position we can put a, or b, or c, or d, *i.e.*, *any one* of the n letters; thus for each way of filling up the 1st position there are n ways of filling up the first two; hence the total number of ways of filling up the first two positions $= n \times n = n^2$.

Again, when the first two positions have been filled up in any one way, there are evidently n ways of filling up the third (for instance a a occupying the first two positions, the 3rd position may be occupied by *any* of the n letters); thus for each way of filling up the first two positions there are n ways of filling up the first three; hence the total number of ways of filling up the first three positions $= n^2 \times n = n^3$.

Continuing this chain of reasoning it is evident that the number of ways in which r positions can be filled up $= n^r$.

NOTE. In working out examples of this nature it is instructive to work out the result in every particular case by the mode of reasoning pointed out above; to apply the formula is more or less mechanical.

Example 1. If there be x things to be given to n persons shew that n^x will represent the whole number of different ways in which they may be given.

Any one of the things can be given away in n ways, for it can be given to any of the n persons; and for *each* of these ways any of the remaining things can also be given away in n ways, for the person who has already got the first thing may also get this. Hence there are altogether n^2 ways of giving away two things.

Again, for each way of giving away two things there being n ways of giving away a third, the total number of ways of giving away three things $= n^2 \times n = n^3$.

Proceeding thus we find that the total number of ways of giving away x things $= n^x$.

Example 2. If there be two kinds of balls, black and white, and at least 4 of each kind, in how many different ways can a ball be put in each of 4 different boxes?

In the 1st box we can put either a white ball or a black ball, but whichever is put we can also then put in the 2nd box either a white or a black ball: thus there are 2×2 or 2^2 ways of putting balls in the first two boxes.

Again, for each way of putting balls in the first two boxes, there being 2 ways of putting a ball in the third, the total number of ways of putting balls in the first three boxes $= 2^2 \times 2 = 2^3$.

Hence, since for each of these ways again there are two ways of putting a ball in the 4th, the required number of ways $= 2^4 = 16$.

Exercise (71).

1. In how many ways can 3 prizes be given away to 7 boys when each boy is eligible for any of the prizes?

2. There are 3 candidates for a Professorship, and one is to be elected by the votes of 5 men; in how many ways can the votes be given?

(Bombay University P. E. Paper, 1888.)

3. If there be 4 different steamers plying between Calcutta and Chandbali, all of them leaving Calcutta together, in how many ways can a gentleman make 7 journeys from Calcutta to Chandbali?

4. In the preceding example what would be the answer if one of the steamers were in dock for repairs when the gentleman made his 3rd journey and if two of them were in dock when he made his 5th journey, on every other occasion all the 4 steamers plying together?

5. Sixteen men compete with one another in running, swimming and riding. How many prize-lists could be made if there were altogether 6 prizes of different values, one for running, 2 for swimming and 3 for riding?

6. A guardian with 6 wards wishes every one of them to study either Law or Medicine or Engineering; in how many ways can he make up his mind with regard to the education of his wards, if every one of them be fit for any of those branches of study?

7. There are m men and n monkeys, n being greater than m. If a man may have any number of monkeys, in how many ways may every monkey have a master?

(Bombay University P. E. Paper, 1891.)

9. To find the number of combinations of n things taken r at a time when each may occur once, twice, thrice, &c., up to r times in any combination.

Let ${}^{n}H_{r}$ denote the number of such combinations, and let the n letters a, b, c, d, &c., represent the n things.

If we write down all the ${}^{n}H_{r}$ combinations, among them the number of those that contain any particular letter, say a *at least once*, will be $= {}^{n}H_{r-1}$; for, each such combination consisting of an a and a group of $r-1$ letters besides, there are altogether as many such combinations as there are ways of forming a group of $r-1$ letters out of the n letters. . . . . (a)

Now, when all the ${}^{n}H_{r}$ combinations are written down, it is evident that the total number of letters written down is $r \times {}^{n}H_{r}$; also since no special favour has been shown to any of the n letters, it is clear that each must have been written down the same number of times, and therefore that number $= (r \times {}^{n}H_{r}) \div n$.

Hence the total number of a's in these combinations clearly $= (r \times {}^{n}H_{r}) \div n$. . . . . (β)

But the total number of a's may also be counted in a different way:—

Of the ${}^{n}H_{r}$ combinations take those that contain a at least once, and remove from each the single letter a which is common to them all. We thus get, as is clear from (α), ${}^nH_{r-1}$ a's ; and there are left all possible combinations of the n letters, taken $r-1$ at a time, the aggregate number of a's in which, by (β), $= \{(r-1)\times {}^nH_{r-1}\}\div n$.

Hence the total number of a's may also be put down

$$= {}^nH_{r-1} + \frac{r-1}{n}\times {}^nH_{r-1}$$

$$= \frac{n+r-1}{n}\times {}^nH_{r-1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\gamma)$$

From (β) and (γ) therefore we must have

$$\frac{r}{n}\times {}^nH_r = \frac{n+r-1}{n}\times {}^nH_{r-1}$$

or,

$${}^nH_r = \frac{n+r-1}{r}\times {}^nH_{r-1}.$$

Similarly,

$$\begin{aligned} {}^nH_{r-1} &= \frac{n+r-2}{r-1}\times {}^nH_{r-2}, \\ {}^nH_{r-2} &= \frac{n+r-3}{r-2}\times {}^nH_{r-3}, \\ &\ldots\ldots\ldots\ldots \\ {}^nH_3 &= \frac{n+2}{3}\times {}^nH_2, \\ {}^nH_2 &= \frac{n+1}{2}\times {}^nH_1. \end{aligned}$$

Now ${}^nH_1$ is obviously $= n$.

Hence, multiplying and cancelling factors common to both sides, we have

$$\begin{aligned} {}^nH_r &= \frac{n(n+1)(n+2)\ldots(n+r-1)}{\underline{|r}} \\ &= \frac{\underline{|n+r-1}}{\underline{|r}\times\underline{|n-1}}. \end{aligned}$$

Obs. Thus ${}^nH_r = {}^{n+r-1}C_r$, that is the number of combinations of n things taken r at a time when repetitions are allowed is the same as the number of combinations of $n+r-1$ things taken r at a time when repetitions are not allowed.

Exercise (72).

1. In how many ways can a group of 5 letters be formed out of 5 a's, 5 b's, 5 c's and 5 d's ?

2. In how many ways can you select six coins out of 20 rupees, 10 eight-anna pieces and 7 four-anna pieces ?

3. If there be three different kinds of mangoes for sale in a market, in how many ways can you purchase 25 mangoes ?

4. In how many ways can a party of six men be selected out of 6 Hindus, 6 Brahmos, 6 Christians and 6 Mahommedans ?

10. To find the number of permutations or combinations of n things taken r at a time when the things are not all different.

The method can be best illustrated by solving a few particular cases.

Example 1. To find the number of combinations of the letters of the word *proportion* taken 6 at a time.

We have altogether got 10 letters of which 2 are p, 2 are r, 3 are o, and the rest different ; thus the letters are :—

$$\begin{array}{cccccc} (1) & (2) & (3) & (4) & (5) & (6) \\ [p, p]; & [r, r]; & [o, o, o]; & [t]; & [i]; & [n]. \end{array}$$

Now in forming combinations of these letters taking 6 at a time, it is evident, that, in some combinations all the letters will be different whereas in others not so ; hence the combinations can be divided into the following classes :—

(i) Combinations in which all the 6 letters are different.

(ii) Combinations in which 4 letters are different and the other two alike.

(iii) Combinations in which 3 letters are different and the other 3 alike.

(iv) Combinations in which 2 letters are different and of the other 4 two are alike of one kind and two alike of another.

(v) Combinations in which there is one letter different from any of the rest which consist of 3 letters alike of one kind and 2 alike of another.

(vi) Combinations consisting of 3 different pairs of like letters.

It is easy to see that the above classification is *exhaustive*. Now to find out the number of combinations in each of these classes.

(i) Since we have altogether got 6 different letters, namely, p, r, o, t, i, n, there is only one combination in which all the letters are different, *i.e.*, the number of combinations in this class = 1.

(ii) Combinations of this class are formed by taking (p, p) with any 4 of the 5 letters r, o, t, i, n; taking (r, r) with any 4 of the 5 letters p, o, t, i, n; and taking (o, o) with any 4 of the 5 letters p, r, t, i, n. Hence the number of combinations in this class $= 3 \times {}^5C_4 = 3 \times {}^5C_1 = 3 \times 5 = 15$.

(iii) Combinations of this class are formed by taking the single group (o, o, o) with any three of the 5 letters p, r, t, i, n. Hence the number of combinations in this class $= {}^5C_3 = {}^5C_2 = 10$.

(iv) Combinations of this class are formed by taking $(p, p; r, r)$ with any two of the 4 letters o, t, i, n; taking $(p, p; o, o)$ with any two of the 4 letters r, t, i, n; and taking $(r, r; o, o)$ with any two of the 4 letters p, t, i, n. Hence the number of combinations in this class $= 3 \times {}^4C_2 = 3 \times 6 = 18$.

(v) Combinations of this class are formed by taking $(o, o, o; p, p)$ with any one of the 4 letters r, t, i, n; and taking $(o, o, o; r, r)$ with any one of the 4 letters p, t, i, n. Hence the number of combinations in this class $= 2 \times {}^4C_1 = 2 \times 4 = 8$.

(vi) There is only one combination in this class (namely, $p, p; r, r; o, o$).

Hence the total number of combinations

$$= 1 + 15 + 10 + 18 + 8 + 1$$
$$= 53.$$

Example 2. Find the number of permutations of the letters of the word *examination* taken 4 at a time.

We have altogether got 11 letters of which 2 are a, 2 are i, 2 are n, and the rest different; thus the letters are :—

$$\begin{array}{cccccccc} (1) & (2) & (3) & (4) & (5) & (6) & (7) & (8) \\ [a, a]; & [i, i]; & [n, n]; & [e]; & [x]; & [m]; & [t]; & [o]. \end{array}$$

Now, if we form combinations of these letters taking 4 at a time, evidently in some combinations all the 4 letters will be different and in others not so; hence the combinations can be divided into the following classes :—

(i) Combinations in which all the 4 letters are different.

(ii) Combinations in which 2 letters are different and the other two alike.

(iii) Combinations in which 2 letters are alike of one kind and 2 letters alike of another.

It is easy to see that the above classification is *exhaustive*. Now to find the number of combinations in each class and thence the number of permutations.

(i) Combinations of this class are formed by taking any 4 of the 8 letters a, i, n, e, x, m, t, o. Hence the number of combinations in this class $= {}^8C_4$.

But from each of these combinations we get $\underline{|4}$ permutations; $\therefore$ the number of permutations obtained from this class of combinations $= {}^8C_4 \times \underline{|4} = 8.7.6.5 = 1680$.

(ii) Combinations of this class are formed by taking (a, a) with any two of the 7 letters i, n, e, x, m, t, o; taking (i, i) with any two of the 7 letters a, n, e, x, m, t, o; and taking (n, n) with any two of the 7 letters a, i, e, x, m, t, o. Hence the number of combinations in this class $= 3 \times {}^7C_2$.

But each of these combinations gives us $\dfrac{\underline{|4}}{\underline{|2}}$ permutations; hence the number of permutations obtained from combinations of this class $= 3 \times {}^7C_2 \times \dfrac{\underline{|4}}{\underline{|2}} = 3 \times 21 \times 12 = 756$.

(iii) Combinations of this class are evidently, (a, a, i, i); $(a, a, n, n,)$; and (i, i, n, n); *i.e.*, they are altogether 3 in number.

But each of these combinations gives us $\dfrac{\underline{|4}}{\underline{|2}\,\underline{|2}}$ permutations; hence the number of permutations obtained from combinations of this class $= 3 \times \dfrac{\underline{|4}}{\underline{|2}\,\underline{|2}} = 3 \times 6 = 18$.

Hence the total number of permutations $= 1680 + 756 + 18$
$= 2454$.

Example 3. How many words, each consisting of two vowels and two consonants can be made out of the letters of the word 'devastation'? In how many of them will the two t's be together?

(Calcutta University F. A. Paper, 1879).

We have got altogether 6 consonants, namely, *d*, *v*, *s*, *t*, *t*, *n* of which 2 are alike; and 5 vowels, namely, *e*, *a*, *a*, *i*, *o* of which also 2 are alike.

The combinations of the consonants taken two at a time can evidently be divided only into the two following classes :—

(i) The single combination (*t*, *t*) which consists of two like letters.

(ii) Combinations formed by taking any two of the 5 letters *d*, *v*, *s*, *t*, *n*, (*i.e.*, those in which the letters are different); the number of such combinations being evidently $= {}^5C_2 = 10$.

And the combinations of the vowels taken 2 at a time are also divisible into the two following classes :—

(1) The single combinations (*a*, *a*) which consists of two like letters.

(2) Combinations formed by taking any two of the 4 letters *e*, *a*, *i*, *o*, (*i.e.*, those in which the letters are different); the number of such combinations being evidently $= {}^4C_2 = 6$.

Now, in finding the number of words that can be formed with 2 consonants and 2 vowels we must proceed as follows :—

1st, taking (*t*, *t*) with (*a*, *a*) we get one combination of 2 consonants and 2 vowels; and the number of words that we get from this combination is clearly $= \dfrac{\underline{|4}}{\underline{|2}\,\underline{|2}} = 6$.

2ndly, taking (*t*, *t*) with any of the 6 vowel-collections in (2) we get 6 combinations of 2 consonants and 2 vowels; each of these combinations, containing only 2 like letters, evidently gives us $\dfrac{\underline{|4}}{\underline{|2}}$ words. Hence, the total number of words obtained from these combinations

$$= 6 \times \frac{\underline{|4}}{\underline{|2}} = 72.$$

3rdly, taking each of the 10 consonant-collections in (ii) with (*a*, *a*) we get 10 combinations of 2 consonants and 2 vowels; each of these combinations having only two letters alike evidently gives us $\dfrac{\underline{|4}}{\underline{|2}}$ words. Hence the total number of words formed from these combinations $= 10 \times \dfrac{\underline{|4}}{\underline{|2}} = 120$.

4thly, taking each of the 10 consonant-collections in (ii) with each of the 6 vowel-collections in (2), we get altogether 60 combinations of 2 consonants and 2 vowels; each of these combinations consisting of 4 different letters evidently gives us $\underline{|4}$ words. Hence the total number of words obtained from these combinations $= 60 \times \underline{|4} = 1440$.

Hence, the total number of words

$$= 6 + 72 + 120 + 1440$$
$$= 1638.$$

To find the number of words in which the two *t*'s are together.

From the 1st and 2nd of the above four cases we see that the group of letters (*t*, *t*) occurs altogether in 7 combinations; in one of these combinations, namely, (*t*, *t*, *a*, *a*) the vowels are like and in each of the other 6 the vowels are unlike. Hence considering the two *t*'s as one letter, the number of words obtained from the 1st combination $= \dfrac{\underline{|3}}{\underline{|2}} = 3$; and the number of words obtained from the other six $= 6 \times \underline{|3} = 36$.

Hence the total number of words in which the two *t*'s are together $= 3 + 36 = 39$.

Exercise (73).

1. Find the number of permutations which can be formed from the letters of the word *thatch* taken 3 at a time.

2. Find the number of different groups of 4 letters that can be formed from the letters of the word *succeeding*.

3. Find the number of ways in which the letters of the word *toleration* may be arranged taking 4 at a time.

4. Find the number of permutations that can be formed out of the letters of the word *legitimate*, taken 5 at a time.

5. Find the number of combinations of the letters of the word *alliteration* taken four at a time.

(Calcutta University F. A. Paper, 1889.)

6. Find the number of different collections of 7 letters that can be formed from the letters of the word *accommodation*.

7. How many words each consisting of 7 letters can be formed from the letters of the word in the preceding example?

8. Find the number of words each consisting of 3 consonants and 3 vowels that can be formed from the letters of the word *circumference.*

9. In how many of the words found in the preceding example will the 3 *c*'s be together ?

10. Find how many numbers greater than 1000 can be formed from the digits 112340, taken four at a time.

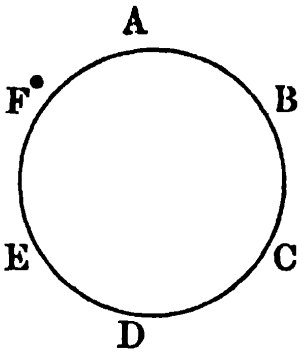
(Madras University F. A. Paper, 1889.)

11. Miscellaneous Examples.

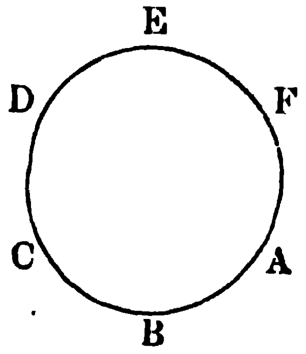
Example 1. In how many ways can *n* persons form a ring ? In how many ways can 7 Englishmen and 7 Americans sit down at a round table, no two Americans being together ?

(i) In placing a number of things round a circle we regard two arrangements as different only if they are different as regards the *relative positions* of the things. Thus if A, B, C, D, E, F denote a number of things, the following two arrangements are *not* different :—

(1)

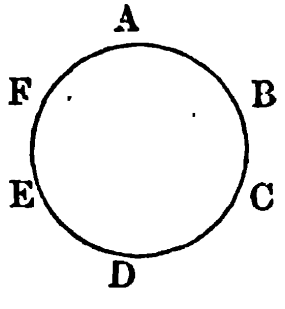


(2)

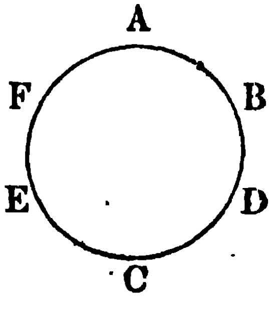


but the following two are—

(3)



(4)



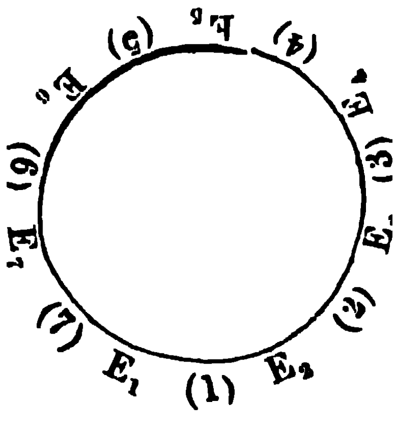
In (1) and (2) if we start from A and travel round the circle in the same direction as that in which the hands of a watch revolve, we meet with the same letters precisely in the *same order*, and hence, these two arrangements are regarded as the same (although the letters do not occupy the same absolute positions in each); but in (3) and (4) this is not the case, and so they are regarded as different.

Hence if we want to place a number of things round a circle we obtain the different arrangements by putting one of the things in a fixed position and changing the positions of the rest in all possible ways.

Hence, it is clear that the number of ways in which n persons can form a ring $= \underline{|n-1}$, for if one of them be placed in a fixed position the remaining $n-1$ persons can be seated in the remaining $n-1$ positions in $\underline{|n-1}$ ways.

(ii) Put one of the Englishmen in a fixed position and then arrange the other six in all possible ways. Thus the number of ways in which the Englishmen may be seated $= \underline{|6} = 720$.

Let one arrangement of the Englishmen be as shewn in the figure.



Then evidently there are only 7 positions (marked (1), (2), (3), &c. in the figure) for the Americans, consistently with the given condition.

Hence for each way of seating the Englishmen there are $\underline{|7}$ ways of seating the Americans.

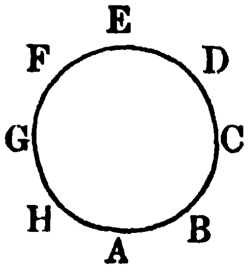
Hence, the total number of ways of seating the Englishmen and the Americans $= \underline{|6} \times \underline{|7}$

$= 3628800.$

Example 2. In how many ways can 8 persons be seated at a round table, so that all shall not have the same neighbours in any two arrangements?

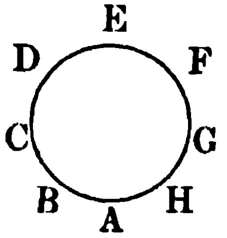
The total number of ways in which the 8 persons can be seated, one of them occupying a fixed position $= \underline{|7} = 5040$.

Let one of these arrangements be



then A remaining fixed, evidently of the 5040 arrangements

another will be



and

it is evident that in both of these arrangements *all* the 8 persons have got the same neighbours (for instance, the neighbours of D, in the 1st as well as in the 2nd of these arrangements, are C and E, although the left-hand neighbour in the 1st arrangement becomes the right-hand neighbour in the 2nd, and *vice versa*; similarly, *every one* of the 8 persons have got the same neighbours in these two arrangements).

It is evident then that in whatever order B, C, D, E, F, G, H may be arranged from the *right to the left* of A, exactly in the same order they may be arranged from his *left to his right*; and it is clear, as shewn above, that only in two such arrangements all the 8 persons have got the same neighbours.

Hence, *on the given condition*, two such arrangements must be counted as one, and ∴ the required number of arrangements

$$= \frac{1}{2} \times 5040$$
$$= 2520.$$

Example 3. A gentleman invites a party of $m+n$ friends to dinner, and places m at one table and n at another, the tables being round. Find the number of ways in which he can arrange the guests.

Every time that he chooses a group of m friends for one of the tables, there is left a party of n friends for the other. Hence the total number of ways in which the friends can be divided into parties of m and n $= {}^{m+n}C_m = \dfrac{|\underline{m+n}}{|\underline{m}\,|\underline{n}}$.

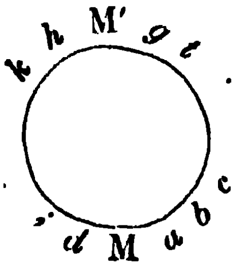
Now, after having chosen the parties in any one way he can evidently place the m friends at one table in $|\underline{m-1}$ ways, and the n persons at the other table in $|\underline{n-1}$ ways; hence *for each way of choosing the parties* there are $|\underline{m-1} \times |\underline{n-1}$ ways of arranging the guests.

Hence the total number of ways of arranging the guests

$$= \frac{|\underline{m+n}}{|\underline{m}\,|\underline{n}} \times \left\{|\underline{m-1} \times |\underline{n-1}\right\}$$

$$= \frac{|\underline{m+n}}{mn}.$$

Exapmle 4. There are $2n$ guests at a dinner party. Supposing that the master and mistress of the house have fixed seats opposite one another, and that there are two specified guests who must not be placed next to one another, find the number of ways in which the company can be placed.



In the accompanying diagram let M, M′ represent seats of the master and mistress respectively, and let a, b, c, &c., represent the $2n$ seats.

Let the guests who must not be placed next to one another be called G_1 and G_2.

Now put G_1 at a, and G_2 at any other position, say, c; then the remaining $2n-2$ guests can be arranged in the remaining $2n-2$ positions in $|\underline{2n-2}$ ways.

But when G_1 is at a, G_2 can have any other position, excepting the position b. Hence there will be altogether $(2n-2) \times |\underline{2n-2}$ arrangements of the guests when G_1 is at a.

The same number of arrangements may be made when G_1 is at g, or h, or p. Hence for these positions (a, g, h, p) of G_1 there are altogether $4(2n-2)\underline{|2n-2}$ ways of arranging the guests ... ... (1)

Again, put G_1 at b and G_2 at any other position, say g; then the remaining $2n-2$ guests can be arranged in the remaining $2n-2$ positions in $\underline{|2n-2}$ ways.

But when G_1 is at b there are altogether $2n-3$ positions for G_2, namely, *every other* position *excepting* a and c. Hence there will be altogether $(2n-3)\underline{|2n-2}$ arrangements of the guests when G_1 is at b.

The same number of arrangements can be made when G_1 is at *any other* position *excepting the four positions* a, g, h, p.

Hence, for these $2n-4$ positions of G_1 there will be altogether $(2n-4)(2n-3)\underline{|2n-2}$ arrangements of the guests. ...(2)

Hence, from (1) and (2), the total number of ways of arranging the guests

$$\begin{aligned}&= 4(2n-2)\underline{|2n-2} + (2n-4)(2n-3)\underline{|2n-2}\\ &= \{4(2n-2)+(2n-4)(2n-3)\}\underline{|2n-2}\\ &= (4n^2-6n+4)\underline{|2n-2}.\end{aligned}$$

NOTE. The answer may also be found by subtracting the number of arrangements in which G_1 and G_2 are together, from the total number of arrangements that can be made, and this is left as an exercise for the student.

Example 5. Prove that ${}^4C_1 + {}^4C_2 + {}^4C_3 + {}^4C_4 = 2^4 - 1$.

Let the 4 things be denoted by a_1, a_2, a_3, a_4.

Evidently then the combinations of the things taken singly are a_1, a_2, a_3, a_4; combinations of them taken 2 at a time are $a_1a_2, a_1a_3, a_1a_4, a_2a_3, a_2a_4, a_3a_4$; combinations of them taken 3 at a time are $a_1a_2a_3, a_1a_2a_4, a_1a_3a_4, a_2a_3a_4$; and lastly the combination of them taken all together is $a_1a_2a_3a_4$.

Hence, it is clear that the terms, with the exception of the 1st term which is 1, of the product $(1+a_1)(1+a_2)(1+a_3)(1+a_4)$ give us all possible combinations of the 4 letters a_1, a_2, a_3, a_4.

Hence the total *number* of combinations

$$\begin{aligned}&= \text{number of terms in the above product } \textit{minus} \text{ one}\\ &= (2\times2\times2\times2)-1\end{aligned}$$

i.e., ${}^4C_1 + {}^4C_2 + {}^4C_3 + {}^4C_4 = 2^4 - 1$.

NOTE. The student can easily verify the result by actual calculation.

Cor. Arguing as above it may be shewn that ${}^nC_1 + {}^nC_2 + {}^nC_3 + \&c. + {}^nC_n = 2^n - 1$.

Example 6. How many different sums can be formed with the following coins : a farthing, a penny, a six-pence, a shilling, a half-crown, a crown, a half-sovereign and a sovereign ?

We have altogether got 8 coins.

Hence, the number of different sums that can be formed

$$= {}^8C_1 + {}^8C_2 + {}^8C_3 + \quad . \; . \; . \; . \; . \; . \quad + {}^8C_7 + {}^8C_8$$
$$= 2^8 - 1$$
$$= 255.$$

Example 7. Out of 3 Mathematical books, 4 Scientific books and 5 Literary books how many different collections can be made, each collection consisting of (i) one book on each subject (ii) *at least* one book on each subject ?

(i) With any one of the Mathematical books we can take any one of the Scientific books ; thus there are altogether 3×4 ways of making a set containing one Mathematical book and one Scientific book.

Now with any of these 3×4 sets we can take any one of the Literary books ; hence the total number of ways of choosing a set containing one Mathematical book, one Scientific book and one Literary book $= (3 \times 4) \times 5 = 60$.

(ii) Since *at least* one book on each subject must be taken to form a set we can choose one, two, or more books from the book on each subject.

Now the total number of ways in which we can select one or more of the Mathematical books $= 2^3 - 1 = 7$; the total number of ways in which we can select one or more of the Scientific books $= 2^4 - 1 = 15$; and the total number of ways in which we can select one or more of the Literary books $= 2^5 - 1 = 31$.

Now to form a set containing at least one book on each subject we must gather together any one of the 7 different selections of the Mathematical books, any one of the 15 different selections of the Scientific books, and any one of the 31 different selections of the Literary books ; hence the total number of ways of forming such a set $= 7 \times 15 \times 31 = 3255$.

Example 8. Find the sum of all the numbers which can be formed with all the digits 1, 2, 3, 4, 5, in the scale of 10.

Evidently one of the numbers formed with the 5 digits a, b, c, d, e, is

$$10^4.a + 10^3.b + 10^2.c + 10.d + e$$

and similarly every other number can be written.

From this it is clear that a will have 10^4 as its co-efficient in as many numbers as there are ways of putting the other digits with the other powers of 10.

Hence $10^4.a$ will occur altogether in $\underline{|4}$ numbers.

Similarly, each of $10^4.b$, $10^4.c$, $10^4\,d$, $10^4.e$ will occur in $\underline{|4}$ numbers.

Hence, if all the numbers formed with the digits be written one below the other, thus :—

$$10^4.a + 10^3.b + 10^2.c + 10.d + e$$
$$10^4.a + 10^3.c + 10^2.b + 10.d + e$$
&c. &c. &c.

the sum of the first column will be

$$= \underline{|4} \times 10^4.(a+b+c+d+e) \quad \ldots \quad \ldots \quad (1)$$

Again, since $10^3.a$, $10^3.b$, $10^3.c$, $10^3.d$, $10^3.e$ will each occur in $\underline{|4}$ numbers, the sum of the 2nd column will be

$$= \underline{|4} \times 10^3.(a+b+c+d+e) \quad \ldots \quad \ldots \quad (2)$$

Similarly, the sum of the 3rd column

$$= \underline{|4} \times 10^2.(a+b+c+d+e) \quad \ldots \quad \ldots \quad (3)$$

the sum of the 4th column

$$= \underline{|4} \times 10.(a+b+c+d+e) \quad \ldots \quad \ldots \quad (4)$$

and the sum of the 5th column

$$= \underline{|4} \times (a+b+c+d+e) \quad \ldots \quad \ldots \quad \ldots \quad (5)$$

Hence, from (1), (2), (3), (4), (5), the sum of all the numbers that can be formed with the five digits

$$= \underline{|4} \times (a+b+c+d+e) \times (10^4 + 10^3 + 10^2 + 10 + 1).$$

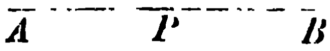
Hence, the required sum

$$= \underline{|4} \times (1+2+3+4+5) \times (10{,}000 + 1000 + 100 + 10 + 1)$$
$$= 24 \times 15 \times 11111$$
$$= 3999960.$$

Example 9. There are n lines in a plane, no two of which are parallel, and no three pass through the same point. Their points of intersection are joined. Show that the number of fresh lines thus introduced is

$$\frac{n(n-1)(n-2)(n-3)}{8}.$$

Let AB be *any* one of the n straight lines and suppose it is intersected by some other straight line CD at P.



Then it is clear that AB contains $n-1$ of the points of intersection, because it is intersected by the remaining $n-1$ straight lines in $n-1$ different points. For a similar reason *each* of the other straight lines contains $n-1$ points. Hence the aggregate number of points contained in the n straight lines $= n(n-1)$. But in making up this aggregate each point has evidently been counted twice; for instance, the point P has been counted once among the points situated on AB and again among those on CD.

Hence the actual number of points $= \frac{n(n-1)}{2}$. Now to find the number of *new* lines formed by joining these points:—

The number of *new* lines passing through P is evidently equal to the number of points lying outside the lines AB and CD, for we get a *new* line by joining P with each of *those points* only.

Now since each of the lines AB and CD contains $n-2$ points *besides* the point P, the number of points situated on AB and $CD = 2(n-2)+1 = 2n-3$. Therefore the number of points *outside* AB and CD

$$= \frac{n(n-1)}{2} - (2n-3).$$

Hence the number of *new* lines passing through P, and similarly through each of the other points,

$$= \frac{n(n-1)}{2} - (2n-3);$$

and ∴ the aggregate number of new lines passing through the points

$$= \frac{n(n-1)}{2}\left\{\frac{n(n-1)}{2} - (2n-3)\right\}.$$

But in making up this aggregate, *every* new line is counted *twice*; for instance, if Q be one of the points outside AB and CD, the line PQ is counted *once* among the lines passing through P and *again* among those passing through Q.

Hence, the actual number of fresh lines introduced

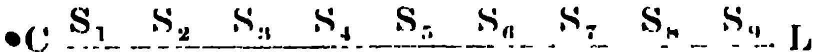
$$= \tfrac{1}{2}\left[\frac{n(n-1)}{2}\left\{\frac{n(n-1)}{2}-(2n-3)\right\}\right]$$

$$= \frac{n(n-1)}{4}\cdot\frac{n^2-5n+6}{2}$$

$$= \frac{n(n-1)(n-2)(n-3)}{8}.$$

N. B. The example has been worked out by the help of most elementary principles and the student is strongly recommended to see that he comprehends each step fully before proceeding to the next. The mode of reasoning followed in the solution is expected to give the learner clearer view of the matter than any mechanical application of formulæ.

Example 10. A train in going from Cambridge to London stops at nine intermediate stations. Six persons enter the train during the journey with six different tickets. How many different sets of tickets may they have had?



The figure explains itself: S_1, S_2, S_3, &c., S_9 representing the nine intermediate stations.

Let us first find the number of different tickets altogether available *during the journey*.

Evidently at S_1, 9 different tickets are available one for each of the remaining 9 stations; similarly at S_2, 8 different tickets are available; and so on.

Hence it is clear that the total number of different tickets available during the journey $= 9+8+7+6+5+4+3+2+1$
$= 45.$

Hence the six different tickets must be *any* six of these 45; and there are evidently as many different sets of 6 tickets as there are combinations of 45 things taken 6 at a time.

Hence, the required number

$$= {}^{45}C_6$$

$$= \frac{45.44.43.42.41.40}{1.2.3.4\,5.6}$$

$$= 3\times 11\times 43\times 7\times 41\times 20$$

$$= 8145060.$$

NOTE. The tickets are all supposed to be of *the sam class*. If different classes of tickets were considered, the total number of different tickets would be 45 × the number of classes of tickets.

Exercise (74).

1. Ten candidates pass an examination in the first division. Three scholarships of different values are to be given among these men irrespective of their places in the examination. How many different scholarships' lists could be made? How many could be made if the scholarships were of equal value?

(Calcutta University F. A. Paper, 1873.)

2. In how many ways can 10 Mahomedans and 8 Hindus sit down at a round table, no two Hindus being together.

3. In how many different ways can 10 beads of 10 different colours be strung on a thread to form a ring?

4. In how many ways can 6 persons be seated at a round table so that all shall not have the same neighbours in any two arrangements?

5. A gentleman invites a party of 12 friends to dinner, and places 6 at one table and 6 at another, the tables being round. Find the number of ways in which he can arrange the guests.

6. There are 12 guests at a dinner party. Supposing that the master and mistress of the house have fixed seats opposite one another, and that there are two specified guests who must always be placed next to one another; find the number of ways in which the company can be placed.

7. Find the number of combinations of 16 things, 8 of which are alike, taken 8 at a time.

8. A gentleman has got 5 sorts of note paper, 8 different pens, and 4 different inkstands; in how many ways can he begin to write a letter?

9. Suppose in a certain Algebraical Exercise book there are 15 examples on Arithmetical Progression, 20 examples on Permutations and Combinations, and 18 examples on the Binomial Theorem; in how many ways can a teacher select for his pupils not more than 2 examples from each of these sets?

10. Suppose in a certain town there are 3 H. E. Schools, in the 1st of which there are 12 teachers, in the 2nd 10, and in the 3rd 8; if not less than 2 and not more than 3 teachers from each school must assist the Principal of the local College in superintending the University Examinations, find the number of ways in which supervisors can be selected from the schools.

11. How many different collections of letters can be made by taking at least 4 letters from each of the words *length*, *stroke*, and *number*?

12 How many different selections of letters can be made by taking at least one letter from each of the words *random*, *noble* and *integral*?

13. In how many ways is it possible to draw a sum of money from a bag containing a sovereign, a half-sovereign, a crown, a florin, a shilling, a penny, and a farthing?

14. Find in how many ways groups of 8 persons, 4 ladies and 4 gentlemen, can be formed out of 8 ladies and 9 gentlemen, subject to the condition that two particular gentlemen are not *simultaneously* to be in the same group with a particular lady.

(Madras University F. A. Paper, 1885.)

15. A telegraph has m arms, and each arm is capable of n distinct positions; find the total number of signals which can be made with the telegraph, supposing that all the arms are to be used to form a signal.

16. A telegraph has five arms, and each arm is capable of 4 distinct positions, *including the position of rest*; what is the total number of signals that can be made?

[When all the arms are in positions of rest no signal can be made.]

17. There are $n-1$ sets containing $2a$, $3a$, &c., na things respectively: shew that the number of combinations which can be formed by taking a out of the first, $2a$ out of the second, and so on, for each combination, is $\dfrac{\underline{|na}}{(\underline{|a})^n}$.

18. You are given three classes of letters:—(a_1, a_2, a_3, a_4); (b_1, b_2, b_3); (c_1, c_2). Prove that the total number of combinations which can be made with these letters when no two of the same class enter into any combination $= 5 \times 4 \times 3 - 1$.

[The a's can be disposed of in 5 ways, for we may take any of the four or *none*. Similarly, the b's and c's can be disposed of in 4 and 3 ways respectively. Hence the number of different ways of disposing of all the letters $= 5 \times 4 \times 3$. But this includes the case in which no letter is taken from any class. Hence &c.]

19. A drawer is fitted with n compartments, and each compartment contains n counters, no two of which are marked alike. Prove that the total number of combinations which can be made with these counters, when no two out of the same compartment enter into any combination, is $(n+1)^n - 1$.

20. If P_r denote the number of permutations of n different things taken r at a time, shew that

$$P_1 + \frac{P_2}{\underline{|2}} + \frac{P_3}{\underline{|3}} + \cdot \cdot \cdot + \frac{P_n}{\underline{|n}} = 2^n - 1.$$

(Madras University F. A. Paper, 1880.)

[See example 5 worked out on page 301.]

21. Shew that 154 numbers less than 1000 and divisible by 5 can be formed with the ten digits each digit not occurring more than once in each number.

(Madras University F. A. Paper, 1883.)

22. Find the sum of all numbers greater than 10,000 formed by using the digits 1, 3, 5, 7, 9.

23. If n points in a plane be joined in all possible ways by indefinite straight lines, and if no two of the straight lines be coincident or parallel, and no three pass through the same point (with the exception of the n original points), then the number of points of intersection, exclusive of the n points will be

$$\frac{n(n-1)(n-2)(n-3)}{8}.$$

24. A mail train from Calcutta to Goalundo stops at 12 intermediate stations. Seventy-five persons enter an intermediate carriage during the journey with 75 different tickets. How many different sets of tickets may they have had?

25. Twenty-five passengers arrive at a Railway station and proceed to the neighbouring village. At the station, there are 2 coaches accommodating 4 each, and 3 carts accommodating 3 each. Find the number of ways in which they can proceed to the village, assuming (1) that the conveyances are always fully occupied, and (2) that the conveyances are all distinguishable from each other. (Madras University F. A. Paper, 1881.)

[The passengers can be divided into parties of 17 and 8 in ${}^{25}C_{17}$ ways, and for each way of accommodating the first party there will be ${}^{8}C_{4}$ ways of accommodating the second.]

26. Prove by the theory of combinations that the product of any n consecutive integers is divisible by $\underline{|n}$.

27. Show that

$$ {}^{n+1}C_r = {}^nC_r + {}^nC_{r-1}. $$

28. Shew that 2.6.10.14 . . . . to n factors is equal to $(n+1)(n+2)(n+3)$ . . . to n factors.

(Madras University F. A. Paper, 1888.)

[We have $(n+1)(n+2)(n+3)$ . . . . to n factors $= \dfrac{\lfloor 2n}{\lfloor n} =$ &c. See example 3 worked out on page 271.]

29. Prove, without assuming the formula for the number of permutations, that

$$ {}^nP_r = {}^{n-1}P_r + r \times {}^{n-1}P_{r-1}. $$

30. In how many ways can eight different things be divided between two persons ?

31. In how many ways can the following books be divided between two students :—3 copies of Todhunter's Algebra, 5 copies of Hamblin Smith's Trigonometry and 7 copies of Milton's Paradise Lost ?

32 If X_r denote the number of permutations of x different things taken r together, shew that

$$ (M+N)_r = M_r + r\,M_{r-1}.N_1 + \frac{r(r-1)}{1.2}\,M_{r-2}N_2 + \;.\;.\;.\;. $$
$$.\;.\;.\; + r.M_1N_{r-1} + N_r. $$

(Madras University F. A. Paper, 1879.)

[Any $(m+n)$ things can be divided into two groups, one containing m things and the other, n things. Hence it is easy to see that any selection of r things out of the $(m+n)$ things can be placed under one or other of the following heads of classification :—(1) r things all from the 1st group, (2) $(r-1)$ things from the first group and 1 thing from the 2nd, (3) $(r-2)$ things from the first group and 2 things from the 2nd, (4) $(r-3)$ things from the 1st group and 3 things from the 2nd, and so on.]

33. There are four sets of balls of four different colours, red, white, indigo, black ; each set consisting of four balls marked 1, 2, 3, 4. Find the number of ways in which the balls can be put into four different boxes, each box containing four balls marked 1, 2, 3, 4, all of different colours.

[Let (R_1, R_2, R_3, R_4), (W_1, W_2, W_3, W_4), (I_1, I_2, I_3, I_4), and (B_1, B_2, B_3, B_4) denote the respective balls of the different colours. Let one of the groups of balls that can be put into the first box be $R_1\ W_2\ I_3\ B_4$. Then R_2, R_3, R_4, can be put into the other three boxes in $\lfloor 3$ ways ; and it can be easily seen that for each of these ways there are only *four* ways of distributing the remaining 9 balls among those boxes.]

34. In how many ways can a pack of 52 cards be distributed equally among four persons so that,—(*a*) each may have ace, king, queen and knave of the *same suit*,—(*b*) each may have ace, king, queen and knave *all of different suits.*

(Madras University F. A. Paper, 1878.)

[In solving (*b*) take the help of the last example.]

CHAPTER XVI.

MATHEMATICAL INDUCTION.

This is a method of proof by which the truth of a general theorem is deduced from that of a particular case of it. First of all we find by actual verification that the theorem is true in certain simple cases. We then proceed to shew that *if* the theorem be true in *any* one case *whatever*, it must be true in the *next* case This fact being established, the truth of the case next to the one last verified becomes obvious, and thence that of the case next following, and so on. Thus, proceeding from case to case the theorem is found to be true universally.

The method will be best illustrated by the following examples.

Example 1. Shew by the method of Induction that the sum of n terms of the series

$$1 + 3 + 6 + 10 + 15 + \&c.$$

is $\frac{1}{6}n(n+1)(n+2)$.

(Bombay University P. E. Paper, 1891.)

The sum of two terms of the series is obviously 4 : also when $n = 2$ we have $\frac{1}{6}n(n+1)(n+2) = \frac{1}{6}.2.3.4. = 4$. Thus the theorem is true when $n = 2$.

Let t_n denote the n^{th} term of the series and S the sum of n terms. Then we have

$$S = 1 + 3 + 6 + 10 + 15 + \quad . \quad . \quad . \quad + t_n$$

$$\text{also} \quad S = 0 + 1 + 3 + \ 6 + 10 + \quad . \quad . \quad . \quad + t_{n-1} + t_n.$$

Hence, by subtraction,

$$0 = (1+2+3+4+5+ \ldots \text{ to } n \text{ terms}) - t_n ;$$

$$\therefore \quad t_n = 1+2+3+4+\&c. \text{ to } n \text{ terms}$$

$$= \frac{n(n+1)}{2}.$$

Evidently therefore the r^{th} term of the series $= \dfrac{r(r+1)}{2}$,

and the $(r+1)^{\text{th}}$ term $\dfrac{(r+1)(r+2)}{2}$

where r is *any* number.

Now *assume* the theorem to be true when $n = r$; that is, *suppose*

$$1+3+6+10+15+ \ldots + \frac{r(r+1)}{2} = \tfrac{1}{6}r(r+1)(r+2).$$

Then we must have

$$1+3+6+10+15 \quad \ldots + \frac{r(r+1)}{2} + \frac{(r+1)(r+2)}{2}$$

$$= \tfrac{1}{6}r(r+1)(r+2) + \frac{(r+1)(r+2)}{2}$$

$$= \frac{(r+1)(r+2)}{6}(r+3)$$

$$= \tfrac{1}{6}(r+1)(r+2)(r+3),$$

which shews that the theorem is true when $n = r+1$.

Thus it is proved that *if* the theorem be true when we take a certain number of terms, *whatever that number may be*, it is true when we increase that number by one.

But we know that the theorem *is* true when 2 terms are taken; therefore it is true when 3 terms are taken, and therefore it is true when 4 terms are taken; and so on. Thus the theorem is true universally.

Example 2. Prove by the method of Induction that every even power of every odd number when divided by 8 leaves 1 for a remainder. (Bombay University P. E. Paper, 1890.)

First to prove that the square of every odd number when divided by 8 leaves 1 for a remainder.

Let n be any positive integer; then $2n+1$ is an odd number, and the next odd number is $2n+3$.

Now *assume* that $(2n+1)^2$ when divided by 8 leaves 1 for a remainder; that is, *suppose*

$$(2n+1)^2 = 8Q+1,$$

where Q is an integer.

Then, since
$$\begin{aligned}(2n+3)^2 &= (2n+1)^2\\ &= (4n+4).2\\ &= 8(n+1),\end{aligned}$$
we have
$$\begin{aligned}(2n+3)^2 &= (2n+1)^2+8(n+1)\\ &= 8(Q+n+1)+1,\end{aligned}$$

which shews that $(2n+3)^2$ when divided by 8 leaves 1 for a remainder.

Thus it is proved that *if* the square of *any* odd number when divided by 8 leaves 1 for a remainder, the square of the *next* odd number does the same.

But we know that this is true of 3^2; therefore it is true of 5^2; and therefore it is true of 7^2; and so on. Thus it is true universally.

Hence it is proved that $(2n+1)^2 = 8Q+1$. . . . (a)

Let us now *assume* that $(2n+1)^{2m}$, where m is any positive integer, when divided by 8 leaves 1 for a remainder; that is, *suppose*

$$(2+1)^{2m} = 8Q'+1,$$

where Q' is an integer.

Then we have

$$\begin{aligned}(2n+1)^{2m+2} &= (2n+1)^{2m}(2n+1)^2\\ &= (8Q'+1)(8Q+1)\\ &= 8(8QQ'+Q+Q')+1,\end{aligned}$$

which shows that $(2n+1)^{2m+2}$ when divided by 8 leaves 1 for a remainder.

Thus it is shewn that *if* any even power of $2n+1$ when divided by 8 leaves 1 for a remainder, the next even power of $2n+1$ does the same.

But we know from (a) that this is true of $(2n+1)^2$; therefore it is true of $(2n+1)^4$; and therefore it is true of $(2n+1)^8$; and so on. Thus it is true universally, which proves the proposition.

Exercise (75).

Prove by Induction :—

1. $1+3+5+$ &c. to n terms $= n^2$.

2. $1^2+2^2+3^2+$ &c. to n terms $= \frac{1}{6}n(n+1)(2n+1)$.

(Bombay University P. E. Paper, 1886.)

3. $1^3+2^3+3^3+$ &c. to n terms $= \left\{\frac{n(n+1)}{2}\right\}^2$.

4. $1^3+3^3+5^3+$ &c. to n terms $= n^2(2n^2-1)$.

5. $1+5+12+22+35+$ &c. to n terms $= \frac{n^2(n+1)}{2}$.

6. $\frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}+$ &c. to n terms $= \frac{}{3(2n+3)}$.

7. x^n-a^n is divisible by $x-a$ when n is any whole number.

8. x^n-a^n is divisible by $x+a$ if n be an *even* whole number.

CHAPTER XVII.

BINOMIAL THEOREM. POSITIVE INTEGRAL EXPONENT.

1. **To prove that** $(1+x)^n = 1+{}^nC_1\,x+{}^nC_2.x^2+{}^nC_3.x^3+{}^nC_4.x^4+\ldots+{}^nC_{n-1}.x^{n-1}+{}^nC_n.x^n$, where n is any positive integer.

We know that

$$(1+x)^2 = 1+2x+x^2$$
$$\text{and } \therefore\ = 1+{}^2C_1.x+{}^2C_2.x^2,$$
$$(1+x)^3 = 1+3x+3x^2+x^3$$
$$\text{and } \therefore\ = 1+{}^3C_1\,x+{}^3C_2.x^2+{}^3C_3.x^3,$$
$$(1+x)^4 = 1+4x+6x^2+4x^3+x^4$$
$$\text{and } \therefore\ = 1+{}^4C_1.x+{}^4C_2\,x^2+{}^4C_3.x^3+{}^4C_4.x^4.$$

These particular cases lead us to be inclined in favour of the truth of the general theorem, which we shall now *prove* by the method of Induction

Assume the theorem to be true when $n = m$; that is, *suppose*

$$(1+x)^m = 1 + {}^mC_1.x + {}^mC_2.x^2 + {}^mC_3\,x^3 + {}^mC_4.x^4 + \dots + {}^mC_{m-1}.x^{m-1} + {}^mC_m.x^m.$$

Then we must have

$$\begin{aligned}(1+x)^{m+1} &= (1+x)\{1 + {}^mC_1.x + {}^mC_2.x^2 + {}^mC_3\,x^3 + {}^mC_4.x^4 + \dots + {}^mC_{m-1}.x^{m-1} + {}^mC_m\,x^m\}\\ &= 1 + ({}^mC_1 + 1)x + ({}^mC_2 + {}^mC_1)x^2 + ({}^mC_3 + {}^mC_2)x^3 + ({}^mC_4 + {}^mC_3)x^4 + \dots + ({}^mC_m + {}^mC_{m-1})x^m + {}^mC_m.x^{m+1}.\end{aligned}$$

$$\begin{aligned}\text{But } {}^mC_r + {}^mC_{r-1} &= \frac{|\underline{m}}{|\underline{r}\,|\underline{m-r}} + \frac{|\underline{m}}{|\underline{r-1}\,|\underline{m-r+1}}\\ &= \frac{|\underline{m}}{|\underline{r}\,|\underline{m-r+1}}\left\{(m-r+1)+r\right\}\\ &= \frac{|\underline{m+1}}{|\underline{r}\,|\underline{m+1-r}} = {}^{m+1}C_r;\end{aligned}$$

$$\text{and} \quad {}^mC_m = {}^{m+1}C_{m+1}.$$

$$\text{Hence,}\ (1+x)^{m+1} = 1 + {}^{m+1}C_1.x + {}^{m+1}C_2.x^2 + {}^{m+1}C_3.x^3 + {}^{m+1}C_4.x^4 + \dots + {}^{m+1}C_m.x^m + {}^{m+1}C_{m+1}.x^{m+1},$$

which shews that the theorem is true when $n = m+1$.

Thus it is proved that *if* the theorem be true when n has *any* integral value, it is also true when that value is increased by one.

But we know that the theorem is true when $n = 4$; therefore it is true when $n = 5$, and therefore it is true when $n = 6$, and therefore it is true when $n = 7$; and so on. Thus the theorem is true universally; *i.e.*, for *all integral values* of n we have

$$(1+x)^n = 1 + {}^nC_1.x + {}^nC_2.x^2 + {}^nC_3.x^3 + \dots + {}^nC_{n-1}.x^{n-1} + {}^nC_n.x^n.$$

Note. This theorem is known as the **Binomial Theorem**, and the series on the right-hand side is called the *expansion* of $(1+x)^n$.

Obs. The number of terms in the expansion is $n+1$.

***Cor.* 1.** Putting $-x$ for x we have

$$\{1+(-x)\}^n = 1 + {}^nC_1\,(-x) + {}^nC_2.(-x)^2 + {}^nC_3.(-x)^3 + \dots\dots + {}^nC_n.(-x)^n,$$

i.e., $(1-x)^n = 1 - {}^nC_1.x + {}^nC_2.x^2 - {}^nC_3.x^3 + \dots + (-x)^n.$

Thus, in the expansion of $(1-x)^n$, the terms are *alternately* positive and negative, and the last term $= \pm x^n$ according as n is even or odd.

***Cor.* 2.** Since $(x+a)^n = \left\{x\left(1+\frac{a}{x}\right)\right\}^n = x^n\left(1+\frac{a}{x}\right)^n$,

$$\therefore\quad (x+a)^n = x^n\left\{1 + {}^nC_1.\frac{a}{x} + {}^nC_2.\frac{a^2}{x^2} + {}^nC_3.\frac{a^3}{x^3} + \dots + {}^nC_n.\frac{a^n}{x^n}\right\}$$

$$= x^n + {}^nC_1.x^{n-1}a + {}^nC_2.x^{n-2}a^2 + {}^nC_3.x^{n-3}a^3 + \dots\dots + {}^nC_n.a^n.$$

Similarly, $(x-a)^n = x^n - {}^nC_1\,x^{n-1}a + {}^nC_2.x^{n-2}a^2 - {}^nC_3.x^{n-3}a^3 + \dots + {}^nC_n\,(-a)^n.$

Example 1. Expand $(1-2x)^{10}$.

$$\begin{aligned}(1-2x)^{10} &= 1 + {}^{10}C_1(-2x) + {}^{10}C_2(-2x)^2 + {}^{10}C_3(-2x)^3\\ &+ {}^{10}C_4(-2x)^4 + {}^{10}C_5(-2x)^5 + {}^{10}C_6(-2x)^6\\ &+ {}^{10}C_7(-2x)^7 + {}^{10}C_8(-2x)^8 + {}^{10}C_9(-2x)^9\\ &+ {}^{10}C_{10}(-2x)^{10}\end{aligned}$$

$$= 1 - 10.2x + \frac{10.9}{1.2}4x^2 - \frac{10.9.8}{1.2.3}.8x^3 + \frac{10.9.8.7}{1.2.3.4}.16x^4$$

$$- \frac{10.9.8.7.6}{1.2.3.4.5}32x^5 + \frac{10.9.8.7}{1.2.3.4}.64x^6 - \frac{10.9.8}{1.2.3}.128x^7$$

$$+ \frac{10.9}{1.2}256x^8 - 10.512x^9 + 1024x^{10}$$

$$= 1 - 20x + 180x^2 - 960x^3 + 3360x^4 - 8064x^5 + 13440x^6 - 15360x^7 + 11520x^8 - 5120x^9 + 1024x^{10}.$$

Example 2. Expand $(\frac{1}{2}x - 2y)^7$.

$$(\tfrac{1}{2}x - 2y)^7 = \left\{\frac{x}{2}\left(1 - \frac{4y}{x}\right)\right\}^7$$

$$= \left(\frac{x}{2}\right)^7\left(1 - \frac{4y}{x}\right)^7$$

$$= \frac{x^7}{128}\left\{1 + {}^7C_1\left(-\frac{4y}{x}\right) + {}^7C_2\left(-\frac{4y}{x}\right)^2\right.$$

$$+ {}^7C_3\left(-\frac{4y}{x}\right)^3 + {}^7C_4\left(-\frac{4v}{x}\right)^4 + {}^7C_5\left(-\frac{4y}{x}\right)^5$$

$$\left. + {}^7C_6\left(-\frac{4y}{x}\right)^6 + {}^7C_7\left(-\frac{4y}{x}\right)^7\right\}$$

$$= \frac{x^7}{128}\left\{1 - 7.\left(\frac{4y}{x}\right) + \frac{7.6}{1.2}\left(\frac{4y}{x}\right)^2 - \frac{7.6.5}{1.2.3}\left(\frac{4y}{x}\right)^3\right.$$

$$\left. + \frac{7.6.5}{1.2.3}\left(\frac{4y}{x}\right)^4 - \frac{7.6}{1.2}\left(\frac{4y}{x}\right)^5 + 7.\left(\frac{4y}{x}\right)^6 - \left(\frac{4y}{x}\right)^7\right\}$$

$$= \frac{x^7}{128}\left\{1 - \frac{7.4.y}{x} + \frac{21\ 4^2.y^2}{x^2} - \frac{35.4^3.y^3}{x^3}\right.$$

$$\left. + \frac{35.4^4.y^4}{x^4} - \frac{21.4^5.y^5}{x^5} + \frac{7.4^6.y^6}{x^6} - \frac{4^7\ y^7}{x^7}\right\}$$

$$= \frac{x^7}{128} - \frac{7}{32}x^6y + \frac{21}{8}x^5y^2 - \frac{35}{2}x^4y^3 + 70x^3y^4$$

$$- 168x^2y^5 + 224xy^6 - 128y^7.$$

Exercise (76).

Expand the following :—

1. $\left(a - 3x\right)^6$.   2. $\left(5 - \frac{x}{6}\right)^6$.   3. $\left(x^4 - x^2y^2\right)^5$.

4. $\left(\sqrt[3]{x} - \sqrt[3]{y}\right)^6$.   5. $\left(2x - 3y\right)^5$.   6. $\left(\frac{2x}{3} - \frac{3}{2x}\right)^6$.

Find the value of :—

7. $\{a + \sqrt{a^2 - 1}\}^6 + \{a - \sqrt{a^2 - 1}\}^6$.

8. $\{2 - \sqrt{1 - x}\}^6 + \{2 + \sqrt{1 - x}\}^6$.

9. $\{x + \sqrt{x^2 - 1}\}^7 + \{x - \sqrt{x^2 - 1}\}^7$.

10. Expand $\left(x + 1 - \frac{1}{x}\right)^3$.

2. To find an expression for the general term in the expansion of $(x+a)^n$.

The second term $= {}^nC_1.x^{n-1}.a$,

„ 3rd „ $= {}^nC_2.x^{n-2}.a^2$,

„ 4th „ $= {}^nC_3.x^{n-3}.a^3$,

„ 5th „ $= {}^nC_4.x^{n-4}.a^4$,

and so on.

It is thus observed that the suffix of C in any term is one less than the number of the term, that the index of a is the same as the suffix of C, and that the sum of the indices of x and a is n.

Hence, the $(r+1)$th term

$$= {}^nC_r.x^{n-r}.a^r.$$

This is called the *general term*, as by giving suitable numerical values to r, any term of the expansion can be found.

Example 1. Find the 7th term of the expansion of $(a^3+3ab)^9$.

The required term $= {}^9C_6.(a^3)^3.(3ab)^6$

$= {}^9C_3.(a^3)^3.(3ab)^6$

$= \frac{9.8.7}{1.2.3}.a^9.3^6.a^6.b^6$

$= 61236a^{15}b^6$.

Example 2. Write down the 49th term of $(a-x)^{50}$.

The required term $= {}^{50}C_{48}.a^2.(-x)^{48}$

$= {}^{50}C_2.a^2.(-x)^{48}$

$= \frac{50.49}{2}.a^2.(-x)^{48}$

$= 1225a^2x^{48}$.

Example 3. Write down the co-efficient of y in the expansion of $\left(y^2+\frac{c^3}{y}\right)^5$.

Suppose y occurs in the $(r+1)$th term of the expansion.

Now, since the $(r+1)^{th}$ term $= {}^5C_r\left(y^2\right)^{5-r}.\left(\frac{c^3}{y}\right)^r$

$= {}^5C_r.c^{3r}.y^{10-3r}$,

if it contains y we must have

$$10 - 3r = 1$$
$$\text{and} \quad \therefore r = 3.$$

Hence, the required co-efficient

$$= {}^5C_r\, c^{3r}$$
$$= {}^5C_3.c^{3.3}$$
$$= {}^5C_3.c^9$$
$$= 10c^9.$$

Example 4. Find the co-efficient of x^{32} and x^{-17} in $\left(x^4 - \frac{1}{x^3}\right)^{15}$.

(i) Suppose that x^{32} occurs in the $(r+1)$th term.

Then since the $(r+1)$th term

$$= {}^{15}C_r.\left(x^4\right)^{15-r}.\left(-\frac{1}{x^3}\right)^r$$
$$= {}^{15}C_r.x^{60-4r}.\frac{(-1)^r}{x^{3r}}$$
$$= {}^{15}C_r\,(-1)^r\,x^{60-7r},$$

we must have $\quad 60 - 7r = 32,$

$$\text{or,} \quad 7r = 28, \quad \therefore \quad r = 4$$

Hence, the required co-efficient

$$= {}^{15}C_r\,(-1)^r$$
$$= {}^{15}C_4.(-1)^4$$
$$= \frac{15.14.13.12}{1.2.3.4.} \times 1$$
$$= 1365.$$

(ii) Suppose that x^{-17} occurs in the $(r+1)$th term.

Then, as before, we must have

$$60 - 7r = -17$$
$$\text{or,} \quad 7r = 77, \quad \therefore \quad r = 11.$$

Hence, the required co-efficient

$$= {}^{15}C_r.(-1)^r$$
$$= {}^{15}C_{11}.(-1)^{11}$$
$$= {}^{15}C_4.(-1)$$
$$= -\frac{15 \times 14 \times 13 \times 12}{1.2.3.4}$$
$$= -1365.$$

Example 5. Find the term independent of x in $\left(x-\dfrac{1}{x^2}\right)^{3n}$.

Suppose that the $(r+1)$th term does not contain x.

Then, since the $(r+1)$th term

$$= {}^{3n}C_r . x^{3n-r} . \left(-\frac{1}{x^2}\right)^r$$
$$= {}^{3n}C_r . (-1)^r . x^{3n-3r},$$

we must have $3n-3r = 0$ and $\therefore\ r = n$

Hence, the required term

$$= {}^{3n}C_r . (-1)^r$$
$$= {}^{3n}C_n . (-1)^n$$
$$= (-1)^n . \frac{\lfloor 3n}{\lfloor n \lfloor 2n}.$$

Example 6. Prove that if the term x^r occurs in the expansion of $\left(x+\dfrac{1}{x}\right)^n$ the co-efficient of the term $= \dfrac{\lfloor n}{\lfloor \frac{1}{2}(n-r) \lfloor \frac{1}{2}(n+r)}$.

Suppose that x^r occurs in the $(p+1)$th term.

Then, since the $(p+1)$th term

$$= {}^nC_p . x^{n-p} . \left(\frac{1}{x}\right)^p$$
$$= {}^nC_p . x^{n-2p},$$

we must have $n-2p = r$ and $\therefore\ p = \frac{1}{2}(n-r)$.

Hence, the required co-efficient

$$= {}^nC_p = \frac{\lfloor n}{\lfloor p \lfloor n-p}$$
$$= \frac{\lfloor n}{\lfloor \frac{1}{2}(n-r) \lfloor \frac{1}{2}(n+r)}.$$

Exercise (77).

1. Find the 5th term of $(a^2-c^2)^{12}$.
2. Find the 9th term of $(\frac{1}{3}a-\frac{1}{2}b)^{12}$.

(Calcutta University F. A. Paper, 1868.)

3. Find the 8th term of $\left(x^{\frac{3}{2}}y^{\frac{1}{2}} - x^{\frac{1}{2}}y^{\frac{3}{2}}\right)^{10}$.

4. Find the 19th term of $\left(2x^{\frac{1}{2}} - y^{\frac{1}{3}}\right)^{20}$.

(Calcutta University F. A. Paper, 1870.)

5. Find the middle term of the expansion of $(a^m + x^m)^{12}$.

6. Find the two middle terms of $(a+x)^{13}$.

7. Write down the 4th term of $\left(\frac{x}{a} - \frac{a}{x}\right)^{10}$.

(Calcutta University F. A. Paper, 1888.)

8. Find the co-efficient of x^{18} in $(ax^4 - bx)^9$.

9. Find the middle term of $\left(1 - \frac{x^2}{2}\right)^{11}$.

10. Write down the co-efficients of x^{17} and x^{18} in the expansion of $(a^4 - bx^3)^{10}$.

(Calcutta University F. A. Paper, 1876.)

11. Write down the co-efficient of x^{11} in $(x - 2y)^{13}$.

(Calcutta University F. A. Paper, 1883.)

12. Write down the co-efficient of x^{2r+1} in the expansion of $\left(x - \frac{1}{x}\right)^{2n+1}$.

13. Find the term independent of x in

$$\left(\frac{3}{2}x^2 - \frac{1}{3x}\right)^9.$$

14. Shew that the middle term in the expansion of

$$(1+x)^{2n} = \frac{1.3.5.\&c.(2n-1)}{|\underline{n}} \cdot 2^n.x^n.$$

[Since there are altogether $2n+1$ terms in the expansion, evidently the $(n+1)th$ term is the middle term, for it has n terms before it and n terms after it. For the subsequent part of the solution refer to the latter part of Example 3 worked out on page 271 in the chapter on Permutations and Combinations.]

15. If there be a term independent of x in $\left(x^2 - \frac{1}{x}\right)^{2n}$, find it.

16. If x^{4r} occurs in the expansion of $\left(x - \frac{1}{x^2}\right)^{4n}$, prove that its co-efficient is $\dfrac{|\underline{4n}}{|\underline{\tfrac{4}{3}(n-r)}\,|\underline{\tfrac{4}{3}(2n+r)}}$.

3. In the expansion of $(1+x)^n$ the co-efficients of terms equidistant from the beginning and end are equal.

The co-efficient of the $(r+1)^{th}$ term from the beginning $= {}^nC_r$. And since there are altogether $n+1$ terms in the expansion, the $(r+1)^{th}$ term from the end has $(n+1)-(r+1)$, or $n-r$ terms, before it; hence it is the $(n-r+1)^{th}$ term *from the beginning*, and $\therefore$ its co-efficient is $= {}^nC_{n-r}$.

But ${}^nC_r = {}^nC_{n-r}$; hence the co-efficient of the $(r+1)^{th}$ term from the beginning is the same as the co-efficient of the $(r+1)^{th}$ term from the end.

4. To find the greatest co-efficient in the expansion of $(1+x)^n$.

The co-efficients are ${}^nC_0, {}^nC_1, {}^nC_2, {}^nC_3, \ldots {}^nC_n$; to find which of them is the greatest.

We know from the chapter on Permutations and Combinations that if n be even ${}^nC_r$ is greatest when $r = \frac{n}{2}$, and if n be odd ${}^nC_r$ is greatest when $r = \frac{n-1}{2}$ or $\frac{n+1}{2}$.

Hence when n is even the greatest co-efficient is ${}^nC_{\frac{n}{2}}$, and when n is odd it is ${}^nC_{\frac{n-1}{2}}$ or ${}^nC_{\frac{n+1}{2}}$, both of them being equal.

5. To find the greatest term in the expansion of $(x+a)^n$.

Let the rth and $(r+1)$th terms of the expansion be denoted by T_r and T_{r+1} respectively.

$$\begin{aligned}
\text{Then}\quad T_r &= {}^nC_{r-1}.x^{n-r+1}.a^{r-1},\\
\text{and}\quad T_{r+1} &= {}^nC_r.x^{n-r}.a^r;\\
\text{hence,}\quad \frac{T_{r+1}}{T_r} &= \frac{n-r+1}{r}.\frac{a}{x},\\
\therefore\quad T_{r+1} &= T_r \times \left(\frac{n+1}{r}-1\right)\frac{a}{x}.
\end{aligned}$$

Hence, $T_{r+1} >$ = or $< T_r$,

according as $\left(\frac{n+1}{r}-1\right)\frac{a}{x} >$ = or < 1, *i.e.*, according as $\frac{n+1}{r} - 1 > = \text{ or } < \frac{x}{a}$,

i.e., according as $\frac{n+1}{r} > = \text{ or } < \frac{x}{a} + 1$,

i.e., according as $\frac{n+1}{\frac{x}{a}+1} > = \text{ or } < r$,

i.e., according as $r < = \text{ or } > \frac{n+1}{\frac{x}{a}+1}$

(i) If $\frac{n+1}{\frac{x}{a}+1}$ be an integer, denote it by p;

then for all the values of r from 1 to $p-1$, $T_{r+1} > T_r$, *i.e.*, of the terms T_1, T_2, T_3, &c., T_{p-1}, T_p, each is greater than the preceding;

if $r = p$, $T_{r+1} = T_r$, *i.e.*, $T_{p+1} = T_p$;

if $r > p$, $T_{r+1} < T_r$, *i.e.*, of the terms T_{p+1}, T_{p+2}, T_{p+3}, &c., T_n, T_{n+1}, each is less than the preceding;

hence, in this case the p^{th} term of the expansion is equal to the $(p+1)^{th}$ term and these are greater than any other term.

(ii) If $\frac{n+1}{\frac{x}{a}+1}$ be not an integer, denote its integral part by q; then for all values of r from 1 to q, $T_{r+1} > T_r$, *i.e.*, of the terms T_1, T_2, T_3, &c., T_q, T_{q+1}, each is greater than the preceding;

and for all values of r from $q+1$ to n, $T_{r+1} < T_r$, *i.e.*, of the terms T_{q+1}, T_{q+2}, T_{q+3}, &c., T_n, T_{n+1}, each is less than the preceding;

hence, in this case the $(q+1)^{th}$ term is the greatest.

N. B. By a similar mode of reasoning we can find out the *numerically greatest term* in the expansion of $(x-a)^n$; in this case we have to reject the sign of the multiplying factor and proceed as above.

It is always best however to work out every numerical example independently of the general formula. The following examples will show how to deal with any particular case.

Example 1. Find the greatest term in the expansion of $(2+3x)^{12}$ when $x = \frac{5}{6}$.

Let the r^{th} and $(r+1)^{th}$ terms be denoted by T_r and T_{r+1} respectively.

$$\text{Then, } T_{r+1} = \left(\frac{12-r+1}{r}\cdot\frac{3x}{2}\right)\times T_r$$

$$= \left(\frac{13-r}{r}\cdot\frac{5}{4}\right)\times T_r.$$

Hence, $T_{r+1} > = \text{ or } < T_r$,

according as $\frac{13-r}{r}\times\frac{5}{4} > = \text{ or } < 1$,

i.e., according as $65-5r > = \text{ or } < 4r$,

i.e., according as $65 > = \text{ or } < 9r$,

i.e., according as $r < = \text{ or } > 7\frac{2}{9}$.

Hence for all values of r up to 7, T_{r+1} is greater than T_r and for greater values, less; hence the greatest term is the 8th and its value

$$= {}^{12}C_7 . 2^5 . (3x)^7$$
$$= {}^{12}C_5 . 2^5 . (\tfrac{5}{2})^7$$
$$= \frac{12.\ 11.\ 10.\ 9.\ 8}{1.\ 2.\ 3.\ 4.\ 5}\times 2^5\times\frac{5^7}{2^7}$$
$$= 15468750.$$

Example 2. Find the greatest term in the expansion of $(3-5x)^{11}$ when $x = \frac{1}{5}$.

We have *numerically*

$$T_{r+1} = \left(\frac{11-r+1}{r}\cdot\frac{5x}{3}\right)\times T_r$$

$$= \left(\frac{12-r}{r}\cdot\frac{1}{3}\right)\times T_r.$$

Hence, $T_{r+1} > = \text{ or } < T_r$,

according as $\frac{12-r}{r}\times\frac{1}{3} > = \text{ or } < 1$,

i.e., according as $12-r > = \text{ or } < 3r$,

i.e., according as $12 > = \text{ or } < 4r$,

i.e., according as $r < = \text{ or } > 3$.

Thus when $r = 1$ or 2, T_{r+1} is greater than T_r; when $r = 3$, $T_{r+1} = T_r$ and for greater values of r, T_{r+1} is less than T_r.

Hence the 3rd and 4th terms are *numerically* equal to each other and greater than any other term; and $\therefore$ the required term

$$= {}^{11}C_2 . 3^9 . (5x)^2$$

$$= \frac{11.10}{2} \times 3^9 = 1082565.$$

Exercise (78).

Find which is the greatest term in the expansion of each of the following :—

1. $\left(\frac{x}{2} - \frac{y}{3}\right)^{16}$ when $x = 8$, $y \quad 9$.
2. $(2a + 5b)^{34}$ when $a = 3$, $b = 2$.
3. $(4 + 3x)^{25}$ when $x = 4$.
4. $(\frac{3}{4}m - \frac{2}{3}n)^{10}$ when $m = 8$, $n = 3$.

Find the value of the greatest term in the expansion of each of the following :—

5. $(2 + 3x)^{14}$ when $x = \frac{2}{3}$.
6. $(a + x)^n$ when $a = \frac{1}{2}$, $x = \frac{1}{3}$, $n = 9$.
7. $(3 + 5x)^8$ when $x = \frac{1}{2}$.

6. To find the sum of the co-efficients of the terms in the expansion of $(1 + x)^n$.

Since $(1 + x)^n = 1 + {}^nC_1 x + {}^nC_2 x^2 + \;.\;.\;.\;.\;.\; + {}^nC_n x^n$, putting $x = 1$ we have

$$2^n = 1 + {}^nC_1 + {}^nC_2 + {}^nC_3 + \;.\;.\;.\;.\;.\; {}^nC_n,$$

i.e., the sum of the co-efficients $= 2^n$.

Cor. ${}^nC_1 + {}^nC_2 + {}^nC_3 + \;.\;.\;.\;.\;.\; + {}^nC_n = 2^n - 1$; *i.e.*, the *total number of combinations of n things* is $2^n - 1$.

7. To prove that in the expansion of $(1 + x)^n$, the sum of the co-efficients of the odd terms is equal to the sum of the co-efficients of the even terms.

Since $(1 + x)^n = 1 + {}^nC_1 x + {}^nC_2 x^2 + {}^nC_3 x^3 + \;.\;.\;.\;.\; + {}^nC_n x^n$, putting $x = -1$ we have

$$0 = 1 - {}^nC_1 + {}^nC_2 - {}^nC_3 + {}^nC_4 - {}^nC_5 + \&c.;$$

$$\therefore \quad 1 + {}^nC_2 + {}^nC_4 + \&c. = {}^nC_1 + {}^nC_3 + {}^nC_5 + \&c.$$

Cor. Each of these sums = half the sum of all the co-efficients

$$= \tfrac{1}{2} \times 2^n = 2^{n-1}.$$

8. Miscellaneous Examples.

Example 1. If A be the sum of the odd terms and B the sum of the even terms in the expansion of $(x+a)^n$, prove that

$$A^2 - B^2 = (x^2 - a^2)^n.$$

Let the terms in the expansion of $(x+a)^n$ be denoted by t_0, t_1, t_2, t_3, &c.

$$\begin{aligned} \text{Then} \quad (x+a)^n &= t_0 + t_1 + t_2 + t_3 + \quad . \quad . \quad . \quad . \quad + t_n \\ &= (t_0 + t_2 + t_4 + \&\text{c.}) + (t_1 + t_3 + t_5 + \&\text{c.}) \\ &= A + B; \\ \text{and} \quad (x - a)^n &= t_0 - t_1 + t_2 - t_3 + t_4 - t_5 + \&\text{c.} \\ &= (t_0 + t_2 + t_4 + \&\text{c.}) - (t_1 + t_3 + t_5 + \&\text{c.}) \\ &= A - B. \end{aligned}$$

$$\begin{aligned} \text{Hence,} \quad (x+a)^n \times (x-a)^n &= (A+B) \times (A-B) \\ \textit{i.e.,} \quad (x^2 - a^2)^n &= A^2 - B^2. \end{aligned}$$

Example 2. If t_0, t_1, t_2, t_3, &c. represent the terms of the expansion of $(x+a)^n$, shew that

$$(t_0 - t_2 + t_4 - \&\text{c.})^2 + (t_1 - t_3 + t_5 - \&\text{c.})^2 = (x^2 + a^2)^n.$$

$$\begin{aligned} \text{We have } (x+a)^n &= x^n + {}^nC_1 x^{n-1}.a + {}^nC_2 x^{n-2}.a^2 \\ &\quad + {}^nC_3 x^{n-3}.a^3 + \; . \; . \; . \; . \; + {}^nC_n a^n \\ &= t_0 + t_1 + t_2 + t_3 + \; . \; . \; . \; . \; + t_n . \end{aligned}$$

Hence, since $(x+ai)^n$

$$\begin{aligned} &= x^n + {}^nC_1 x^{n-1}.ai + {}^nC_2 x^{n-2}.a^2 i^2 \\ &\qquad + {}^nC_3 x^{n-3}.a^3 i^3 + {}^nC_4 x^{n-4}.a^4 i^4 + \&\text{c.} \\ &= x^n + {}^nC_1 x^{n-1}.ai - {}^nC_2 x^{n-2}.a^2 - {}^nC_3 x^{n-3}.a^3 i \\ &\qquad + {}^nC_4 x^{n-4}.a^4 + {}^nC_5 x^{n-5}\, a^5 i - {}^nC_6 x^{n-6}.a^6 \\ &\qquad\qquad - {}^nC_7 x^{n-7}.a^7 i + \&\text{c.}, \end{aligned}$$

$$\begin{aligned} \therefore \quad (x+ai)^n &= t_0 + t_1 i - t_2 - t_3 i \\ &\quad + t_4 + t_5 i - t_6 - t_7 i + t_8 + t_9 i - \&\text{c.} \\ &= (t_0 - t_2 + t_4 - t_6 + t_8 - \&\text{c.}) \\ &\quad + (t_1 - t_3 + t_5 - t_7 + t_9 - \&\text{c.})i \qquad \ldots \qquad (1) \end{aligned}$$

Also, since $(x - ai)^n$

$$= x^n - {}^nC_1 x^{n-1}.ai + {}^nC_2 x^{n-2}.a^2 i^2$$
$$- {}^nC_3 x^{n-3}.a^3 i^3 + {}^nC_4 x^{n-4}.a^4 i^4 - \&c.$$
$$= x^n - {}^nC_1 x^{n-1}.ai - {}^nC_2 x^{n-2}.a^2 + {}^nC_3 x^{n-3}.a^3 i$$
$$+ {}^nC_4 x^{n-4}.a^4 - {}^nC_5 x^{n-5} a^5 i + \&c.,$$

$$\therefore \quad (x - ai)^n = t_0 - t_1 i - t_2 + t_3 i + t_4 - t_5 i$$
$$- t_6 + t_7 i + t_8 - \&c.$$
$$= (t_0 - t_2 + t_4 - t_6 + \&c.)$$
$$- (t_1 - t_3 + t_5 - t_7 + \&c.)i \quad \ldots \quad (2)$$

Hence from (1) and (2),

$(x + ai)^n \times (x - ai)^n$

$$= \{(t_0 - t_2 + t_4 + \&c.) + (t_1 - t_3 + t_5 - \&c.)i\}$$
$$\times \{(t_0 - t_2 + t_4 - \&c.) - (t_1 - t_3 + t_5 - \&c.)i\},$$

i.e., $(x^2 + a^2)^n = (t_0 - t_2 + t_4 - t_6 + \&c.)^2 + (t_1 - t_3 + t_5 - t_7 + \&c.)^2.$

Example 3. If a, b, c, d be any consecutive co-efficients of an expanded binomial, show that $(bc + ad)(b - c) = 2(ac^2 - b^2 d)$.

Let a, b, c, d be the co-efficients of the $(r-1)^{\text{th}}$, r^{th}, $(r+1)^{\text{th}}$ and $(r+2)^{\text{th}}$ terms respectively of the expansion of $(1 + x)^n$.

Then since $\dfrac{{}^nC_r}{{}^nC_{r-1}} = \dfrac{n - r + 1}{r}$, we must have

$$\text{(i)} \quad \frac{b}{a} = \frac{{}^nC_{r-1}}{{}^nC_{r-2}} = \frac{n - (r-1) + 1}{r - 1} = \frac{n - r + 2}{r - 1}$$

$$\text{(ii)} \quad \frac{c}{b} = \frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n - r + 1}{r};$$

$$\text{(iii)} \quad \frac{d}{c} = \frac{{}^nC_{r+1}}{{}^nC_r} = \frac{n - (r+1) + 1}{r + 1} = \frac{n - r}{r + 1}.$$

From (i) $\quad \dfrac{b}{a}(r - 1) = n - r + 2 \quad \ldots \quad \ldots \quad (\alpha)$

" (ii) $\quad \dfrac{c}{b} r = n - r + 1 \quad \ldots \quad \ldots \quad (\beta)$

" (iii) $\quad \dfrac{d}{c}(r + 1) = n - r \quad \ldots \quad \ldots \quad (\gamma)$

Now, from (α) and (β), by subtraction,

$$\frac{b}{a}(r-1) - \frac{c}{b}r = 1,$$

$$\text{or,} \quad r\left(\frac{b}{a} - \frac{c}{b}\right) = \frac{b}{a} + 1,$$

$$\therefore \quad r = \frac{b(a+b)}{b^2 - ac};$$

similarly from (β) and (γ),

$$r\left(\frac{c}{b} - \frac{d}{c}\right) = \frac{d}{c} + 1,$$

$$\therefore \quad r = \frac{b(c+d)}{c^2 - bd}.$$

$$\text{Hence,} \quad \frac{c+d}{c^2 - bd} = \frac{a+b}{b^2 - ac};$$

$$\text{or,} \quad b^2c - ac^2 + b^2d - acd = ac^2 - abd + bc^2 - b^2d,$$

$$\text{or,} \quad b^2c + abd - bc^2 - acd = 2(ac^2 - b^2d),$$

$$\text{or,} \quad (bc + ad)(b - c) = 2(ac^2 - b^2d).$$

Example 4. If $(1+x)^n = a_0 + a_1x + a_2x^2 + a_3x^3 + \&c. + a_nx^n$, find the value of $a_0 + \frac{a_1}{2} + \frac{a_2}{3} + \frac{a_3}{4} + \&c. + \frac{a_n}{n+1}$.

The given expression

$$= 1 + \frac{{}^nC_1}{2} + \frac{{}^nC_2}{3} + \frac{{}^nC_3}{4} + \quad \dots \quad + \frac{{}^nC_n}{n+1}$$

$$= \frac{1}{n+1}\cdot\left\{(n+1) + \frac{(n+1).{}^nC_1}{2} + \frac{(n+1).{}^nC_2}{3} + \frac{(n+1).{}^nC_3}{4} + \quad \dots \quad + \frac{(n+1).{}^nC_n}{n+1}\right\}$$

$$= \frac{1}{n+1}\cdot\left\{{}^{n+1}C_1 + {}^{n+1}C_2 + {}^{n+1}C_3 + {}^{n+1}C_4 + \quad \dots \quad + {}^{n+1}C_{n+1}\right\}$$

$$= \frac{1}{n+1}\cdot\left(2^{n+1} - 1\right). \quad \left[\text{Cor. Art. 6}\right].$$

Example 5. If $(1+x)^n = p_0 + p_1 x + p_2 x^2 + \ldots\ldots + p_n x^n$, find the value of $p_1 + 2p_2 + 3p_3 + \&c. + np_n$.

(Calcutta University F. A. Paper, 1882.)

The given expression

$$= {}^nC_1 + 2.{}^nC_2 + 3.{}^nC_3 + 4.{}^nC_4 + \quad . \quad . \quad . \quad + n.{}^nC_n.$$

But $r.{}^nC_r = r.\dfrac{\underline{|n}}{\underline{|r}\,\underline{|n-r}} = \dfrac{n\underline{|n-1}}{\underline{|r-1}\,\underline{|n-r}} = n.{}^{n-1}C_{r-1}$;

$\therefore$ the given expression

$$= n + n.{}^{n-1}C_1 + n.{}^{n-1}C_2 + n.{}^{n-1}C_3 + \quad . \quad . \quad . \quad + n.{}^{n-1}C_{n-1}$$

$$= n\{1 + {}^{n-1}C_1 + {}^{n-1}C_2 + {}^{n-1}C_3 + \quad . \quad . \quad . \quad . \quad + {}^{n-1}C_{n-1}\}$$

$$= n.2^{n-1}.$$

Example 6. I $(1+x)^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots\ldots + a_n x^n$, find the value of $a_0{}^2 + a_1{}^2 + a_2{}^2 + \ldots\ldots + a_n{}^2$.

Since $(1+x)^n = a_0 + a_1 x + a_2 x^2 + \ldots\ldots + a_n x^n$,

$\therefore$ putting $\dfrac{1}{x}$ for x, we have,

$$\left(1+\frac{1}{x}\right)^n = a_0 + \frac{a_1}{x} + \frac{a_2}{x^2} + \quad \ldots\ldots \quad + \frac{a_n}{x^n}.$$

Therefore, $\{a_0 + a_1 x + a_2 x^2 + \&c. + a_n x^n\}$

$$\times \left\{a_0 + \frac{a_1}{x} + \frac{a_2}{x^2} + \&c. \quad + \frac{a_n}{x^n}\right\}$$

$$= \left(1+x\right)^n \times \left(1+\frac{1}{x}\right)^n$$

$$= \frac{(1+x)^{2n}}{x^n}. \qquad \ldots \quad (1)$$

Hence, equating the terms independent of x on both sides, we have,

$$a_0{}^2 + a_1{}^2 + a_2{}^2 + \ldots\ldots + a_n{}^2$$

$$= \text{term independent of } x \text{ in } \frac{1}{x^n}(1+x)^{2n}$$

$$= \text{co-efficient of } x^n \text{ in } (1+x)^{2n}$$

$$= {}^{2n}C_n$$

$$= \frac{\underline{|2n}}{\underline{|n}\,\underline{|n}}.$$

N. B. It is assumed in the above solution that the equation (1) being *identical*, (*i.e.*, true for every value of x) the co-efficient of any power of x on one side is equal to the co-efficient of the same power of x on the other. A formal proof of this theorem will be given later on.

Example 7. If generally n_r be the co-efficient of the $(r+1)^{\text{th}}$ term of $(1+x)^n$ show that—

$$(n+p)_r = n_r + n_{r-1}p_1 + n_{r-2}p_2 + \ldots\ldots + n_1 p_{r-1} + p_r.$$

Since n_r is the co-efficient of the $(r+1)^{\text{th}}$ term of $(1+x)^n$, therefore also p_r is the co efficient of the $(r+1)^{\text{th}}$ term of $(1+x)^p$ and $(n+p)_r$ is the co-efficient of the $(r+1)^{\text{th}}$ term of $(1+x)^{n+p}$; hence we have

$$(1+x)^n = 1 + n_1 x + n_2 x^2 + n_3 x^3 + \ \ldots \ + n_r x^r + \&c.,$$

$$(1+x)^p = 1 + p_1 x + p_2 x^2 + p_3 x^3 + \ \ldots \ + p_r x^r + \&c.,$$

and

$$(1+x)^{n+p} = 1 + (n+p)_1 x + (n+p)_2 x^2 + \ \ldots \ + (n+p)_r x^r + \&c.$$

Therefore *identically*

$$\begin{aligned}1 + (n+p)_1 x + (n+p)_2 x^2 + \&c. + (n+p)_r x^r + \&c. \\ = (1 + n_1 x + n_2 x^2 + \&c. + n_r x^r + \&c.) \\ \times (1 + p_1 x + p_2 x^2 + \&c. + p_r x^r + \&c.).\end{aligned}$$

Hence, equating the co-efficient of x^r on the left-hand side with that on the right-hand side we have

$$(n+p)_r = n_r + n_{r-1}p_1 + n_{r-2}p_2 + \ \ldots \ + n_1 p_{r-1} + p_r.$$

Example 8. Find the value of $a - (a+b)n + (a+2b)\frac{n(n-1)}{1.2}$

$$- (a+3b)\frac{n(n-1)(n-2)}{1.2.3} + \&c.$$

The given expression

$$= a\left\{1 - n + \frac{n(n-1)}{1.2} - \frac{n(n-1)(n-2)}{1.2.3} + \ \ldots \text{ to } (n+1) \text{ terms}\right\}$$

$$- b\left\{n - n(n-1) + \frac{n(n-1)(n-2)}{1.2} - \ \ldots \text{ to } n \text{ terms}\right\}$$

$$= a(1-1)^n - bn(1-1)^{n-1}$$

$$= 0.$$

Exercise (79).

1. Find the value of

$$ {}^{20}C_1 + {}^{20}C_2 + {}^{20}C_3 + \ldots + {}^{20}C_{20}. $$

2. Find the value of

$$ {}^{26}C_1 + {}^{26}C_3 + {}^{26}C_5 + {}^{26}C_7 + \ldots + {}^{26}C_{25}. $$

3. If C_r denote the number of combinations of n things taken r at a time, shew that

$$ C_0 + C_2 + C_4 + \ldots = C_1 + C_3 + C_5 + \ldots $$

(Calcutta University F. A. Paper, 1878.)

4. If the co-efficient of the $(p+1)^{th}$ term of the expansion of $(1+x)^{2n}$ be equal to that of the $(p+3)^{th}$ term, shew that $p = n-1$.

5. Given that the co-efficient of the pth term of the expansion of $(1+x)^n = P$, and that of the $(p+1)^{th} = P'$; find n.

6. Prove that four times the product of the sums of the odd and even terms of the expansion of $(a+b)^n = (a+b)^{2n} - (a-b)^{2n}$.

[Expand $(a+b)^n$ and $(a-b)^n$ and take the sum and difference of the results.]

7. Find the rth term from the beginning, the rth term from the end, and the middle term of $\left(x - \dfrac{1}{x}\right)^{2n}$.

8. Find the $(r+2)^{th}$ term from the end in $\left(x - \dfrac{1}{x}\right)^{2n+1}$.

9. If a, b, c be three consecutive co-efficients in the expansion of a power of $1+x$, prove that the index of the power is $\dfrac{2ac + b(a+c)}{b^2 - ac}$ and that the number of the term of which a is the co-efficient is $\dfrac{a(b+c)}{b^2 - ac}$.

(Bombay University P. E. Paper, 1890.)

$\left[\text{We have } \dfrac{b}{a} = \dfrac{n-r}{r+1} \text{ and } \dfrac{c}{b} = \dfrac{n-r-1}{r+2}\right]$.

10. Prove that the co-efficient of the $(r+1)$th term of the expansion of $(1+x)^{n+1}$ is equal to the sum of the co-efficients of the rth and $(r+1)$th terms of $(1+x)^n$.

[Let $(1+x)^n = a_0 + a_1x + a_2x^2 + \&c. + a_nx^n$; multiply both sides by $1+x$ and then equate co-efficients of x^r.]

If $(1+x)^n = a_0 + a_1x + a_2x^2 + \&c. + a_nx^n$, find the values of :—

11. $a_0a_1 + a_1a_2 + a_2a_3 + \dots\dots + a_{n-1}a_n$.

12. $a_0a_2 + a_1a_3 + a_2a_4 + \dots\dots + a_{n-2}a_n$.

13. $a_0 + 2a_1 + 3a_2 + 4a_3 + \dots + (n+1)a_n$.

[The given expression $= (a_0 + a_1 + a_2 + \&c. + a_n) + (a_1 + 2a_2 + 3a_3 + \&c. + na_n)$; now refer to example 5 worked out on page 328.]

14. $\dfrac{a_1}{a_0} + \dfrac{2a_2}{a_1} + \dfrac{3a_3}{a_2} + \dots\dots + \dfrac{na_n}{a_{n-1}}$.

15. Prove that

$$1 - n\cdot\frac{1+x}{1+nx} + \frac{n(n-1)}{1.2}\cdot\frac{1+2x}{(1+nx)^2} - \frac{n(n-1)(n-2)}{1.2.3}\cdot\frac{1+3x}{(1+nx)^3} + \&c.$$
$$= 0.$$

CHAPTER XVIII.

BINOMIAL THEOREM ; ANY EXPONENT.

1. To prove the Binomial Theorem for any Exponent.

We know that *when m is a positive integer*

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{1.2}x^2 + \frac{m(m-1)(m-2)}{1.2.3}x^3 + \&c.$$

To prove that this equality also holds when m is either a positive fraction or any negative quantity.

Let the symbol $f(m)$ stand for the series $1 + mx + \dfrac{m(m-1)}{1.2}x^2 + \dfrac{m(m-1)(m-2)}{1.2.3}x^3 + \&c.$ for *all values* of m whether positive or negative, integral or fractional ;

then $f(n)$ will stand for the series

$$1 + n^x + \frac{n(n-1)}{1.2}x^2 + \frac{n(n-1)(n-2)}{1.2.3}x^3 + \&c.,$$

and $f(m+n)$ will stand for the series

$$1+(m+n)x+\frac{(m+n)(m+n-1)}{1.2}x^2$$
$$+\frac{(m+n)(m+n-1)(m+n-2)}{1.2.3}x^3+\&c.$$

Evidently then *when m and n are positive integers* we can put $(1+x)^m=f(m)$, $(1+x)^n=f(n)$ and $(1+x)^{m+n}=f(m+n)$, and it is to be now proved that we can also do this when m and n are fractional or negative.

Now, since by actual multiplication

$$\left\{1+mx+\frac{m(m-1)}{1.2}x^2+\frac{m(m-1)(m-2)}{1.2.3}x^3+\&c.\right\}$$
$$\times\left\{1+nx+\frac{n(n-1)}{1.2}x^2+\frac{n(n-1)(n-2)}{1.2.3}x^3+\&c\right\}$$
$$=1+(m+n)x+\left\{\frac{m(m-1)}{1.2}+mn+\frac{n(n-1)}{1.2}\right\}x^2$$
$$+\left\{\frac{m(m-1)(m-2)}{1.2.3}+\frac{mn(m-1)}{1.2}+\frac{mn(n-1)}{1.2}+\frac{n(n-1)(n-2)}{1.2.3}\right\}x^3$$
$$+\&c.$$
$$=1+(m+n)x+\frac{(m+n)^2-(m+n)}{1.2}x^2$$
$$+\frac{m(m^2-3m+2)+3mn(m+n-2)+n(n^2-3n+2)}{1.2.3}x^3+\&c.$$
$$=1+(m+n)x+\frac{(m+n)(m+n-1)}{1.2}x^2$$
$$+\frac{(m^3+n^3)-3(m^2+n^2+2mn)+3mn(m+n)+2(m+n)}{1.2.3}x^3+\&c.$$
$$=1+(m+n)x+\frac{(m+n)(m+n-1)}{1.2}x^2$$
$$+\frac{(m+n)\{(m^2-mn+n^2)-3(m+n)+3mn+2\}}{1.2.3}x^3+\&c.$$
$$=1+(m+n)x+\frac{(m+n)(m+n-1)}{1.2}x^2$$
$$+\frac{(m+n)\{(m+n)^2-3(m+n)+2\}}{1.2.3}x^3+\&c.$$

$$= 1 + (m+n)x + \frac{(m+n)(m+n-1)}{1.2}x^2 + \frac{(m+n)(m+n-1)(m+n-2)}{1.2.3}x^3 + \&c.,$$

it is clear that *for all values of* m *and* n we have $f(m) \times f(n) = f(m+n)$.

Hence, $f(m) \times f(n) \times f(p) = f(m+n) \times f(p)$
$= f(m+n+p)$.

Proceeding thus we have

$$f(m) \times f(n) \times f(p) \times \&c. \text{ to } r \text{ factors} = f(m+n+p+\&c. \text{ to } r \text{ terms}).$$

Let each of the quantities m, n, p, &c., be equal to $\frac{s}{r}$, where s and r are positive integers;

$$\therefore \left\{f\left(\frac{s}{r}\right)\right\}^r = f(s);$$

but since s is a positive integer, $f(s) = (1+x)^s$;

$$\therefore (1+x)^s = \left\{f\left(\frac{s}{r}\right)\right\}^r,$$

$$\therefore (1+x)^{\frac{s}{r}} = f\left(\frac{s}{r}\right).$$

But $f\left(\frac{s}{r}\right)$ stands for the series

$$1 + \frac{s}{r}x + \frac{\frac{s}{r}\left(\frac{s}{r}-1\right)}{1.2}x^2 + \&c.;$$

$$\therefore (1+x)^{\frac{s}{r}} = 1 + \frac{s}{r}x + \frac{\frac{s}{r}\left(\frac{s}{r}-1\right)}{1.2}x^2 + \&c.;$$

which proves the Binomial Theorem for any positive fractional index.

Again, since $f(m) \times f(n) = f(m+n)$ for all values of m and n, putting $-n$ for m (where n is positive) we have

$$f(-n) \times f(n) = f(-n+n) = f(0) = 1,$$

since every term of the series denoted by $f(0)$ except the first vanishes;

$$\therefore \quad \frac{1}{f(n)} = f(-n),$$

but $f(n) = (1+x)^n$ for *any positive* value of n,

$$\therefore \quad \frac{1}{(1+x)^n} = f(-n)$$

$$\text{or,} \quad (1+x)^{-n} = f(-n).$$

But $f(-n)$ stands for the series,

$$1 + (-n)x + \frac{(-n)(-n-1)}{1.2}x^2 + \&c.\ ;$$

$$\therefore \quad (1+x)^{-n} = 1 + (-n)x + \frac{(-n)(-n-1)}{1.2}x^2 + \&c.,$$

which proves the Binomial Theorem for any negative index.

Thus the Theorem is completely established.

Cor. When n is fractional or negative the number of terms in the expansion is unlimited

Example 1. Give the first four terms of the expansion of $\dfrac{1}{(2-3x^2)^{\frac{4}{5}}}$. (Calcutta University F. A. Paper, 1876.)

$$\frac{1}{(2-3x^2)^{\frac{4}{5}}} = \frac{1}{2^{\frac{4}{5}}\left(1-\frac{3x^2}{2}\right)^{\frac{4}{5}}}$$

$$= \frac{1}{2^{\frac{4}{5}}}\left(1-\frac{3x^2}{2}\right)^{-\frac{4}{5}}$$

$$= \frac{1}{2^{\frac{4}{5}}}\left\{1 + \left(-\tfrac{4}{5}\right).\left(-\frac{3x^2}{2}\right) + \frac{\left(-\frac{4}{5}\right)\left(-\frac{4}{5}-1\right)}{1.2}\left(-\frac{3x^2}{2}\right)^2 \right.$$

$$\left. + \frac{\left(-\frac{4}{5}\right)\left(-\frac{4}{5}-1\right)\left(-\frac{4}{5}-2\right)}{1.2.3}\left(\frac{-3x^2}{2}\right)^3 + \&c.\right\}$$

$$= \frac{1}{2^{\frac{4}{5}}}\left\{1 + \frac{4}{5}.\frac{3x^2}{2} + \frac{4.9}{5^2.2}.\frac{9x^4}{4} + \frac{4.9.14}{5^3.2.3}.\frac{27x^6}{8} + \&c.\right\}$$

$$= \frac{1}{\sqrt[5]{16}}\left\{1 + \frac{6}{5}x^2 + \frac{81}{50}x^4 + \frac{567}{250}x^6 + \&c.\right\}.$$

Example 2. Find the $(r+1)^{th}$ term in the expansion of $(1+x)^{\frac{1}{2}}$.

$$\text{The } (r+1)\text{th term} = \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)\ldots\ldots(\frac{1}{2}-r+1)}{\underline{|r}}x^r$$

$$= \frac{1(-1)(-3)\ldots\ldots(-2r+3)}{2^r\underline{|r}}x^r$$

$$= \left(-1\right)^{r-1}\frac{1.3.5.\&c.(2r-3)}{2^r\underline{|r}}x^r,$$

taking out -1 from each of the $r-1$ negative factors in the numerator.

Example 3. Find the $(r+1)^{th}$ term in the expansion of $(1-x)^{-3}$.

$$\text{The } (r+1)\text{th term} = \frac{(-3)(-3-1)(-3-2).\&c.(-3-r+1)}{\underline{|r}}(-x).$$

$$= \frac{(-3)(-4)(-5).\&c.(-r-2)}{\underline{|r}}(-x)$$

$$= (-1)^r.\frac{3.4.5.\ \&c.\ (r+2)}{\underline{|r}}(-1)^r.x^r$$

$$= (-1)^{2r}.\frac{3\ 4\ 5\ \&c.\ (r+2)}{1.2.3.\ \&c.\ r}x^r$$

$$= \frac{(r+1)(r+2)}{1.2}x^r, \text{ by cancelling like factors}$$

in the numerator and denominator.

N. B. It should be borne in mind that when n is fractional or negative we *cannot put* ${}^nC_r$ for the co-efficient of the $(r+1)$th term.

Example 4. Write down the rth term of the expansion of $\left(1-x\right)^{\frac{3}{2}}$.

(Calcutta University F. A. Paper, 1878.)

$$\text{The } r\text{th term} = \frac{\frac{3}{2}(\frac{3}{2}-1)(\frac{3}{2}-2).\&c.(\frac{3}{2}-r+2)}{\underline{|r-1}}.(-x)^{r-1}$$

$$\frac{3.1(-1)(-3).\&c.(-2r+7)}{2^{r-1}.\underline{|r-1}}(-x)^{r-1}$$

$$= (-1)^{r-3}.\frac{3.1.1.3.5.\ \&c.\ (2r-7)}{2^{r-1}|\underline{r-1}}.(-1)^{r-1}.x^{r-1}$$

for values of r greater than 3 (∵ of the $r-1$ factors in the numerator 2 only are positive and the rest negative)

$$= (-1)^{2r-4}.\frac{3\ 1.1.3.5.\ .\ .\ .\ (2r-7)}{2^{r-1}|\underline{r-1}}.x^{r-1}$$

$$= 3\times\frac{1.3.5.\ .\ .\ .\ .\ (2r-7)}{2^{r-1}|\underline{r-1}}.x^{r-1}.$$

Example 5. Show that the co-efficient of x^r in the expansion of $(1+x)^{\frac{5}{3}}$ is $\frac{10}{3^r}.\frac{1.4.7.\ .\ .\ .\ .\ (3r-8)}{|\underline{r}}.(-1)^{r-2}$.

(Calcutta University F. A. Paper, 1884.)

Since x^r occurs in the $(r+1)$th term,

∴ the required co-efficient

$$= \frac{\frac{5}{3}(\frac{5}{3}-1)(\frac{5}{3}-2)\ .\ .\ .\ .\ (\frac{5}{3}-r+1)}{|\underline{r}}$$

$$= \frac{5.2.(-1)(-4)(-7)\ .\ .\ .\ (-3r+8)}{3^r|\underline{r}}$$

$$= (-1)^{r-2}.\frac{5.2.1\ 4.7.\ .\ .\ .\ .\ (3r-8)}{3^r|\underline{r}}$$

(∵ of the r factors in the numerator 2 only are positive and the rest negative)

$$= \frac{10}{3^r}.\frac{1.4.7.\ .\ .\ .\ .\ (3r-8)}{|\underline{r}}.(-1)^{r-2}.$$

Exercise (80).

1. Expand $(1-x)^{-1}$ to 10 terms.
2. Expand $(1-x)^{-2}$ to 8 terms.
3. Expand $(1-x)^{-3}$ to 7 terms.
4. Expand $(1-x)^{-\frac{1}{2}}$ to 5 terms.
5. Expand $(1-2x)^{-3}$ to 5 terms.
6. Expand $\left(1+\frac{2x}{3}\right)^{\frac{3}{2}}$ to 4 terms.
7. Expand $(9-6x)^{-\frac{3}{2}}$ to 4 terms.

8. Expand $(1-x^2)^{\frac{7}{3}}$ to 5 terms.

9. Expand $(1-x)^{-\frac{2}{3}}$ to 4 terms.

10. Expand $\left(1-\frac{x}{4}\right)^{-4}$ to 5 terms.

11. Expand $(3a^{-1}-2x)^{-4}$ to 5 terms.

12. Expand $\dfrac{3a}{(ax-x^2)^{\frac{1}{3}}}$ to 4 terms.

13. Write down the co-efficients of x^6 and x^{10} in the expansion of $(a^3+3bx^2)^{-5}$.

(Calcutta University F. A. Paper, 1878.)

14. Write down the 11th term of $(1-2x^3)^{\frac{11}{2}}$.

15. Write down the $(r+1)$th term in the expansion of $(1-px)^{\frac{1}{p}}$.

16. Write down the $(r+1)$th term in the expansion of $\dfrac{1}{\sqrt[4]{1-x}}$.

17. Find the $(r+1)^{\text{th}}$ term in the expansion of $(1-2x)^{-\frac{7}{2}}$.

18. Find the r^{th} term of $(a^2+x^2)^{-2}$.

19. Find the co-efficient of x^{12} in $(a^5-b^3x^2)^{\frac{5}{2}}$.

20. Find the 14^{th} term of $(2^{10}-2^7.x)^{\frac{13}{2}}$.

21. Expand $\dfrac{3a}{(a^3-x^2)^{\frac{1}{3}}}$ to 5 terms and write down the $(r+1)^{\text{th}}$ term of the expansion.

22. Write down the r^{th} term of the expansion of $(a-x)^{-\frac{1}{n}}$.

(Calcutta University F. A. Paper, 1885.)

23. Find the $(r+1)^{\text{th}}$ term of $(1-2x)^{-\frac{3}{2}}$.

(Calcutta University F. A. Paper, 1889.)

2. The expansion of $(1+x)^n$ by the Binomial Theorem is not always arithmetically intelligible.

For instance, since

$$(1-x)^{-1} = 1+x+x^2+x^3+x^4+\dots \quad (1)$$

putting $x = 2$, we have

$$(-1)^{-1} = 1+2+2^2+2^3+2^4+\dots$$

This curious result certainly justifies us in holding that

$$1+nx+\frac{n(n-1)}{1.2}x^2+\frac{n(n-1)(n-2)}{1.2.3}x^3+\dots$$

is not in all cases the true arithmetical equivalent of $(1+x)^n$.

Now from the formula for the sum of a Geometrical Progression, we know that the sum of the first r terms of the series (1)

$$= \frac{1-x^r}{1-x} = \frac{1}{1-x} - \frac{x^r}{1-x}$$

and when x is numerically less than 1, by taking r sufficiently large we can make $\frac{x^r}{1-x}$ as small as we please; that is by taking a sufficient number of terms the sum can be made to differ as little as we please from $\frac{1}{1-x}$. But when x is numerically greater than 1, the value of $\frac{x^r}{1-x}$ increases with r and therefore no such approximation to the value of $\frac{1}{1-x}$ is obtained by taking any number of terms of the series

$$1+x+x^2+x^3+x^4+\dots.$$

Thus the expansion of $(1-x)^{-1}$ in ascending powers of x is arithmetically intelligible *only when x is less than* 1.

N. B. It is easy to see however that we can always expand $(x+y)^n$ by the Binomial Theorem, for we can write the expression in either of the two following forms:—

$$x^n\left(1+\frac{y}{x}\right)^n,\ y^n\left(1+\frac{x}{y}\right)^n;$$

and use the first or the second of them according as x is greater or less than y.

3. To find the numerically greatest term in the expansion of $(1+x)^n$, for any rational value of n.

A general discussion of this is not quite necessary as any particular case can be dealt with exactly in the method illustrated in Art. 5 of the last chapter.

A few particular cases are solved below :—

Example 1. Which is the greatest term in the expansion of $(1+x)^{\frac{21}{2}}$, when $x = \frac{2}{3}$?

$$T_{r+1} = \left(\frac{\frac{21}{2}-r+1}{r}.x\right)\times T_r$$
$$= \left(\frac{\frac{23}{2}-r}{r}\cdot\frac{2}{3}\right)\times T_r.$$

$$\begin{array}{lll} \text{Hence,} & T_{r+1} & > = \text{ or } < T_r \\ \text{according as} & \frac{\frac{23}{2}-r}{r}\cdot\frac{2}{3} & > = \text{ or } < 1, \\ \textit{i.e.}, \text{ according as} & 23-2r & > = \text{ or } < 3r, \\ \textit{i.e.}, \text{ according as} & 23 & > = \text{ or } < 5r, \\ \textit{i.e.}, \text{ according as} & r & < = \text{ or } > 4\frac{3}{5}. \end{array}$$

Thus for all values of r up to 4, T_{r+1} is greater than T_r and for greater values of r, T_{r+1} is less than T_r. Hence, the 5th term is the greatest.

Example 2. Which is the greatest term in the expansion of $(1+x)^{-n}$ when $x = \frac{5}{7}$ and $n = 3$?

$$T_{r+1} = \left(\frac{-n-r+1}{r}.x\right)\times T_r$$
$$= \left(\frac{n+r-1}{r}x\right)\times T_r \textit{ numerically}$$
$$= \left(\frac{2+r}{r}\cdot\frac{5}{7}\right)\times T_r.$$

$$\begin{array}{lll} \text{Hence} & T_{r+1} & > = \text{ or } < T_r \\ \text{according as} & \frac{2+r}{r}\cdot\frac{5}{7} & > = \text{ or } < 1, \\ \textit{i.e.}, \text{ according as} & 10+5r & > = \text{ or } < 7r, \\ \textit{i.e.}, \text{ according as} & 10 & > = \text{ or } < 2r, \\ \textit{i.e.}, \text{ according as} & r & < = \text{ or } > 5. \end{array}$$

Thus for all values of r up to 4, T_{r+1} is greater than T_r; when $r = 5$, $T_{r+1} = T_r$; and for greater values of r, T_{r+1} is less than T_r.

Hence the 5th and 6th terms are the greatest.

Exercise (81).

Which is the greatest term in each of the following expansions?—

1. $(1+x)^{-n}$ when $x = \frac{1}{5}$ and $n = 12$.
2. $(1-7x)^{-\frac{11}{4}}$ when $x = \frac{1}{8}$.
3. $(1+x)^{\frac{3}{2}}$ when $x = \frac{5}{8}$.
4. $(5-4x)^{-7}$ when $x = \frac{1}{2}$.
5. Which is the greatest co-efficient in $(1+x)^{\sqrt{\frac{11}{2}}}$?

4. A few important applications of the Binomial Theorem.

Example 1. Evaluate $\sqrt{24}$ by means of the Binomial Theorem to 5 places of decimals.

(Calcutta University F. A. Paper, 1872.)

$$\begin{aligned}\sqrt{24} &= (24)^{\frac{1}{2}} = (5^2-1)^{\frac{1}{2}} \\ &= 5\left(1-\frac{1}{5^2}\right)^{\frac{1}{2}} \\ &= 5\left[1+\tfrac{1}{2}\cdot\left(-\frac{1}{5^2}\right)+\frac{\frac{1}{2}(\frac{1}{2}-1)}{1.2}\cdot\left(-\frac{1}{5^2}\right)^2\right. \\ &\qquad\left.+\frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{1.2.3}\cdot\left(-\frac{1}{5^2}\right)^3+\&c.\right] \\ &= 5\left[1-\frac{1}{2.5^2}-\frac{1}{2^3.5^4}-\frac{1}{2^4.5^6}-\&c.\right] \\ &= 5\left[1-\frac{2}{10^2}-\frac{2}{10^4}-\frac{2^2}{10^6}-\&c.\right] \\ &= 5[1-(\cdot 02+\cdot 0002+\cdot 000004)-\&c.]\end{aligned}$$

$$= 5(1 - \cdot 020204) \quad \text{[neglecting the other terms]}$$
$$= 5 \times \cdot 979796$$
$$= 4 \cdot 89898.$$

Example 2. Find the value of $\sqrt[5]{3128}$ to 5 places of decimals

$$\sqrt[5]{3128} = (3125+3)^{\frac{1}{5}}$$
$$= (5^5+3)^{\frac{1}{5}}$$
$$= 5\left(1+\frac{3}{5^5}\right)^{\frac{1}{5}}$$
$$= 5\left[1+\frac{1}{5}\cdot\frac{3}{5^5}+\frac{\frac{1}{5}(\frac{1}{5}-1)}{1.2}\cdot\frac{9}{5^{10}}+\&c.\right.$$
$$= 5\left[1+\frac{3}{5^6}-\frac{1.4}{1.2}\cdot\frac{9}{5^{12}}+\&c.\right]$$
$$1+\frac{3.2^6}{10^6}-\frac{18.2^{12}}{10^{12}}+\&c.$$
$$= 5\left(1+\frac{192}{1000,000}\right)$$

(the next term is neglected as more than five figures after the decimal point will be zeros)

$$= 5 \times 1 \cdot 000192$$
$$= 5 \cdot 00096.$$

Example 3. If x be small compared with unity, shew that

$$\frac{\sqrt{1+x}+\sqrt[3]{(1-x)^2}}{1+x+\sqrt{1+x}} = 1-\frac{5x}{6} \text{ nearly.}$$

Since x is small compared with unity we can neglect x^2, x^3, &c., and so for an approximate value it will be sufficient to retain only the first two terms in the expansion of each binomial.

Hence the given expression

$$= \frac{(1+x)^{\frac{1}{2}}+(1-x)^{\frac{2}{3}}}{1+x+(1+x)^{\frac{1}{2}}}$$
$$= \frac{(1+\frac{1}{2}x)+(1-\frac{2}{3}x)}{1+x+(1+\frac{1}{2}x)}$$

$$\frac{2-\frac{1}{6}x}{2+\frac{3}{2}x}$$

$$\frac{1-\frac{1}{12}x}{1+\frac{3}{4}x}$$

$$= (1-\tfrac{1}{12}x)(1+\tfrac{3}{4}x)^{-1}$$

$$= (1-\tfrac{1}{12}x)(1-\tfrac{3}{4}x)$$

$$= 1-\tfrac{1}{12}x-\tfrac{3}{4}x \qquad \text{(the term involving } x^2 \text{ being neglected)}$$

$$= 1-\tfrac{5}{6}x.$$

Example 4. If c be a quantity so small that c^3 may be neglected in comparison with l^3, shew that $\sqrt{\frac{l}{l+c}}+\sqrt{\frac{l}{l-c}}$ is *very nearly* equal to $2+\frac{3c^2}{4l^2}$. (Calcutta University F. A. Paper, 1888.)

We have

$$\sqrt{\frac{l}{l+c}}=\sqrt{\frac{1}{1+\frac{c}{l}}}$$

$$=\left(1+\frac{c}{l}\right)^{-\frac{1}{2}}$$

$$=1-\frac{1}{2}\cdot\frac{c}{l}+\frac{(-\frac{1}{2})(-\frac{3}{2})}{1.2}\frac{c^2}{l^2}$$

(neglecting the subsequent terms since each of them contains $\frac{c^3}{l^3}$ as a factor and is therefore very small)

$$=1-\frac{1}{2}\cdot\frac{c}{l}+\frac{3}{8}\cdot\frac{c^2}{l^2};$$

and in like manner

$$\sqrt{\frac{l}{l-c}}=\left(1-\frac{c}{l}\right)^{-\frac{1}{2}}$$

$$=1+\frac{1}{2}\cdot\frac{c}{l}+\frac{3}{8}\cdot\frac{c^2}{l^2}.$$

Hence, the given expression

$$=2+\frac{3}{4}\cdot\frac{c^2}{l^2}.$$

Example 5. Find the first two terms in negative powers of x of the expansion of

$$\frac{x-h}{\left\{(x-h)^2+k^2\right\}^{\frac{3}{2}}}-\frac{x+h}{\left\{(x+h)^2+k^2\right\}^{\frac{3}{2}}}$$ where x is large compared with either h or k.

(Calcutta University F. A. Paper, 1880.)

$$\frac{x-h}{\left\{(x-h)^2+k^2\right\}^{\frac{3}{2}}}=\frac{x-h}{(x-h)^3.\left\{1+\frac{k^2}{(x-h)^2}\right\}^{\frac{3}{2}}}$$

$$=\frac{1}{(x-h)^2\left\{1+\frac{3}{2}\cdot\frac{k^2}{(x-h)^2}\right\}}$$

(neglecting higher powers of $\frac{k^2}{(x-h)^2}$ which are all very small, because $x-h$ is large compared with k)

$$=\frac{1}{(x-h)^2}\left\{1+\frac{3}{2}\cdot\frac{k^2}{(x-h)^2}\right\}^{-1}$$

$$=\frac{1}{(x-h)^2}\left\{1-\frac{3}{2}\cdot\frac{k^2}{(x-h)^2}\right\}$$

$$=\frac{1}{(x-h)^2}-\frac{3k^2}{2}\cdot\frac{1}{(x-h)^4}$$

$$=\frac{1}{x^2}\left(1-\frac{h}{x}\right)^{-2}-\frac{3k^2}{2x^4}\left(1-\frac{h}{x}\right)^{-4}$$

$$=\frac{1}{x^2}\left(1+2.\frac{h}{x}+3.\frac{h^2}{x^2}+4.\frac{h^3}{x^3}+\&c.\right)$$

$$-\frac{3k^2}{2x^4}\left(1+4\frac{h}{x}+\frac{4.5}{1.2}\cdot\frac{h^2}{x^2}+\&c.\right) \quad \ldots\ldots \quad (1)$$

In like manner

$$\frac{x+h}{\left\{(x+h)^2+k^2\right\}^{\frac{3}{2}}}=\frac{x+h}{(x+h)^3\left\{1+\frac{k^2}{(x+h)^2}\right\}^{\frac{3}{2}}}$$

$$=\frac{1}{(x+h)^2\left\{1+\frac{3}{2}\cdot\frac{k^2}{(x+h)^2}\right\}}$$

$$= \frac{1}{(x+h)^2}\cdot\left\{1 - \frac{3}{2}\cdot\frac{k^2}{(x+h)^2}\right\}$$

$$= \frac{1}{x^2}\left(1+\frac{h}{x}\right)^{-2} - \frac{3k^2}{2x^4}\left(1+\frac{h}{x}\right)^{-4}$$

$$= \frac{1}{x^2}\left(1 - 2.\frac{h}{x} + 3.\frac{h^2}{x^2} - 4.\frac{h^3}{x^3} + \&c.\right)$$

$$- \frac{3k^2}{2x^4}\left(1 - 4.\frac{h}{x} + \frac{4.5}{1.2}.\frac{h^2}{x^2} - \&c\right) \quad \ldots \quad (2)$$

Hence, subtracting (2) from (1), the given expression

$$= \frac{4h}{x^3} + \frac{1}{x^5}\left(8h^3 - 12k^2h\right) + \&c.$$

$$= 4hx^{-3} + (8h^3 - 12k^2h)x^{-5} + \&c.$$

Exercise (82).

1. Evaluate the seventh root of 127 to five places of decimals.

2. Extract the cube root of 1·03 to four places of decimals.

3. Find the fifth root of 244 correct to three places of decimals. (Calcutta University F. A. Paper, 1887.)

4. Find to four places of decimals the value of $\sqrt[7]{108}$.

Find to five places of decimals the value of :—

5. $\sqrt[3]{31}$. **6.** $\sqrt[3]{126}$. **7.** $\sqrt[3]{998}$.

If x be so small that its square and higher powers may be neglected, find the value of :—

8. $\dfrac{\sqrt{1-3x} + \sqrt[3]{(1-x)^5}}{\sqrt{4+x}}$. **9.** $\dfrac{(1-x)^{\frac{1}{3}} + (1-5x)^2}{\sqrt[4]{16-x}}$.

10. $\dfrac{(8+3x)^{\frac{2}{3}}}{(2+3x)\sqrt{4-5x}}$. **11.** $\dfrac{\sqrt[3]{27-11x} - \sqrt[7]{1-3x}}{\sqrt[7]{(1-4x)^3} + \sqrt{9+\frac{x}{3}}}$.

5. Miscellaneous Examples.

Example 1. Write down the sum of

$$1 + \frac{1}{4} + \frac{1.3}{4.8} + \frac{1.3.5}{4.8.12} + \&c.$$

The given expression

$$= 1 + \tfrac{1}{2}\cdot(\tfrac{1}{2}) + \frac{1.3}{2.4}\cdot(\tfrac{1}{2})^2 + \frac{1.3.5}{2.4.6}\cdot(\tfrac{1}{2})^3 + \&c.$$

$$1 + \tfrac{1}{2}\cdot(\tfrac{1}{2}) + \frac{\frac{1}{2}\cdot\frac{3}{2}}{1.2}\cdot(\tfrac{1}{2})^2 + \frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}}{1.2.3}\cdot(\tfrac{1}{2})^3 + \&c.$$

$$= (1 - \tfrac{1}{2})^{-\frac{1}{2}} = (\tfrac{1}{2})^{-\frac{1}{2}} = 1 \div (\tfrac{1}{2})^{\frac{1}{2}} = 1 \div \frac{1}{\sqrt{2}} = \sqrt{2}.$$

Example 2. Find the value of

$$1 + \frac{3}{4} + \frac{3.5}{4.8} + \frac{3.5.7}{4.8.12} + \&c.$$

The given expression

$$= 1 + \tfrac{3}{2}\cdot(\tfrac{1}{2}) + \frac{3.5}{2.4}\cdot(\tfrac{1}{2})^2 + \frac{3.5.7}{2.4.6}\cdot(\tfrac{1}{2})^3 + \&c.$$

$$= 1 + \tfrac{3}{2}\cdot(\tfrac{1}{2}) + \frac{\frac{3}{2}\cdot\frac{5}{2}}{1.2}\cdot(\tfrac{1}{2})^2 + \frac{\frac{3}{2}\cdot\frac{5}{2}\cdot\frac{7}{2}}{1.2.3}\cdot(\tfrac{1}{2})^3 + \&c.$$

$$= (1 - \tfrac{1}{2})^{-\frac{3}{2}} = (\tfrac{1}{2})^{-\frac{3}{2}} = 1 \div (\tfrac{1}{2})^{\frac{3}{2}} = 2^{\frac{3}{2}} = \sqrt{8}.$$

Example 3. Prove that

$$\sqrt{2} = \frac{7}{5}\left(1 + \frac{1}{10^2} + \frac{1.3}{1.2}\cdot\frac{1}{10^4} + \frac{1.3.5}{1.2.3}\cdot\frac{1}{10^6} + \&c.\right)$$

(Madras University F. A. Paper, 1888.)

We have

$$1 + \frac{1}{10^2} + \frac{1.3}{1.2}\cdot\frac{1}{10^4} + \frac{1.3.5}{1.2.3}\cdot\frac{1}{10^6} + \&c$$

$$= 1 + \frac{1}{10^2} + \frac{\frac{1}{2}\cdot\frac{3}{2}}{1.2}\cdot\frac{2^2}{10^4} + \frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}}{1.2.3}\cdot\frac{2^3}{10^6} + \&c.$$

$$= 1 + \tfrac{1}{2}\cdot\frac{2}{10^2} + \frac{\frac{1}{2}\cdot\frac{3}{2}}{1.2}\cdot\left(\frac{2}{10^2}\right)^2 + \frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}}{1.2.3}\cdot\left(\frac{2}{10^2}\right)^3 + \&c.$$

$$= \left(1 - \frac{2}{10^2}\right)^{-\frac{1}{2}} = \left(1 - \frac{1}{50}\right)^{-\frac{1}{2}} = \left(\frac{49}{50}\right)^{-\frac{1}{2}} = \left(\frac{50}{49}\right)^{\frac{1}{2}}$$

Hence, the right-hand side

$$= \frac{7}{5}\left(\frac{5}{7}\sqrt{2}\right) = \sqrt{2}.$$

Example 4. Prove that

$$\left(\frac{1+x}{1-x}\right)^n = 1 + n\frac{2x}{1+x} + \frac{n(n+1)}{1.2}\cdot\frac{2^2x^2}{(1+x)^2} + \&c.$$

(Calcutta University F. A. Paper, 1880.)

$$\left(\frac{1+x}{1-x}\right)^n = \left(\frac{1-x}{1+x}\right)^{-n} = \left(1 - \frac{2x}{1+x}\right)^{-n}$$

$$= 1 + n\frac{2x}{1+x} + \frac{n(n+1)}{1.2}\left(\frac{2x}{1+x}\right)^2 + \&c.$$

Example 5. What is the co-efficient of x^r in the expansion of

$$\frac{1+4x^2+x^4}{(1-x)^4}?$$

(Calcutta University F. A. Paper, 1881.)

The given expression

$$= (1+4x^2+x^4)(1-x)^{-4}$$

$$= (1+4x^2+x^4)(1+p_1x+p_2x^2+\&c.+p_rx^r+\&c.)$$

suppose.

Hence, the co-efficient of x^r

$$= p_r + 4p_{r-2} + p_{r-4}.$$

But $p_r = \dfrac{4.5.6.\ \&c.\ (3+r)}{1.2.3.4.5.6.\ \&c.\ r} = \dfrac{(r+3)(r+2)(r+1)}{1.2.3}.$

Hence, the required co-efficient

$$= \frac{(r+3)(r+2)(r+1)}{1.2.3} + 4\frac{(r+1)r(r-1)}{1.2.3} + \frac{(r-1)(r-2)(r-3)}{1.2.3}$$

$$= \frac{(r^3+6r^2+11r+6)+4(r^3-r)+(r^3-6r^2+11r-6)}{6}$$

$$= \frac{6r^3+18r}{6} = r^3+3r.$$

Example 6. Find the co-efficient of x^n in the expansion of

$$\frac{2+x+x^2}{(1+x)^3}.$$

The given expression

$$= (2+x+x^2)(1+x)^{-3}$$

$$= (2+x+x^2)(1+p_1x+p_2x^2+\&c.+p_nx^n+\&c.)$$

suppose.

Hence, the co-efficient of x^n

$$= 2p_n + p_{n-1} + p_{n-2}\,.$$

But $p_n = \dfrac{(-3)(-3-1).\&c.(-3-n+1)}{|\underline{n}}$

$$= (-1)^n.\frac{3.4.5.\&c.(n+2)}{1.2.3.4.5.\&c.n}$$

$$= (-1)^n.\frac{(n+1)(n+2)}{1.2}\,.$$

Hence, the required co-efficient

$$= (-1)^n.2\frac{(n+1)(n+2)}{2} + (-1)^{n-1}.\frac{n(n+1)}{2} + (-1)^{n-2}.\frac{(n-1)n}{2}$$

$$= \frac{(-1)^n}{2}\left\{2(n+1)(n+2) - n(n+1) + (n-1)n\right\}$$

$$\left[\because (-1)^{n-1} = \frac{(-1)^n}{-1} = -(-1)^n, \text{ and } (-1)^{n-2} = \frac{(-1)^n}{(-1)^2} = (-1)^n\right]$$

$$= \frac{(-1)^n}{2}.(2n^2 + 4n + 4)$$

$$= (-1)^n(n^2 + 2n + 2).$$

Example 7. In the expansion of $\dfrac{3x-8}{4-4x+x^2}$ in ascending powers of x, prove that the co-efficient of x^4 is $-\frac{1}{4}$, and find the co-efficient of x^r.

(Bombay University P. E. Paper, 1890.)

We have the given expression

$$= \frac{3x-8}{(2-x)^2} = \frac{3x-8}{4(1-\frac{x}{2})^2}$$

$$= \frac{1}{4}\left(3x-8\right)\left(1-\frac{x}{2}\right)^{-2}$$

$$= \frac{1}{4}\left(3x-8\right)\left(1 + 2.\frac{x}{2} + 3.\frac{x^2}{2^2} + 4.\frac{x^3}{2^3} + \right.$$

$$\left. \ldots . + r.\frac{x^{r-1}}{2^{r-1}} + (r+1).\frac{x^r}{2^r} + \&c.\right).$$

Hence, the co-efficient of x^r

$$= \frac{1}{4}\left[3.\frac{r}{2^{r-1}} - 8.\frac{r+1}{2^r}\right]$$

$$= \frac{3r}{2^{r+1}} - \frac{8(r+1)}{2^{r+2}}$$

$$= \frac{6r - 8(r+1)}{2^{r+2}}$$

$$= -\frac{2r+8}{2^{r+2}} = -\frac{r+4}{2^{r+1}}$$

Hence, the co-efficient of x^4

$$= -\frac{4+4}{2^5} = -\frac{8}{32} = -\frac{1}{4}.$$

Example 8. If $(1-x)^{-n} = p_0 + p_1x + p_2x^2 + \&c. + p_rx^r + \&c.$, find the value of $p_0 + p_1 + p_2 + \&c. + p_r$.

We have

$$(1-x)^{-n} = p_0 + p_1x + p_2x^2 + \&c. + p_rx^r + \&c. \quad \ldots \quad \ldots \quad (1)$$

$$\text{also } (1-x)^{-1} = 1 + x + x^2 + \&c. + x^r + \&c. \quad \ldots \quad \ldots \quad (2)$$

Now, if we multiply together the series on the right-hand sides of (1) and (2), we have the co-efficient of x^r in the product $= p_0 + p_1 + p_2 + \&c. + p_r$; hence this expression

$$= \text{co-efficient of } x^r \text{ in } (1-x)^{-n} \times (1-x)^{-1}$$

$$= \text{co-efficient of } x^r \text{ in } (1-x)^{-(n+1)}$$

$$= \frac{(n+1)(n+2).\&c.(n+r)}{\lfloor r}.$$

Example 9. If $p_r = \frac{1.3.5.\&c.(2r-1)}{2.4.6.\&c.\,2r}$, prove that

$$p_{2n+1} + p_1p_{2n} + p_2p_{2n-1} + \&c. + p_{n-1}p_{n+2} + p_np_{n+1}$$

We have $p_r = \dfrac{\frac{1}{2}.\frac{3}{2}.\frac{5}{2}.\&c.\frac{2r-1}{2}}{1.2.3.\&c.\,r}$

$$= \frac{\frac{1}{2}(\frac{1}{2}+1)(\frac{1}{2}+2).\&c.(\frac{1}{2}+r-1)}{\lfloor r}$$

and $\therefore = $ co-efficient of x^r in $(1-x)^{-\frac{1}{2}}$.

Hence, p_1, p_2, p_3, &c., denote the co-efficients of x, x^2, x^3, &c., in $(1-x)^{-\frac{1}{2}}$.

Thus we have $(1-x)^{-\frac{1}{2}}$

$$= 1 + p_1x + p_2x^2 + \&c. + p_{2n}x^{2n} + p_{2n+1}x^{2n+1} + \&c.$$

Now, if we square both sides, evidently the co-efficient of x^{2n+1} on the right-hand side will be

$$2(p_{2n+1}+p_1p_{2n}+p_2p_{2n-1}+\&c.+p_{n-1}p_{n+2}+p_np_{n+1});$$

and $\therefore$ this expression $=$ the co-efficient of

$$x^{2n+1} \text{ in } \left\{(1-x)^{-\frac{1}{2}}\right\}^2 \; i.e., \text{ in } (1-x)^{-1} = 1.$$

$$\text{Hence, } p_{2n+1}+p_1p_{2n}+\&c.+p_{n-1}p_{n+2}+p_np_{n+1} = \tfrac{1}{2}.$$

Example 10. If n be any positive integer, shew that the integral part of $(3+\sqrt{5})^n$ is an *odd* number.

Let I denote the integral and f the fractional part of $(3+\sqrt{5})^n$.

$$\text{Then } \mathrm{I}+f = 3^n+{}^nC_1.3^{n-1}.\sqrt{5}+{}^nC_2.3^{n-2}.5$$
$$+{}^nC_3.3^{n-3}.(\sqrt{5})^3+\&c. \quad \ldots \quad \ldots \quad (1)$$

Now, $3-\sqrt{5}$ is positive and less than 1, therefore $(3-\sqrt{5})^n$ is a proper fraction; denote it by f';

$$\text{Then } f' = 3^n-{}^nC_1.3^{n-1}.\sqrt{5}+{}^nC_2.3^{n-2}.5$$
$$-{}^nC_3.3^{n-3}.(\sqrt{5})^3+\&c. \quad \ldots \quad \ldots \quad (2)$$

Now add (1) and (2); the *irrational* terms on the right disappear, and we have

$$\mathrm{I}+f+f' = 2\{3^n+{}^nC_2.3^{n-2}.5+\&c.\}$$
$$= \text{an } \textit{even} \text{ integer}.$$

Clearly then $f+f'$ is an integer, and since each of them is a proper fraction, their sum must be less than 2; hence $f+f' = 1$.

$\therefore$ I $=$ an even integer *minus* one $=$ an *odd* integer.

Exercise (83).

1. Write down the rth terms in the expansions of $(4x-3y)^n$ and $(1-3x^2)^{\frac{3}{7}}$, and the co-efficients of x^{11} and x^{12} in the expansion of

$$(1+x+x^2+x^3+\&c. \text{ to infinity})^{-11}.$$

2. Find the co-efficient of x^7 in the expansion of

$$(1+2x+3x^2+4x^3+\&c. \text{ to infinity})^{-3}.$$

3. Find the co-efficient of x^{24} in the expansion of

$$(1+3x+6x^2+10x^3+\&c. \text{ to infinity})^{\frac{2}{3}}.$$

4. Find the co-efficient of x^r in the expansion of

$$(1+2x+3x^2+\&c.\text{ to infinity})^2.$$

5. Find the value of

$$1+\frac{2}{3}\cdot\frac{1}{2}+\frac{2.5}{3.6}\cdot\frac{1}{2^2}+\frac{2.5.8}{3.6.9}\cdot\frac{1}{2^3}+\&c.$$

6. Find the value of

$$1-\frac{1}{2}\cdot\frac{1}{3}+\frac{1.3}{2.4}\cdot\frac{1}{3^2}-\frac{1.3.5}{2.4.6}\cdot\frac{1}{3^3}+\&c.$$

7. Show that

$$\sqrt{3}=1+\frac{1}{3}+\frac{1.3}{3.6}+\frac{1.3.5}{3.6.9}+\frac{1.3.5.7}{3.6.9.12}+\&c.$$

(Bombay University P. E. Paper, 1889.)

8. Prove that

$$1+\frac{1}{6}+\frac{1.4}{6.12}+\frac{1.4.7}{6.12.18}+\frac{1.4.7.10}{6.12.18.24}+\&c.=\sqrt[3]{2}$$

9. Find the value of

$$1+3.\frac{3}{16}+3^2.\frac{3.7}{16.32}+3^3.\frac{3.7.11}{16.32.48}+\&c.$$

10. Show that

$$(a-b)^n=a^n\left\{1-n\left(\frac{b}{a-b}\right)+\frac{n(n+1)}{1.2}\left(\frac{b}{a-b}\right)^2-\&c.\right\}$$

$$\left[(a-b)^n=\left(\frac{1}{a-b}\right)^{-n}=a^{-n}\cdot\left(\frac{a}{a-b}\right)^{-n}=a^n\cdot\left(1+\frac{b}{a-b}\right)^{-n}\right].$$

11. Show that

$$\left(\frac{1+2x}{1+x}\right)^n=1+n\left(\frac{x}{1+2x}\right)+\frac{n(n+1)}{1.2}\left(\frac{x}{1+2x}\right)^2+\&c$$

12. Prove that

$$(1+x)^n=2^n\left\{1-n\cdot\frac{1-x}{1+x}+\frac{n(n+1)}{1.2}\left(\frac{1-x}{1+x}\right)^2-\&c.\right\}.$$

13. Prove that

$$3^n\left\{1+\frac{2n}{5}+\frac{2n(2n+2)}{5.10}+\frac{2n(2n+2)(2n+4)}{5.10.15}+\&c.\right\}$$

$$=2^n\left\{1+\frac{3n}{5}+\frac{3n(3n+3)}{5.10}+\frac{3n(3n+3)(3n+6)}{5.10.15}+\&c.\right\}$$

14. Prove that the expansion of $(1-x^3)^n$ may be put into the form $(1-x)^{3n}+3nx(1-x)^{3n-2}+\frac{3n(3n-3)}{1.2}x^2(1-x)^{3n-4}+\&c.$

15. Find the co-efficient of x^n in the expansion of $\frac{a+bx^2}{(1-cx)^3}$.

16. Find the co-efficient of x^n in the expansion of $\frac{(1-2x)^2}{(1+x)^3}$.

17. Show that the co-efficient of x^{2n} in the expansion of $\frac{(1-2x)^3}{(1-3x^2)^4}$ is $3^{n-1}.\frac{(n+1)(n+2)(5n+3)}{2}$.

18. Find the co-efficients of x^{2n} and x^{2n+1} in the expansion of

$$\frac{(1+x)^2}{(1-x^2)^3}.$$

19. Show that the co-efficient of x^r in the expansion of $(1-4x)^{-\frac{1}{2}}$ is $\frac{1.3.5.\&c.(2r-1)}{\underline{|r}}.2^r$.

20. Show that the co-efficient of x^n in the expansion of $(1-2x+3x^2-4x^3+\&c.)^{-n}$ is $\frac{1.3.5.\&c.\ (2n-1)}{\underline{|n}}.2^n$

21. If generally m_r be the co-efficient of the $(r+1)$th term of $(1-x)^{-m}$, shew that $m_r+(m+1)_{r-1}=(m+1)_r$.

[$(1-x)^{-m}=(1-x)^{-(m+1)}\times(1-x)$; hence equating the co-efficient of x^n, &c.]

22. If n be a positive integer, show that the integral part of $(5+2\sqrt{6})^n$ is an odd number.

23. If $(7+4\sqrt{3})^n=p+\beta$, where n and p are positive integers, and β a proper fraction, show that $(1-\beta)(p+\beta)=1$.

24. Find the co-efficients of x^{2r} and x^{2r+1} in the expansion of $\left(\frac{a+x}{a-x}\right)^{\frac{1}{2}}$.

$$\left[\left(\frac{a+x}{a-x}\right)^{\frac{1}{2}}=\frac{a+x}{(a^2-x^2)^{\frac{1}{2}}}=(a+x)(a^2-x^2)^{-\frac{1}{2}},\ \&c.\right]$$

25. Show that the n^{th} co-efficient in the expansion of $(1-x)^{-n}$ is always the double of the $(n-1)^{th}$.

26. Prove that if n be an *even* positive integer,

$$\frac{1}{\underline{|1}\,\underline{|n-1}} + \frac{1}{\underline{|3}\,\underline{|n-3}} + \frac{1}{\underline{|5}\,\underline{|n-5}} + \&c. + \frac{1}{\underline{|n-1}\,\underline{|1}}$$

$$= \frac{2^{n-1}}{\underline{|n}}.$$

[We have the given expression

$$\frac{1}{\underline{|n}}\left\{ n + \frac{n(n-1)(n-2)}{\underline{|3}} + \&c. \right\}\Bigg]$$

27. If n be any positive integer, prove that

$$\frac{2^n}{n+1} = 1 + \frac{1}{3}\cdot\frac{n(n-1)}{\underline{|2}} + \frac{1}{5}\cdot\frac{n(n-1)(n-2)(n-3)}{\underline{|4}} + \&c.$$

28. Prove that if M differ from N^2 by a small quantity, the square root of M is approximately equal to

$$\frac{3}{2}N - \frac{(3N^2 - M)^2}{8N^3}.$$

(Madras University F. A. Paper, 1891.)

[If $M = N^2 + x$, we have $\sqrt{M} = N + \frac{x}{2N} - \frac{x^2}{8N^3}$ approximately ; now put $M - N^2$ for x.]

CHAPTER XIX.

LOGARITHMS.

1. Definition—The logarithm of any number to a given base is the index of the power to which the base must be raised to be equal to that number.

Thus if $a^x = n$, x is called the logarithm of n to the base a.

Since $5^3 = 125$, 3 is the logarithm of 125 to the base 5.

NOTE. The logarithm of n to the base a is written $\log_a n$; thus the equations $a^x = n$ and $x = \log_a n$ have the same meaning.

2. General propositions :—

(i) The Logarithm of 1 to any base is 0.

For $a^0 = 1$ whatever the value of a may be.

(ii) The logarithm of a product is equal to the sum of the logarithms of its factors.

If m, n, p be any three numbers and a the given base, to prove that $\log_a(mnp) = \log_a m + \log_a n + \log_a p$.

Let $x = \log_a m$, $y = \log_a n$, $z = \log_a p$.

Then $m = a^x$, $n = a^y$, $p = a^z$,

and ∴ we have
$$\begin{aligned} mnp &= a^x \times a^y \times a^z \\ &= a^{x+y+z}. \end{aligned}$$

Therefore, by definition,
$$\begin{aligned} \log_a(mnp) &= x + y + z \\ &= \log_a m + \log_a n + \log_a p. \end{aligned}$$

(iii) The logarithm of a quotient is equal to the logarithm of the dividend diminished by the logarithm of the divisor.

If m and n be any two numbers and a the given base, to prove that
$$\log_a\left(\frac{m}{n}\right) = \log_a m - \log_a n.$$

Let $x = \log_a m$ and $y = \log_a n$.

Then $m = a^x$ and $n = a^y$;

and ∴ we have $\dfrac{m}{n} = \dfrac{a^x}{a^y} = a^{x-y}$.

Therefore, by definition,
$$\log_a\left(\frac{m}{n}\right) = x - y = \log_a m - \log_a n.$$

(iv) The logarithm of any power (integral or fractional) of a number is equal to the product of the index of the power and the logarithm of the number.

If m be any number and a the given base, to prove that
$$\log_a(m^r) = r \times \log_a m,$$
where r is any number integral or fractional.

Let $x = \log_a m$; then $m = a^x$,

and ∴
$$\begin{aligned} m^r &= \left(a^x\right)^r \\ &= a^{rx}. \end{aligned}$$

Therefore, by definition,

$$\begin{aligned} \log_a(m^r) &= rx \\ &= r \times \log_a m. \end{aligned}$$

3. Illustrations of the foregoing principles :—

Example 1. Find the logarithm of 5832 to the base $3\sqrt{2}$.

Let x be the required logarithm.

Then by definition,

$$\begin{aligned} (3\sqrt{2})^x &= 5832 \\ &= 8 \times 729 \\ &= 8 \times 9 \times 81 \\ &= 2^3 \times 3^6 \\ &= (\sqrt{2})^6 \times 3^6 \\ &= (3\sqrt{2})^6\ ; \\ \therefore\quad x &= 6. \end{aligned}$$

Example 2. Simplify $\log \dfrac{3 \times \sqrt{32}}{40 \times \sqrt[3]{18}}$.

The given expression

$$\begin{aligned} &= \log\left(3 \times 32^{\frac{1}{2}}\right) - \log\left(40 \times 18^{\frac{1}{3}}\right) \\ &= \log 3 + \log (32)^{\frac{1}{2}} - \{\log 40 + \log (18)^{\frac{1}{3}}\} \\ &= \log 3 + \tfrac{1}{2} \log 32 - \log 40 - \tfrac{1}{3} \log 18 \\ &= \log 3 + \tfrac{1}{2} \log (2^5) - \log (5 \times 2^3) - \tfrac{1}{3} \log (2 \times 3^2) \\ &= \log 3 + \tfrac{5}{2} \log 2 - (\log 5 + 3 \log 2) - \tfrac{1}{3}(\log 2 + 2 \log 3) \\ &= (1 - \tfrac{2}{3}) \log 3 + (\tfrac{5}{2} - 3 - \tfrac{1}{3}) \log 2 - \log 5 \\ &= \tfrac{1}{3} \log 3 - \tfrac{5}{6} \log 2 - \log 5. \end{aligned}$$

Example 3. Solve the equation

$$a^{2x} = b^{3-x} \times c^{x+5}.$$

Taking logarithms of both sides, we have

$$\begin{aligned} \log a^{2x} &= \log b^{3-x} + \log c^{x+5}\ ; \\ \therefore\ 2x \log a &= (3-x) \log b + (x+5) \log c, \\ \therefore\ x(2 \log a + \log b - \log c) &= 3 \log b + 5 \log c, \\ \therefore\ x &= \frac{3 \log b + 5 \log c}{2 \log a + \log b - \log c}. \end{aligned}$$

Exercise (84).

Find the logarithms of :—

1. 81 to the base $\sqrt[3]{9}$. 2. 1728 to the base $2\sqrt{3}$.

3. 64000 to the base $2\sqrt[3]{5}$. 4. $\cdot 0001\dot{7}$ to the base $5\sqrt{3}$.

Simplify the following :—

5. $\log(\sqrt[3]{m^2}\times\sqrt[5]{n^3})$. 6. $\log\dfrac{a^3\times b^{-4}}{\sqrt[4]{a^{-5}\times b^3}}$.

7. $\log\dfrac{9\times\sqrt[4]{216}}{4:\sqrt[5]{144}}$.

Solve the following equations :—

8. $a^x\times b^{2x+3}=c^{1-x}$. 9. $2^{x+3}=27\times 5^{4-x}$.

10. If a, b, c be in *G. P.*, prove that $\log_a n$, $\log_b n$, $\log_c n$ are in *H. P.*

4. Characteristic ; Mantissa.

The integral part of any logarithm is called the *characteristic*, and the fractional or decimal part is called the *mantissa.*

For instance, since 257 lies between 100 and 1000, *i.e.*, between 10^2 and 10^3, the logarithm of it to the base 10 must lie between 2 and 3, and ∴ be = 2 + a fraction ; here 2 is the characteristic and the unknown fraction, the mantissa.

Again, since ·0257 lies between ·01 and ·1, *i.e.*, between 10^{-2} and 10^{-1}, its logarithm to the base 10 must lie between −2 and −1, and ∴ be = −2 + a fraction ; here −2 is the characteristic and the unknown fraction, the mantissa.

Note 1. From the above exposition, it is clear that in the case of negative logarithms the integral part is called the characteristic *only when the fractional part is positive.* Thus if the logarithm of a number be given as −2·345 (which is equivalent to −2−·345), −2 is *not* the characteristic ; but as the given logarithm can be written as −3 + 1 − ·345 or −3 + ·655, −3 is its *characteristic* and ·655, the *mantissa.*

Note 2. Negative logarithms like −3 + ·655 are written as $\bar{3}\cdot655$; *i.e.*, the negative sign is placed *over* the characteristic, shewing thereby that the *characteristic alone* is negative. There is thus a clear distinction between −3·655 and $\bar{3}\cdot655$

Note 3. Logarithms to the base 10 are called *Common Logarithms.*

5. Advantages of the Common System of Logarithms.

(i) The characteristic of the logarithm of any number can be determined by inspection.

First, let the number be *greater than unity.*

Then, if the integral part of the number consists of two digits, the number evidently lies between 10 and 10^2 ;

if the integral part consists of three digits, the number lies between 10^2 and 10^3 ;

if the integral part consists of four digits, the number lies between 10^3 and 10^4 ;

and so on.

Hence, if the integral part of the number consists of n digits, the number evidently lies between 10^{n-1} and 10^n ; and $\therefore$ its logarithm must be greater than $n-1$ and less than n ;

$\therefore$ the characteristic $= n-1$.

That is, *the characteristic of the logarithm of any number greater than unity is one less than the number of digits in its integral part.*

[Thus the characteristics of the logarithms of the numbers 583, 67015 and 5600205 are 2, 4 and 6 respectively.]

2ndly, let the number be *less than unity and not less than zero.*

If there be no zero immediately after the decimal point (as in ·2803), the number evidently lies between ·1 and 1, *i.e.*, between 10^{-1} and 10^0 ;

if there be only one zero immediately after the decimal point (as in ·02803) it is easy to see that the number lies between ·01 and ·1, *i.e.*, between 10^{-2} and 10^{-1} ;

if there be only two zeros immediately after the decimal point (as in ·002803) it is easy to see that the number lies between ·001 and ·01, *i.e.*, between 10^{-3} and 10^{-2} ;
and so on.

Hence, if there be only n zeros immediately after the decimal point the number evidently lies between $10^{-(n+1)}$ and 10^{-n} and $\therefore$ its logarithm must be greater than $-(n+1)$ and less than $-n$.

$\therefore$ the characteristic $= -(n+1)$.

That is, *the characteristic of the logarithm of a decimal fraction is negative and numerically greater by one than the number of zeros immediately after the decimal point*

[Thus the characteristic of the logarithms of the numbers ·56, ·000208, and ·00001005 are -1, -4 and -5 respectively.]

(ii) If two numbers N and M be so related that $N = M \times 10^p$ where p is any integer, positive or negative, the logarithms of N and M will have the same mantissa.

Let i denote the characteristic and f the mantissa in the logarithm of M, so that $\log M = i + f$.

$$\begin{aligned} \text{Then, } \log N &= \log (M \times 10^p) \\ &= \log M + \log 10^p \\ &= (i+f)+p \\ &= (i+p)+f. \end{aligned}$$

Hence, since $(i+p)$ is *an integer*, positive or negative, f is the mantissa in $\log N$.

Thus $\log M$ and $\log N$ have the same mantissa.

Note. It is evident therefore that the logarithms of the numbers ·0002504 and 25·04 have the same mantissa, for $\cdot 0002504 = 25 \cdot 04 \times 10^{-6}$. Similarly the mantissa is the same in the logarithms of the numbers ·000567008, 567008, 5670·08, ·00567008, 56 7008, 5·67008 and ·567008.

Example 1. Write down the characteristics of the logarithms of the numbers 4300·567 and ·000050008.

The first number has four digits in its integral part;

∴ the characteristic in its logarithm = 3.

The second number has four zeros immediately after the decimal point;

∴ the characteristic in its logarithm = -5.

Example 2. Given $\log 67005 = 4 \cdot 8261072$, find $\log \cdot 00067005$.

The characteristic of the required logarithm is evidently = -4, for there are three zeros after the decimal point; and the mantissa is clearly the same as that of the given logarithm, (because the two numbers differ only in the position of the decimal point).

$$\begin{aligned} \therefore \quad \log \cdot 00067005 &= -4 + \cdot 8261072 \\ &= \bar{4} \cdot 8261072. \end{aligned}$$

Example 3. Find the number of digits in 2^{64}, having given $\log 2 = \cdot30103$.

$$\begin{aligned} \text{We have } \log(2^{64}) &= 64 \log 2 \\ &= 64 \times \cdot30103 \\ &= 19\cdot26592. \end{aligned}$$

Hence, the required number of digits $= 20$.

Example 4. Given $\log 79003 = 4\cdot8976436$, find $\log \sqrt[7]{\cdot000079003}$, correct to seven places of decimals.

The required logarithm

$$\begin{aligned} &= \frac{1}{7} \log \cdot000079003 \\ &= \frac{1}{7}(-5 + \cdot8976436) \\ &= \frac{1}{7}(-7 + 2\cdot8976436) \\ &= -1 + \cdot41394908 \ldots \\ &= 1\cdot4139491. \end{aligned}$$

Note. The artifices here illustrated ought to be carefully noticed.

Example 5. In the preceding example find the value of $\sqrt[7]{\cdot000079003}$, if $\log 2593875 = 6\cdot4139491$.

Let $x =$ the required value.

$$\begin{aligned} \text{Then } \log x &= \log \sqrt[7]{\cdot000079003} \\ &= 1\cdot4139491 \quad \text{(as before)}. \end{aligned}$$

Hence x is a decimal fraction having no zero immediately after the decimal point, and its logarithm has the same mantissa as that of $\log 2593875$.

$$\therefore \; x = \cdot2593875.$$

Example 6. Given $\log 2 = \cdot30103$ and $\log 3 = \cdot4771213$, find the logarithm of $\cdot00015$.

$$\begin{aligned} \text{We have } \log 15 &= \log(3 \times 5) \\ &= \log 3 + \log\left(\tfrac{10}{2}\right) \\ &= \log 3 + \log 10 - \log 2 \\ &= \cdot4771213 + 1 - \cdot30103 \end{aligned}$$

$$\begin{aligned} &= \cdot4771213 \\ &+ \cdot69897 \\ &= 1\cdot1760913. \end{aligned}$$

$$\begin{aligned} \text{Hence } \log(\cdot00015) &= -4 + 1\cdot1760913 \\ &= \bar{4}\cdot1760913. \end{aligned}$$

Exercise (85).

1. Write down the characteristics of the logarithms of the numbers 375609, 2036, 0000020009, 5·678 and ·9876.

2. Given log 53198 = 4·7283375, find the logarithms of 5·3198, ·053198, ·53198 and 531980000.

3. Given log 2 = ·30103 and log 3 = ·4771213, find the number of digits in $(648)^5$.

Given log 35874 = 4·5547798 :—

4. Find the logarithm of $\sqrt[3]{\cdot0000003587 4}$ correct to 7 places of decimals.

5. Find the logarithm of $\sqrt[5]{\cdot00035874}$.

6. Find the value of $(\cdot35874)^5$, correct to six places of decimals, given log 594154 = 5·773899.

7. Find the number of zeros after the decimal point in the value of $(\cdot0035874)^7$.

8. Given log 2 and log 3, find the logarithms of $5\frac{1}{3}$, 125000, ·1875 and 1·6875.

9. Given log 44092388 = 7·6443636, find the seventh root of ·00324.

Solve the following equations :—

10. $5^{5-3x} = 2^{x+2}$.   **11.** $\left.\begin{aligned} 2^{3x+2y} &= 5 \\ 4^{2x} &= 2^{2y+1} \end{aligned}\right\}$.

6. Given the logarithms of all numbers to a certain base *a*, to find the logarithm of any number to a new base *b*.

Let N be any number whose logarithm to the base b is required.

Let y = the required logarithm.

$$
\begin{aligned}
\text{Then,} \qquad N &= b^y, \\
\text{and} \quad \therefore \quad \log_a N &= \log_a(b^y) \\
&= y \log_a b, \\
\therefore \quad y &= \log_a N \times \frac{1}{\log_a b} \\
\text{i.e., } \log_b N &= \log_a N \times \frac{1}{\log_a b} \quad .. \quad ... \quad (1)
\end{aligned}
$$

Thus it is proved that the logarithms of all numbers to a new base (b) are obtained by multiplying their logarithms to the old base (a) by a constant quantity, *viz*, the reciprocal of the logarithm of b to the base a.

Cor. Putting $N = a$ in equation (1), we have $\log_b a \times \log_a b = 1$.

Example. Given $\log_{10} 2 = \cdot 30103$, find $\log_5 64$.

We must have

$$\log_5 64 = \log_{10} 64 \times \frac{1}{\log_{10} 5}.$$

$$
\begin{aligned}
\text{Now,} \qquad \log_{10} 64 = \log_{10}(2^6) &= 6 \log_{10} 2 \\
&= 6 \times \cdot 30103 \\
&= 1 \cdot 80618\,; \\
\text{and} \qquad \log_{10} 5 = \log_{10}\left(\frac{10}{2}\right) &= 1 - \cdot 30103 \\
&= \cdot 69897.
\end{aligned}
$$

Hence the required logarithm

$$= \frac{1 \cdot 80618}{\cdot 69897} = \frac{180618}{69897} = 2 \cdot 58.$$

Exercise (86).

1. Given $\log_{10} 2 = \cdot 301030$ and $\log_{10} 7 = \cdot 845098$, find the logarithm of $\left(\frac{4}{343}\right)^{\frac{1}{2}}$ to the base 1000.

2. Given $\log_{10} 2 = \cdot 3010300$ and $\log_{10} 3 = \cdot 4771213$, find the logarithm of 81 to the base $\sqrt{8}$, correct to three places of decimals.

CHAPTER XX.

EXPONENTIAL THEOREM AND LOGARITHMIC SERIES.

1. To expand a^x in ascending powers of x.

By the Binomial Theorem, if $n > 1$,

$$\left(1+\frac{1}{n}\right)^{nx} = 1 + nx.\frac{1}{n} + \frac{nx(nx-1)}{\lfloor 2}.\frac{1}{n^2}$$
$$+\frac{nx(nx-1)(nx-2)}{\lfloor 3}.\frac{1}{n^3} + \&c.$$

$$= 1 + x + \frac{x\left(x-\frac{1}{n}\right)}{\lfloor 2} + \frac{x\left(x-\frac{1}{n}\right)\left(x-\frac{2}{n}\right)}{\lfloor 3} + \&c. \ldots (1)$$

Hence, putting $x = 1$, we have

$$\left(1+\frac{1}{n}\right)^{n} = 1 + 1 + \frac{1-\frac{1}{n}}{\lfloor 2} + \frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{\lfloor 3} + \&c. \ldots (2)$$

But $\left(1+\frac{1}{n}\right)^{nx} = \left\{\left(1+\frac{1}{n}\right)^{n}\right\}^{x};$

hence, the series (1) is the xth power of the series (2); that is,

$$1 + x + \frac{x\left(x-\frac{1}{n}\right)}{\lfloor 2} + \frac{x\left(x-\frac{1}{n}\right)\left(x-\frac{2}{n}\right)}{\lfloor 3} + \&c.$$

$$= \left\{1 + 1 + \frac{\left(1-\frac{1}{n}\right)}{\lfloor 2} + \frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{\lfloor 3} + \&c.\right\}^{x};$$

and this is true however great n may be. If therefore n be indefinitely increased, $\frac{1}{n}$ vanishes and we have

$$1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \frac{x^4}{\lfloor 4} + \&c.$$
$$= \left(1 + 1 + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \frac{1}{\lfloor 4} + \&c.\right)^{x}.$$

The series $1 + 1 + \frac{1}{\underline{|2}} + \frac{1}{\underline{|3}} + \frac{1}{\underline{|4}} + \&c.$ is usually denoted by e; hence

$$e^x = 1 + x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \frac{x^4}{\underline{|4}} + \&c. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (3)$$

Now suppose $a = e^c$, so that $c = \log_e a$

Then we have

$$a^x = e^{cx},$$

$$\text{and} \quad \therefore \text{ by } (3) = 1 + cx + \frac{c^2x^2}{\underline{|2}} + \frac{c^3x^3}{\underline{|3}} + \&c.$$

$$= 1 + x\log_e a + \frac{x^2(\log_e a)^2}{\underline{|2}} + \frac{x^3(\log_e a)^3}{\underline{|3}} + \&c.$$

This is the *Exponential Theorem.*

2. To prove that e is incommensurable.

For if not, let $e = \frac{m}{n}$, where m and n are positive integers;

$$\text{then } \frac{m}{n} = 1 + 1 + \frac{1}{\underline{|2}} + \frac{1}{\underline{|3}} + \&c. \ldots + \frac{1}{\underline{|n}} + \frac{1}{\underline{|n+1}} + \&c.$$

Multiply both sides by $\underline{|n}$; then

$$m\underline{|n-1} = \text{an integer} + \frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)(n+3)} + \&c.$$

$$\text{But } \frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)(n+3)} + \&c.$$

is a proper fraction, for it is greater than $\frac{1}{n+1}$ and less than the geometrical series

$$\frac{1}{n+1} + \frac{1}{(n+1)^2} + \frac{1}{(n+1)^3} + \&c., \text{ that is, less than } \frac{1}{n}.$$

Thus an integer is equal to an integer *plus* a fraction, which is absurd; therefore e is incommensurable.

3. Miscellaneous Examples.

Example 1. Find the value of $\frac{1}{2}(e+e^{-1})$.

Since $$e^x = 1+x+\frac{x^2}{\underline{|2}}+\frac{x^3}{\underline{|3}}+\frac{x^4}{\underline{|4}}+\&c.$$

Putting $x = 1$, we have

$$e = 1+1+\frac{1}{\underline{|2}}+\frac{1}{\underline{|3}}+\frac{1}{\underline{|4}}+\&c.\,;$$

and putting $x = -1$,

$$e^{-1} = 1-1+\frac{1}{\underline{|2}}-\frac{1}{\underline{|3}}+\frac{1}{\underline{|4}}-\&c.$$

Hence, $$e+e^{-1} = 2\left(1+\frac{1}{\underline{|2}}+\frac{1}{\underline{|4}}+\frac{1}{\underline{|6}}+\&c.\right),$$

$$\therefore\ \tfrac{1}{2}(e+e^{-1}) = 1+\frac{1}{\underline{|2}}+\frac{1}{\underline{|4}}+\frac{1}{\underline{|6}}+\&c.$$

Example 2. Find the value of $\frac{1}{\sqrt[5]{e}}$ to four places of decimals.

(Calcutta University F. A. Paper, 1888.)

Since $\frac{1}{\sqrt[5]{e}} = e^{-\frac{1}{5}}$, $\therefore$ putting $x = -\frac{1}{5}$ in the series for e^x, we have the required value

$$= 1-\frac{1}{5}+\frac{1}{5^2\underline{|2}}-\frac{1}{5^3\underline{|3}}+\frac{1}{5^4\underline{|4}}-\&c.$$

$$= 1-\frac{2}{10}+\frac{2^2}{10^2\underline{|2}}-\frac{2^3}{10^3\underline{|3}}+\frac{2^4}{10^4\underline{|4}}-\&c.$$

$$= 1-{\cdot}2+\frac{{\cdot}04}{2}-\frac{{\cdot}008}{6}+\frac{{\cdot}0004}{6}-\&c.$$

$$= 1-{\cdot}2+{\cdot}02-{\cdot}001\dot{3}+{\cdot}0000\dot{6}-\&c.$$

$$= 1{\cdot}02-{\cdot}2013\ldots \quad \left(\begin{array}{c}\text{neglecting the 5th term which has 4}\\ \text{zeros after the decimal point}\end{array}\right)$$

$$= {\cdot}8187.$$

Example 3. Show that

$$\frac{1}{\underline{|2}}+\frac{1+2}{\underline{|3}}+\frac{1+2+3}{\underline{|4}}+\frac{1+2+3+4}{\underline{|5}}+\&c. = \frac{e}{2}.$$

(Calcutta University F. A. Paper, 1888.)

The nth term of the given series

$$= \frac{1+2+3+\ldots+n}{|\underline{n+1}}$$

$$= \frac{n(n+1)}{2|\underline{n+1}}$$

$$= \frac{1}{2|\underline{n-1}}.$$

Hence, the 1st term $= \frac{1}{2}\left[\because \frac{1}{|\underline{n-1}} = \frac{n}{|\underline{n}} = 1, \text{ when } n = 1.\right]$

„ 2nd „ $= \frac{1}{2|\underline{1}}$

„ 3rd „ $= \frac{1}{2|\underline{2}}$

„ 4th „ $= \frac{1}{2|\underline{3}}$

and so on.

Hence the sum of the series

$$= \frac{1}{2}\left(1 + \frac{1}{|\underline{1}} + \frac{1}{|\underline{2}} + \frac{1}{|\underline{3}} + \&c.\right)$$

2

Example 4. Prove that

$$\frac{2}{|\underline{1}} + \frac{4}{|\underline{3}} + \frac{6}{|\underline{5}} + \frac{8}{|\underline{7}} + \&c. \text{ to infinity} = e,$$

and that $\frac{2}{|\underline{3}} + \frac{4}{|\underline{5}} + \frac{6}{|\underline{7}} + \frac{8}{|\underline{9}} + \&c. \text{ to infinity} = \frac{1}{e}$.

(Calcutta University F. A. Paper, 1887.)

(i) The nth term $= \frac{2n}{|\underline{2n-1}}$

$$= \frac{(2n-1)+1}{|\underline{2n-1}}$$

$$= \frac{1}{|\underline{2n-2}} + \frac{1}{|\underline{2n-1}}$$

Hence, the 1st term $= 1 + \frac{1}{\underline{|1}}$,

" 2nd " $= \frac{1}{\underline{|2}} + \frac{1}{\underline{|3}}$,

" 3rd " $= \frac{1}{\underline{|4}} + \frac{1}{\underline{|5}}$,

and so on.

Hence the required sum

$$= 1 + \frac{1}{\underline{|1}} + \frac{1}{\underline{|2}} + \frac{1}{\underline{|3}} + \frac{1}{\underline{|4}} + \&c.$$

$$= e.$$

(ii) The nth term $= \frac{2n}{\underline{|2n+1}}$

$$= \frac{(2n+1)-1}{\underline{|2n+1}}$$

$$= \frac{1}{\underline{|2n}} - \frac{1}{\underline{|2n+1}}.$$

Hence, the 1st term $= \frac{1}{\underline{|2}} - \frac{1}{\underline{|3}}$,

" 2nd " $= \frac{1}{\underline{|4}} - \frac{1}{\underline{|5}}$,

" 3rd " $= \frac{1}{\underline{|6}} - \frac{1}{\underline{|7}}$,

and so on.

Therefore the required sum

$$= \frac{1}{\underline{|2}} - \frac{1}{\underline{|3}} + \frac{1}{\underline{|4}} - \frac{1}{\underline{|5}} + \&c.$$

$$= 1 - 1 + \frac{1}{\underline{|2}} - \frac{1}{\underline{|3}} + \frac{1}{\underline{|4}} - \frac{1}{\underline{|5}} + \&c.$$

$$= e^{-1}$$

$$= \frac{1}{e}.$$

Example 5. Sum the series

$$\frac{4}{\underline{|1}} + \frac{11}{\underline{|2}} + \frac{22}{\underline{|3}} + \frac{37}{\underline{|4}} + \frac{56}{\underline{|5}} + \&c. \text{ to infinity.}$$

If t_n denote the n^{th} term of the series, it is easy to see, by the method explained in Example 4, Art. 7, Chapter XII., that

$$t_n = \frac{2n^2+n+1}{|\underline{n}}$$

$$= \frac{2n(n-1)+3n+1}{|\underline{n}}$$

$$= \frac{2}{|\underline{n-2}} + \frac{3}{|\underline{n-1}} + \frac{1}{|\underline{n}}$$

$$\text{Hence, } t_1 = 0 + 3 + \frac{1}{|\underline{1}} \left\{ \begin{array}{l} \because \dfrac{1}{|\underline{n-2}} = \dfrac{n(n-1)}{|\underline{n}} = 0, \text{ when } n = 1 \\ \text{and } \dfrac{1}{|\underline{n-1}} = \dfrac{n}{|\underline{n}} = 1, \text{ when } n = 1 \end{array} \right.$$

$$t_2 = 2 + \frac{3}{|\underline{1}} + \frac{1}{|\underline{2}};$$

$$t_3 = \frac{2}{|\underline{1}} + \frac{3}{|\underline{2}} + \frac{1}{|\underline{3}};$$

$$t_4 = \frac{2}{|\underline{2}} + \frac{3}{|\underline{3}} + \frac{1}{|\underline{4}};$$

and so on.

Now, adding up the vertical clounms separately, we have the required sum

$$= 2\left(1 + \frac{1}{|\underline{1}} + \frac{1}{|\underline{2}} + \frac{1}{|\underline{3}} + \&c.\right)$$

$$+ 3\left(1 + \frac{1}{|\underline{1}} + \frac{1}{|\underline{2}} + \frac{1}{|\underline{3}} + \frac{1}{|\underline{4}} + \&c.\right)$$

$$+ \left(\frac{1}{|\underline{1}} + \frac{1}{|\underline{2}} + \frac{1}{|\underline{3}} + \frac{1}{|\underline{4}} + \&c.\right)$$

$$= 2e + 3e + (e-1)$$

$$= 6e - 1.$$

Exercise (87).

1. Prove that when n is indefinitely increased the limit of $\left(1+\frac{x}{n}\right)^n = e^x$.

2. Find the value of $\dfrac{e^2-1}{2e}$.

3. Show that the error committed by taking the first 6 terms only of the expansion of e is less than $\dfrac{1}{600}$.

4. Find $\dfrac{1}{\sqrt{e}}$ correct to two places of decimals.

(Calcutta University F. A. Paper, 1886.)

5. Show that $e^{-1} = 2\left(\dfrac{1}{\underline{|3}} + \dfrac{2}{\underline{|5}} + \dfrac{3}{\underline{|7}} + \&c.\right)$.

6. Find the co-efficient of x^4 in the expansion of $(e^x-1)^2$.

7. Find the value of

$$1 + \frac{1+a}{\underline{|2}} + \frac{1+a+a^2}{\underline{|3}} + \frac{1+a+a^2+a^3}{\underline{|4}} + \&c\text{, to infinity.}$$

8. Expand $\dfrac{e^{5x}+e^x}{e^{3x}}$ in a series of ascending powers of x.

9. Find the co-efficient of x^r in the expansion of $\dfrac{1-ax-x^2}{e^x}$.

10. Find the co-efficient of x^n in the expansion of $\dfrac{a+bx+cx^2}{e^x}$.

11. Express $\frac{1}{2}(e^{ix}+e^{-ix})$ in ascending powers of x, where $i = \sqrt{-1}$.

12. Express $\dfrac{1}{2i}(e^{ix}-e^{-ix})$ in ascending powers of x, where $i = \sqrt{-1}$

13. Shew that $\dfrac{e-1}{e+1} = \dfrac{\dfrac{1}{\underline{|2}} + \dfrac{1}{\underline{|4}} + \dfrac{1}{\underline{|6}} + \ldots\ldots}{\dfrac{1}{\underline{|1}} + \dfrac{1}{\underline{|3}} + \dfrac{1}{\underline{|5}} + \ldots\ldots}$.

(Calcutta University F. A. Paper, 1889.)

$$\left[\frac{e-1}{e+1} = \frac{(e-1)^2}{e^2-1} = \frac{e^2+1-2e}{e^2-1} = \frac{e+e^{-1}-2}{e-e^{-1}} = \&c.\right]$$

14. Shew that $2e = \dfrac{1^2}{\underline{|1}} + \dfrac{2^2}{\underline{|2}} + \dfrac{3^2}{\underline{|3}} + \dfrac{4^2}{\underline{|4}} + \dfrac{5^2}{\underline{|5}} + \&c.$

15. Shew that $3e = 1 + \frac{3}{\underline{|1}} + \frac{5}{\underline{|2}} + \frac{7}{\underline{|3}} + \frac{9}{\underline{|4}} + \&c.$

16. Shew that $e^2 - 1 = \dfrac{1 + \frac{2^2}{\underline{|2}} + \frac{2^4}{\underline{|3}} + \frac{2^6}{\underline{|4}} + \ldots\ldots}{1 + \frac{1}{\underline{|2}} + \frac{2}{\underline{|3}} + \frac{2^2}{\underline{|4}} + \ldots\ldots}$

17. Show that $\dfrac{e^2 + 1}{e^2 - 1} = \dfrac{1 + \frac{1}{\underline{|2}} + \frac{1}{\underline{|4}} + \frac{1}{\underline{|6}} + \ldots\ldots}{1 + \frac{1}{\underline{|3}} + \frac{1}{\underline{|5}} + \frac{1}{\underline{|7}} + \ldots\ldots}$

18. Prove that $\dfrac{1 + \frac{1}{\underline{|2}} + \frac{2}{\underline{|3}} + \frac{2^2}{\underline{|4}} + \frac{2^3}{\underline{|5}} + \ldots}{1 + \frac{1}{\underline{|2}} + \frac{1}{\underline{|4}} + \frac{1}{\underline{|6}} + \ldots} = \dfrac{e}{2}.$

19. Sum the series $\frac{9}{\underline{|1}} + \frac{19}{\underline{|2}} + \frac{35}{\underline{|3}} + \frac{57}{\underline{|4}} + \frac{85}{\underline{|5}} + \&c.$ to infinity.

20. Sum the series

$$\frac{1.4}{\underline{|0}} + \frac{2.5}{\underline{|1}} + \frac{3.6}{\underline{|2}} + \frac{4.7}{\underline{|3}} + \frac{5.8}{\underline{|4}} + \&c. \text{ to infinity.}$$

4. To expand $\log_e(1 + x)$ in ascending powers of x.

By the exponential Theorem, for *all* positive values of a we have

$$a^x = 1 + x\log_e a + \frac{x^2(\log_e a)^2}{\underline{|2}} + \frac{x^3(\log_e a)^3}{\underline{|3}} + \&c. \ldots\ldots (1)$$

Also, since $a^x = \{1 + (a - 1)\}^x$, it is clear that for all positive values of a *less than* 2 we must have, by the Binomial Theorem,

$$a^x = 1 + x(a - 1) + \frac{x(x - 1)}{1.2}(a - 1)^2 + \frac{x(x - 1)(x - 2)}{1.2.3}(a - 1)^3$$

$$+ \frac{x(x - 1)(x - 2)(x - 3)}{1.2.3.4}(a - 1)^4 + \&c. \ldots\ldots$$

$$= 1 + x\{(a - 1) - \tfrac{1}{2}(a - 1)^2 + \tfrac{1}{3}(a - 1)^3 - \tfrac{1}{4}(a - 1)^4 + \&c.\} +$$

terms containing x^2 and higher powers of x . . . . . (2)

Hence, from (1) and (2), for all positive values of a *less than* 2 we have the identity

$$1 + x\log_e a + \frac{x^2(\log_e a)^2}{\underline{|2}} + \frac{x^3(\log_e a)^3}{\underline{|3}} + \&c. \quad . \ . \ . \ . \ . \ . \ .$$

$$= 1 + x\{(a-1) - \tfrac{1}{2}(a-1)^2 + \tfrac{1}{3}(a-1)^3 - \tfrac{1}{4}(a-1)^4 + \&c.\} + \&c.$$

Hence, *assuming* that the co-efficients of like powers of x are the same on both sides, we have

$$\log_e a = (a-1) - \tfrac{1}{2}(a-1)^2 + \tfrac{1}{3}(a-1)^3 - \tfrac{1}{4}(a-1)^4 + \&c.$$

Putting therefore x for $a-1$, we have for all values of x *numerically less than* unity

$$\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \&c. \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

This is the *Logarithmic series.*

Cor. 1. Putting $-x$ for x in (3), we get

$$\log_e(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \&c. \ldots \quad \ldots \quad \ldots \quad (4)$$

Cor. 2. From (3) and (4), we have

$$\log_e\left(\frac{1+x}{1-x}\right) = \log_e(1+x) - \log_e(1-x)$$

$$= 2\left\{x + \frac{x^3}{3} + \frac{x^5}{5} + \&c.\right\} \quad \ldots \quad \ldots \quad (5)$$

Cor. 3. Writing $\frac{n+1}{n}$ for $\frac{1+x}{1-x}$, *i.e.*, $\frac{1}{2n+1}$ for x in (5),

we have $\log_e(n+1) - \log_e n$

$$= 2\left\{\frac{1}{2n+1} + \frac{1}{3(2n+1)^3} + \frac{1}{5(2n+1)^5} + \frac{1}{7(2n+1)^7} + \&c.\right\} \quad \ldots \quad \ldots \quad (6)$$

Note 1. When we know the logarithm of any number, the series (6) (the terms of which diminish rapidly) enables us to calculate the logarithm of the next consecutive number. Thus, since $\log_e 1 = 0$, putting $n = 1$, we get the value of $\log_e 2$; *then* putting $n = 2$, we get the value of $\log_e 3$; and so on.

Note 2. We know from Art. 6, Chap. XIX., that the logarithm of any number to the base 10 will be obtained by multiplying the logarithm of that number to the base e by the constant factor $\frac{1}{\log_e 10}$. This factor is thus known as the ***modulus*** of the common system and if we denote its value by m we have, from the series (6), multiplying both sides by m,

$\log_{10}(n+1) - \log_{10} n = 2m\left\{\frac{1}{2n+1} + \frac{1}{3(2n+1)^3} + \&c.\right\}$. Thus if we know the value of m,* this formula enables us to calculate and tabulate logarithms of all numbers to the base 10.

Note 3. Logarithms to the base e are known as the *Napierian System* named after Napier, their inventor. They are also called *Natural* logarithms.

Example 1. Show that $\log_e 2 - \frac{1}{2}$

$$= \frac{1}{1.2.3} + \frac{1}{3.4.5} + \frac{1}{5.6.7} + \&c.$$

Putting $x = 1$ in the Logarithmic Series, we have

$$\begin{aligned}\log_e 2 &= 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \tfrac{1}{6} + \&c.\\ &= (1 - \tfrac{1}{2}) + (\tfrac{1}{3} - \tfrac{1}{4}) + (\tfrac{1}{5} - \tfrac{1}{6}) + \&c.\\ &= \frac{1}{1.2} + \frac{1}{3.4} + \frac{1}{5.6} + \&c. \quad \ldots \quad (1)\\ \text{and also } &= 1 - (\tfrac{1}{2} - \tfrac{1}{3}) - (\tfrac{1}{4} - \tfrac{1}{5}) - (\tfrac{1}{6} - \tfrac{1}{7}) - \&c.\\ &= 1 - \frac{1}{2.3} - \frac{1}{4.5} - \frac{1}{6.7} - \&c. \quad \ldots \quad (2)\end{aligned}$$

Hence, from (1) and (2),

$$\begin{aligned}2\log_e 2 &= 1 + \left(\frac{1}{1.2} - \frac{1}{2.3}\right) + \left(\frac{1}{3.4} - \frac{1}{4.5}\right)\\ &\quad + \left(\frac{1}{5.6} - \frac{1}{6.7}\right) + \&c.\end{aligned}$$

$$\text{or,}\quad 2\log_e 2 - 1 = \frac{2}{1.2.3} + \frac{2}{3.4.5} + \frac{2}{5.6.7} + \&c.$$

$$\therefore \log_e 2 - \tfrac{1}{2} = \frac{1}{1.2.3} + \frac{1}{3.4.5} + \frac{1}{5.6.7} + \&c.$$

Example 2. If α and β be the roots of $ax^2 + bx + c = 0$, show that

$$\log_e(a - bx + cx^2) = \log_e a + (\alpha + \beta)x - \left(\frac{\alpha^2 + \beta^2}{2}\right)x^2 + \&c.$$

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* The numerical value of m is very nearly equal to ·43429.

Since α and β are the roots of the equation $ax^2+bx+c = 0$, we must have

$$\alpha+\beta = -\frac{b}{a}, \text{ and } \alpha\beta = \frac{c}{a};$$

$$\begin{aligned} \text{and } \therefore\ a-bx+cx^2 &= a\left(1-\frac{b}{a}.x+\frac{c}{a}.x^2\right) \\ &= a\{1+(\alpha+\beta)x+\alpha\beta x^2\} \\ &= a(1+\alpha x)(1+\beta x). \end{aligned}$$

Hence, $\log_e(a-bx+cx^2)$

$$\begin{aligned} &= \log_e a+\log_e(1+\alpha x)+\log_e(1+\beta x) \\ &= \log_e a+\left\{\alpha x-\frac{\alpha^2x^2}{2}+\frac{\alpha^3x^3}{3}-\&c.\right\} \\ &\qquad +\left\{\beta x-\frac{\beta^2x^2}{2}+\frac{\beta^3x^3}{3}-\&c.\right\} \\ &= \log_e a+(\alpha+\beta)x-\frac{\alpha^2+\beta^2}{2}x^2+\&c. \end{aligned}$$

Example 3. Show that $\log_e(x+2h) = 2\log_e(x+h)-$

$$\log_e x-\left\{\frac{h^2}{(x+h)^2}+\frac{1}{2}.\frac{h^4}{(x+h)^4}+\frac{1}{3}.\frac{h^6}{(x+h)^6}+\&c.\right\}$$

The right-hand expression

$$\begin{aligned} &= \log_e(x+h)^2-\log_e x+\log_e\left\{1-\frac{h^2}{(x+h)^2}\right\} \\ &= \log_e\left\{\frac{(x+h)^2}{x}\right\}+\log_e\left\{\frac{x^2+2xh}{(x+h)^2}\right\} \\ &= \log_e\left\{\frac{(x+h)^2}{x}\times\frac{x(x+2h)}{(x+h)^2}\right\} \\ &= \log_e(x+2h). \end{aligned}$$

Exercise (88).

1. Show that $\log_e(n+1)-\log_e n$

$$= \frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}-\frac{1}{4n^4}+\&c.$$

2. Prove that $\log_e(n+1)-\log_e(n-1)$

$$= 2\left(\frac{1}{n}+\frac{1}{3n^3}+\frac{1}{5n^5}+\&c.\right).$$

3. Shew how you would calculate the logarithms to the base e of the first 12 natural numbers.

4. Find the *modulus* of the common system correct to 3 places of decimals.

5. Find the value of

$$\frac{1}{3} - \frac{1}{3^2.2} + \frac{1}{3^3.3} - \frac{1}{3^4.4} + \&c.$$

6. Prove that $\frac{a-x}{a} + \frac{1}{2}\left(\frac{a-x}{a}\right)^2 + \frac{1}{3}\left(\frac{a-x}{a}\right)^3 + \&c.$

$$= \log_e a - \log_e x.$$

7. Prove that $3 \log_e(1+x) - \log_e x$

$$-\left\{\frac{1}{(1+x)^3} + \frac{1}{2}\cdot\frac{1}{(1+x)^6} + \frac{1}{3}\cdot\frac{1}{(1+x)^9} + \&c. \right\}$$

$$= \log_e\{1 + (1+x) + (1+x)^2\}.$$

8. Prove that $\log_e 4 = 1 + \frac{2}{1.2.3} + \frac{2}{3.4.5} + \frac{2}{5.6.7} + \&c.$

9. If $\log\frac{1}{1 - x - x^2 + x^3}$ be expanded in a series of powers of x, show that the co-efficient of x^n is $\frac{3}{n}$ or $\frac{1}{n}$ according as n is even or odd.

10. Prove that $\log_e\left\{(1+x)^{\frac{1+x}{2}} \times (1-x)^{\frac{1-x}{2}}\right\}$

$$= \frac{x^2}{1.2} + \frac{x^4}{3.4} + \frac{x^6}{5.6} + \&c.$$

11. If α, β be the roots of the equation $ax^2 + bx + c = 0$, prove that

$$\log_e(ax^2 + bx + c) = \log_e a + 2 \log_e x - \frac{1}{x}\left(\alpha + \beta\right)$$

$$- \frac{1}{2x^2}\left(\alpha^2 + \beta^2\right) - \frac{1}{3x^3}\left(\alpha^3 + \beta^3\right) - \&c.$$

12. Sum the series

$$\frac{1}{1.2} - \frac{1}{2.3} + \frac{1}{3.4} - \frac{1}{4.5} + \&c.$$

13. Sum the series

$$\frac{2}{1}\cdot\frac{1}{3} + \frac{3}{2}\cdot\frac{1}{9} + \frac{4}{3}\cdot\frac{1}{27} + \frac{5}{4}\cdot\frac{1}{81} + \&c.$$

14. Sum the series

$$\tfrac{1}{2}+\tfrac{3}{2}\cdot\tfrac{1}{4}+\tfrac{5}{3}\cdot\tfrac{1}{8}+\tfrac{7}{4}\cdot\tfrac{1}{16}+\&c.$$

15. Prove that

$$\begin{aligned}\log_{10}(x+5) = {}& \log_{10}(x+3)+\log_{10}(x-3)+\log_{10}(x+4)\\ &+\log_{10}(x-4)-\log_{10}(x-5)-2\log_{10}x\\ &-\cdot 86859\ldots\left\{\frac{72}{x^4-25x^2+72}+\frac{1}{3}\left(\frac{72}{x^4-25x^2+72}\right)^3+\&c.\right\}\end{aligned}$$

(Calcutta University F. A. Paper, 1890.)

[From the formulæ :—

$$\text{(i)}\quad \log_e\left(\frac{1+x}{1-x}\right)=2\left\{x+\frac{x^3}{3}+\frac{x^5}{5}+\&c.\right\}$$

and $$\text{(ii)}\quad \log_{10}a=\frac{1}{\log_e 10}\times\log_e a=\cdot 43429\,..\,\log_e a,$$

we have, putting z for $\frac{72}{x^4-25x^2+72}$,

$$\cdot 86859\,..\left\{\frac{72}{x^4-25x^2+72}+\&c.\right\}=\cdot 43429\times 2\left\{z+\tfrac{1}{3}z^3+\&c.\right\}$$

$$=\frac{1}{\log_e 10}\times\log_e\left(\frac{1+z}{1-z}\right)=\log_{10}\left(\frac{1+z}{1-z}\right)=\log_{10}\frac{x^4-25x^2+144}{x^4-25x^2}.$$

Hence the right-hand side $=\log_{10}\dfrac{(x^2-9)(x^2-16)}{x^2(x-5)}-\log_{10}\dfrac{x^4-25x^2+144}{x^4-25x^2}$

$= \&c.$]

16. Expand $\log(1+x+x^2+x^3)$ in powers of x, and find the co-efficients of x^{2n} and x^{2n+1}.

(Calcutta University F. A. Paper, 1891.)

CHAPTER XXI.

IDENTITIES, ELIMINATION AND MISCELLANEOUS ARTIFICES.

1. The method of Indeterminate multipliers for solving simultaneous simple equations involving three unknown quantities.

To solve the equations :—

$$\left.\begin{aligned} a_1x+b_1y+c_1z &= d_1 \,\ldots\ldots\ldots\ldots (1)\\ a_2x+b_2y+c_2z &= d_2 \,\ldots\ldots\ldots\ldots (2)\\ a_3x+b_3y+c_3z &= d_3 \,\ldots\ldots\ldots\ldots (3)\end{aligned}\right\}$$

Multiply (2) by l and (3) by m, (the values of l and m being at present undetermined), and add the equations ; thus we have

$$x(a_1 + la_2 + ma_3) + y(b_1 + lb_2 + mb_3) + z(c_1 + lc_2 + mc_3) = d_1 + ld_2 + md_3.$$

Now suppose the value of l and m are such that the coefficients of y and z are each equal to zero.

Thus we have $x = \dfrac{d_1 + ld_2 + md_3}{a_1 + la_2 + ma_3} \;\ldots.\; (4)$

where l and m have the values given by the equations

$$\left.\begin{aligned} b_1 + lb_2 + mb_3 &= 0 \\ c_1 + lc_2 + mc_3 &= 0 \end{aligned}\right\}$$

But, from these equations, by cross multiplication, we have

$$\frac{1}{b_2c_3 - b_3c_2} = \frac{l}{b_3c_1 - b_1c_3} = \frac{m}{b_1c_2 - b_2c_1},$$

and $\therefore\ l = \dfrac{b_3c_1 - b_1c_3}{b_2c_3 - b_3c_2}$, and $m = \dfrac{b_1c_2 - b_2c_1}{b_2c_3 - b_3c_2}$.

Hence, by substitution in (4), we have

$$x = \frac{d_1(b_2c_3 - b_3c_2) + d_2(b_3c_1 - b_1c_3) + d_3(b_1c_2 - b_2c_1)}{a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1)}.$$

Thus x is determined, and in a similar manner the values of y and z may be *separately* found, if necessary.

Example. Solve the equations—

$$\left.\begin{aligned} 2x - 3y + 5z &= 11 \quad \ldots \quad \ldots \ (1) \\ 5x + 2y - 7z &= -12 \quad \ldots \quad \ldots \ (2) \\ -4x + 3y + z &= 5 \quad \ldots \quad \ldots \ (3) \end{aligned}\right\}$$

Multiply (2) by l, and (3) by m, and add ; thus we have
$x(2 + 5l - 4m) + y(-3 + 2l + 3m) + z(5 - 7l + m) = 11 - 12l + 5m.$

Then $x = \dfrac{11 - 12l + 5m}{2 + 5l - 4m} \quad \ldots \quad \ldots (4)$

where l and m have the values given by the equations

$$\left.\begin{aligned} -3 + 2l + 3m &= 0 \\ 5 - 7l + m &= 0 \end{aligned}\right\}.$$

But from these equations we have

$$\frac{1}{2+21} = \frac{l}{15+3} = \frac{m}{21-10};$$

$$\therefore\ l = \frac{18}{23} \text{ and } m = \frac{11}{23}.$$

Hence, substituting these values of l and m in (1), we have

$$\begin{aligned} x &= \frac{11.23 - 12.18 + 5.11}{2.23 + 5.18 - 4.11} \\ &= \frac{253 - 216 + 55}{46 + 90 - 44} \\ &= \frac{92}{92} = 1. \end{aligned}$$

Now substituting the value of x in (2) & (3), we have

$$\left.\begin{aligned} 2y - 7z + 17 &= 0 \\ 3y + z - 9 &= 0 \end{aligned}\right\}.$$

Hence, $$\frac{y}{63 - 17} = \frac{z}{51 + 18} = \frac{1}{2 + 21},$$

$$\left.\begin{aligned} \therefore\ y &= \frac{46}{23} = 2 \\ \text{and } z &= \frac{69}{23} = 3 \end{aligned}\right\}$$

Thus we have $x = 1$, $y = 2$, $z = 3$.

Exercise (89).

Solve the following equations :—

1. $$\left.\begin{aligned} 3x + 2y + 5z &= 32 \\ 2x + 5y + 3z &= 31 \\ 5x + 3y + 2z &= 27 \end{aligned}\right\}.$$

2. $$\begin{aligned} 4x - 3y + 2z &= 8 \\ 3x - 4y + 5z &= 6 \\ -6x + 5y + 7z &= -1 \end{aligned}$$

3. $$\left.\begin{aligned} x + 3y + 5z &= 10 \\ 3x + 5y + 7z &= 14 \\ 5x + 7y + 8z &= 15 \end{aligned}\right\}.$$

4. $$\left.\begin{aligned} 5x - 4y + 9z &= 19 \\ 7x + 6y - 12z &= 16 \\ -9x + 8y + 15z &= -13 \end{aligned}\right\}$$

5. $$\left.\begin{aligned} a^2x + ay + z &= -a^3 \\ b^2x + by + z &= -b^3 \\ c^2x + cy + z &= -c^3 \end{aligned}\right\}.$$

6. $$\left.\begin{aligned} x + y + z &= a + b + c \\ \frac{x}{a} + \frac{y}{b} + \frac{z}{c} &= 1 \\ \frac{x}{a^3} + \frac{y}{b^3} + \frac{z}{c^3} &= 0 \end{aligned}\right\}$$

7. $$\left.\begin{aligned} 4(y - x) &= 5z - 22 \\ 3z + 4x &= 6y + 2 \\ z - 3y &= 14 - 10x \end{aligned}\right\}.$$

8. $$\begin{aligned} 2(x - y) &= 3z - 2 \\ x - 3z &= 3y - 1 \\ 2x + 3z &= 4(1 - y) \end{aligned}$$

9. $$\left.\begin{aligned} 5x - 11y^{\frac{1}{2}} + 13z^{\frac{1}{3}} &= 22 \\ 4x + 6y^{\frac{1}{2}} + 5z^{\frac{1}{3}} &= 31 \\ x - y^{\frac{1}{2}} + z^{\frac{1}{3}} &= 2 \end{aligned}\right\}.$$

2. If any Algebraical Expression* $p_0x^n + p_1x^{n-1} + p_2x^{n-2} + p_3x^{n-3} + \&c. \; \ldots \; + p_{n-1}x + p_n$, where n is a positive integer, be divided by $x - a$, the remainder will be $p_0a^n + p_1a^{n-1} + p_2a^{n-2} + p_3a^{n-3} + \&c. + p_{n-1}a + p_n$.

Divide the given expression by $x - a$ till a remainder is obtained which does not involve x. Let Q be the quotient and R the remainder ; then

$$p_0x^n + p_1x^{n-1} + p_2x^{n-2} + p_3x^{n-3} + \&c. \ldots \; + p_{n-1}x + p_n = Q(x - a) + R.$$

Now since R does not contain x, it will remain unaltered whatever value be given to x. Hence, putting $x = a$, we have

$$p_0a^n + p_1a^{n-1} + p_2a^{n-2} + \&c. \quad + p_{n-1}a + p_n = Q'(a - a) + R$$

where Q' is what Q becomes when a is put for x.

Thus $R = p_0a^n + p_1a^{n-1} + p_2a^{n-2} + \&c. \quad + p_{n-1}a + p_n$.

Cor. If $p_0a^n + p_1a^{n-1} + p_2a^{n-2} + \&c. \quad + p_{n-1}a + p_n = 0$, then $p_0x^n + p_1x^{n-1} + p_2x^{n-2} + \&c. \quad + p_{n-1}x + p_n$ is divisible by $x - a$. That is, if a rational and integral expression in x vanishes when a is put for x, then that expression is divisible by $x - a$.

Example 1. Resolve into factors $x^3 - 4x^2 - 11x + 30$.

By trial we find that this expression vanishes when $x = 2$; hence $x - 2$ must be one of its factors. This gives us a clue as to how the factorisation is to be performed and we proceed as follows :—

$$\begin{aligned} & x^3 - 4x^2 - 11x + 30 \\ & \quad = x^2(x - 2) - 2x(x - 2) - 15(x - 2) \\ & \quad = (x - 2)(x^2 - 2x - 15) \\ & \quad = (x - 2)(x - 5)(x + 3). \end{aligned}$$

---

*Any Algebraical expression of the form $p_0x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots + p_{n-1}x + p_n$, where n is a positive integer, is called a *rational and integral* expression in x.

Example 2. Find the remainder when $x^3 - 4x^2 - 11x + 30$ is divided by $x + 5$.

The remainder is what the given expression becomes when -5 is put for x; hence the required remainder

$$\begin{aligned} &= (-5)^3 - 4(-5)^2 - 11(-5) + 30 \\ &= -125 - 100 + 55 + 30 \\ &= -225 + 85 \\ &= -140. \end{aligned}$$

Exercise (90).

Resolve into factors :—

1. $x^3 + 4x^2 + x - 6.$
2. $x^3 + 5x^2 - 2x - 24.$
3. $x^3 + 9x^2 - x - 105.$
4. $x^4 + 6x^3 - 5x^2 - 42x + 40.$
5. $x^3 - 18x + 28.$
6. $x^3 - 29x + 60.$

Find the remainder when

7. $3x^3 - 4x^2 + 5x - 2$ is divided by $x - 2$.
8. $5x^4 - 20x^2 - 5x + 27$ is divided by $x + 1$.
9. $5x^5 + 4x^4 + 3x^3 + 2x^2 + x$ is divided by $x + 2$.
10. $5x^3 - 20x^2 + 4x - 9$ is divided by $x + 3$.

3. **If any algebraical expression $p_0x^n + p_1x^{n-1} + p_2x^{n-2} + p_3x^{n-3} + \ldots\ldots + p_{n-1}x + p_n$, where n is a positive integer, vanish when x is equal to each of the n quantities $a_1, a_2, a_3, a_4, \ldots\ldots a_n$, then will $p_0x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots\ldots + p_{n-1}x + p_n = p_0(x - a_1)(x - a_2)(x - a_3) \ldots\ldots\ldots (x - a_n)$.**

For the sake of brevity let us denote the given expression by $f(x)$.

Then since $f(x)$ vanishes when $x = a_1$, $x - a_1$ must be one of its factors. Hence we have

$$f(x) = (x - a_1)(p_0x^{n-1} + \&c \ldots\ldots).$$

Again since $f(x)$ vanishes when $x = a_2$, $x - a_2$ also must be a factor of $f(x)$ and therefore of $(x - a_1)(p_0x^{n-1} + \&c. \dots)$.

Hence, since $x - a_2$ cannot be a factor of $x - a_1$, it must be a factor of $p_0x^{n-1} + \&c.$, and therefore we must have

$$f(x) = (x - a_1)(x - a_2)(p_0x^{n-2} + \&c. \quad ..).$$

Thus it is proved that $f(x)$ is divisible by $(x - a_1)(x - a_2)$.

By similar reasoning, repeated as often as necessary, we come to the conclusion that $f(x)$ is divisible by $(x - a_1)(x - a_2)(x - a_3) \dots (x - a_n)$. But in this case the quotient must evidently be equal to p_0.

Hence, $f(x) = p_0(x - a_1)(x - a_2)(x - a_3) \dots (x - a_n)$.

4. If an algebraical expression of the form $p_0x^n + p_1x^{n-1} + p_2x^{n-2} + p_3x^{n-3} + \dots + p_{n-1}x + p_n$, where n is a positive integer, vanish for more than n different values of x, then each of the co-efficients p_0, p_1, p_2, p_3, ………… p_{n-1}, p_n must be zero.

Suppose the given expression vanishes when x is equal to each of the n quantities $a, b, c, d, \dots h, k$.

Then, by the last article, we must have

$$p_0x^n + p_1x^{n-1} + p_2x^{n-2} + p_3x^{n-3} + \quad \dots \quad + p_{n-1}x + p_n$$
$$= p_0(x - a)(x - b)(x - c) \quad \dots\dots \quad (x - h)(x - k).$$

Hence, if β be another value of x for which the given expression vanishes, we must have

$$p_0(\beta - a)(\beta - b)(\beta - c) \quad \dots\dots \quad (\beta - k) = 0.$$

Hence, we have $p_0 = 0$, because β being different from each of the n quantities $a, b, c, d, \dots h, k$, none of the factors $\beta - a$, $\beta - b$, $\beta - c$, &c. is zero.

Thus it is proved that if a rational and integral expression of the nth degree in x vanish for more than n values of x, the co-efficient of the nth power of x must be zero . . . (1)

Now p_0 being zero, the given expression reduces to $p_1x^{n-1} + p_2x^{n-2} + p_3x^{n-3} + \quad .. \quad + p_{n-1}x + p_n$, which is of the $(n - 1)$th degree; and by hypothesis it vanishes (for more than n and therefore) for more than $n - 1$ values of x.

Hence, by (1), we must have $p_1 = 0$.

Similarly we prove that each of the other co-efficients p_2, p_3, &c. . . . . . p_n is equal to zero.

Cor. Hence if a rational and integral expression of the nth degree in x vanish for *all* values of x, each of its co-efficients must be zero. For if the expression vanish for *all* values of x it certainly vanishes for *more than n* values.

5. If two rational and integral expressions of the nth degree in x be equal to one another for all values of x, then will the co-efficient of any power of x in one expression be equal to the co-efficient of the same power of x in the other expression.

Let $p_0x^n + p_1x^{n-1} + p_2x^{n-2} + \quad . \; . \; . \quad + p_{n-1}x + p_n$

$= q_0x^n + q_1x^{n-1} + q_2x^{n-2} + . \; . \; . \; . \quad + q_{n-1}x + q_n$, for all values of x. Then we must have $p_0 = q_0$, $p_1 = q_1$, $p_2 = q_2$, and so on.

For, by transposition, we have

$(p_0 - q_0)x^n + (p_1 - q_1)x^{n-1} + (p_2 - q_2)x^{n-2} + \; \dots \; + (p_{n-1} - q_{n-1})x + (p_n - q_n) = 0$ for *all* values of x.

Hence, by the corollary to the last article, we have

$$\begin{array}{llll} p_0 - q_0 = 0, & \text{or} & p_0 = q_0 \,; \\ p_1 - q_1 = 0, & \text{or} & p_1 = q_1 \,; \\ p_2 - q_2 = 0, & \text{or} & p_2 = q_2 \,; \\ \&\text{c.} \quad \&\text{c.} & \&\text{c.} & \&\text{c.} \\ p_{n-1} - q_{n-1} = 0, & \text{or} & p_{n-1} = q_{n-1} \,; \\ p_n - q_n = 0, & \text{or} & p_n = q_n . \end{array}$$

Note. The Therorem above established is usually referred to as the ***Principle of Indeterminate Co-efficients.*** Its application wil be illustrated by the following examples.

Example 1. Find the sum of the series $1^2 + 2^2 + 3^2 + \&c. + n^2$.

Assume that

$$1^2 + 2^2 + 3^2 + \&c. + n^2$$
$$= A + Bn + Cn^2 + Dn^3 + En^4$$

where, A, B, C, D, E, are quantities, independent of n, whose values have to be determined.

Then we must have the sum of the series to $(n+1)$ terms, namely

$$1^2+2^2+3^2+\&c. \ldots +n^2+(n+1)^2$$
$$=A+B(n+1)+C(n+1)^2+D(n+1)^3+E(n+1)^4.$$

By subtraction,

$$n^2+2n+1 = B+C(2n+1)+D(3n^2+3n+1)$$
$$+E(4n^3+6n^2+4n+1).$$

Now, since this equation is true for all integral values of n, the co-efficients of the respective powers of n on each side must be the same : thus E must be equal to zero (for there is no term containing n^3 on the left-hand side), and we shall have

$$3D = 1\ ;\ 3D+2C = 2\ ;\ D+C+B = 1\ ;$$
$$\text{whence } D = \tfrac{1}{3}\ ;\ C = \tfrac{1}{2}\ ;\ B = \tfrac{1}{6}.$$

Hence the required sum $= A+\dfrac{n}{6}+\dfrac{n^2}{2}+\dfrac{n^3}{3}$

Since this equation is true for all positive integral values of n, we may put $n = 1$; then we have

$$1^2 = A+\tfrac{1}{6}+\tfrac{1}{2}+\tfrac{1}{3}$$
$$= A+1, \quad \therefore A = 0.$$

Hence the required sum

$$= \tfrac{1}{6}n(2n^2+3n+1)$$
$$= \tfrac{1}{6}n(n+1)(2n+1).$$

Example 2. Find the conditions that $a^2x^4+bx^3+cx^2+dx+f^2$ may be a perfect square.

Assume $a^2x^4+bx^3+cx^2+dx+f^2 = (ax^2+lx+f)^2$.

Now, since this relation is true for all values of x, the co-efficients of like powers of x on both sides must be equal.

Hence we must have

$$\left.\begin{aligned} b &= 2al \quad \ldots \ \ldots \ (1)\\ c &= l^2+2af \quad \ldots \ \ldots \ (2)\\ d &= 2lf \quad \ldots \ \ldots \ (3)\end{aligned}\right\}$$

From (1) and (3), $\dfrac{b}{a} = 2l = \dfrac{d}{f}$, $\therefore$ $ad = bf$ ... (A)

Also from (1) and (2),

$$c = \frac{b^2}{4a^2} + 2af, \quad \therefore \quad 4a^2c - b^2 = 8a^3f \ldots \quad \text{(B)}$$

Thus (A) and (B) are the relations that must hold good in order that the given expression may be a perfect square.

Exercise (91).

Find, by the method of *indeterminate co-efficients*, the sum of :—

1. $1^2 + 4^2 + 7^2 + 10^2 + \&c.$ to n terms.

2. $1.2 + 2.3 + 3.4 + 4.5 + \&c.$ to n terms.

3. $1^2 + 3^2 + 5^2 + 7^2 + \&c.$ to n terms.

4. $1.2 + 3.4 + 5.6 + \&c.$ to n terms.

5. $1^3 + 3^3 + 5^3 + \&c.$ to n terms.

6. Shew that if $x^4 + ax^3 + bx^2 + cx + d$ be a perfect square, the co-efficients satisfy the relations

$$8c = a(4b - a^2) \text{ and } (4b - a^2)^2 = 64d.$$

7.* If $ax^3 + bx^2 + cx + d$ is divisible by $x^2 + h^2$, prove that $ad = bc$.

8. Find the relations subsisting between a, b, c, d, when $ax^3 + bx^2 + cx + d$ is a complete cube.

6. A short method of finding the quotient and remainder when any rational and integral expression in x is divided by $x - a$.

Let the expression $p_0x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots\ldots + p_{n-1}x + p_n$ be divided by $x - a$; and let the quotient be $q_0x^{n-1} + q_1x^{n-2} + q_2x^{n-3} + \ldots\ldots + q_{n-2}x + q_{n-1}$, and the remainder R.

To find q_0, q_1, q_2, &c., and R.

Now, since Dividend = Divisor × Quotient + Remainder, we must have

$$p_0x^n + p_1x^{n-1} + p_2x^{n-2} + p_3x^{n-3} + \ldots\ldots + p_{n-1}x + p_n$$
$$= (x - a)(q_0x^{n-1} + q_1x^{n-2} + q_2x^{n-3} + \ldots q_{n-2}x + q_{n-1}) + R.$$

Hence, equating the co-efficients of like powers of x on both sides, we have

$$\begin{array}{rclcrcl}
 & & & & q_0 & = & p_0\,; \\
q_1 - aq_0 & = & p_1, & \text{or} & q_1 & = & aq_0 + p_1\,; \\
q_2 - aq_1 & = & p_2, & \text{or} & q_2 & = & aq_1 + p_2\,; \\
q_3 - aq_2 & = & p_3, & \text{or} & q_3 & = & aq_2 + p_3\,; \\
\&c. & & \&c. & & \&c. & & \&c. \\
q_{n-1} - aq_{n-2} & = & p_{n-1}, & \text{or} & q_{n-1} & = & aq_{n-2} + p_n \\
R - aq_{n-1} & = & p_n, & \text{or} & R & = & aq_{n-1} + p_n.
\end{array}$$

From the equations on the right we see that q_0 is obtained at once, because it is equal to p_0. We then get q_1, by adding aq_0 to p_1; and then q_2 by adding aq_1 to p_2; and so on. Hence the process of calculating successively the co-efficients of the quotient and the remainder, may be arranged as follows :—

$$\begin{array}{ccccccc}
p_0, & p_1, & p_2, & p_3, & \dots\dots & p_{n-1}, & p_n\,; \\
 & aq_0, & aq_1, & aq_2, & \dots\dots & aq_{n-2}, & aq_{n-1}\,; \\
\hline
q_0, & q_1, & q_2, & q_3, & \dots\dots, & q_{n-1} & R.
\end{array}$$

Note. It is easy to see that if the divisor be $x+a$ the multiplier will be *not* a but $-a$.

Example 1. Find the quotient and remainder when $x^5 - 4x^4 + 7x^3 - 11x - 13$ is divided by $x - 5$.

The successive co-efficients of the dividend are 1, -4, 7, 0, -11 and -13; and the multiplier is 5. Hence the calculation will be performed as follows :—

$$\begin{array}{cccccc}
1 & -4 & 7 & 0 & -11 & -13 \\
 & 5 & 5 & 60 & 300 & 1445 \\
 & & & & & \\
1 & 1 & 12 & 60 & 289 & 1432
\end{array}$$

Thus the quotient is $x^4 + x^3 + 12x^2 + 60x + 289$, and the remainder 1432.

Example 2. Find the quotient and remainder when $3x^5 + 23x^4 - 29x^3 + 64x^2 - 79$ is divided by $x + 9$.

The successive co-efficients of the dividend are 3, 23, -29, 64, 0 and -79; and the multiplier is -9. Hence we have

$$\begin{array}{cccccc}
3 & 23 & -29 & 64 & 0 & -79 \\
 & -27 & 36 & -63 & -9 & 81 \\
\hline
3 & -4 & 7 & 1 & -9 & 2
\end{array}$$

Thus the quotient is $3x^4 - 4x^3 + 7x^2 + x - 9$, and the remainder is 2.

Example 3. Find the value of

$x^8 - 18x^7 + 57x^6 - 11x^5 - 43x^4 + 12x^3 + 31x^2 - 39x + 58$, when $x = 14$.

By Article 2, we know that the required value is the remainder when the given expression is divided by $x - 14$. Let us then find out this remainder by the method exemplified above :—

$$\begin{array}{rrrrrrrrr}
1 & -18 & 57 & -11 & -43 & 12 & 31 & -39 & 58 \\
 & 14 & -56 & 14 & 42 & -14 & -28 & 42 & 42 \\
\hline
1 & -4 & 1 & 3 & -1 & -2 & 3 & 3 & 100
\end{array}$$

Thus the remainder and therefore the required value $= 100$.

Exercise (92).

Find the quotient and remainder when

1. $x^4 + 5x^3 + 3x^2 - 5x + 7$ is divided by $x - 2$.
2. $2x^4 + 7x^3 - 57x^2 + 65$ is divided by $x - 4$.
3. $5x^5 + 13x^4 + 9x^3 + 41x^2 - 17x + 28$ is divided by $x + 3$.
4. $4x^5 + 25x^4 + 31x^3 + 37x^2 + 43x + 75$ is divided by $x + 5$.
5. $x^5 + 10x^4 + 21x^3 - 13x^2 + 9x + 705$ is divided by $x + 7$.
6. $7x^6 - 59x^5 + 25x^4 - 63x^2 + 2x - 100$ is divided by $x - 8$.
7. $x^7 + 15x^6 + 34x^5 - 26x^4 - 29x^3 + 700x + 89$ is divided by $x + 12$.
8. $x^9 + 5x^7 - 44x^3 + 7x + 11$ is divided by $x - 2$.
9. $x^7 - 13x^5 - 35x^3 - 8x + 23$ is divided by $x - 4$.
10. $x^7 + 6x^5 + 25x^2 + 765$ is divided by $x + 5$.

Find the value of :—

11. $3x^5 - 17x^4 + 2x^3 - 196x^2 + 5x + 79$, when $x = 7$.
12. $5x^5 - 28x^4 + 4x^3 - 789x^2 + 38$, when $x = 8$.
13. $x^6 + 7x^5 - 15x^4 + 29x^3 + 38x + 275$, when $x = -9$.
14. $2x^5 - 31x^4 + 59x^3 + 81x^2 - 37x + 49$, when $x = 13$.
15. $x^6 - 25x^5 + 49x^4 - 73x^3 + 87x^2 + 113x + 54$, when $x = 23$.

7. Symmetrical and alternating functions.

Any expression involving a letter is called a *function* of that letter. Thus x^3+5x+8 is a function of x ; a^2+ab+b^2 is a function of a and b ; $a^3+b^3+c^3+2abc$ is a function of a, b and c ; and so on.

The letters of which a function consists are called its *variables*. Thus $x^2+5xy+y^2$ is a function of which the variables are x and y.

An expression which is not altered by the interchange of any pair of its letters is said to be *a symmetrical* function. Thus $(a+b)^4-a^4-b^4$ and $x^3+y^3+z^3-3xyz$ are symmetrical functions.

An expression whose sign only is altered by the interchange of any pair of its letters is called an *alternating* function. Thus $a-b$ and $a^2(b-c)+b^2(c-a)+c^2(a-b)$ are alternating functions.

The product of two symmetrical functions is also a symmetrical function. For instance, let us take the product of $a+b+c$ and $bc+ca+ab$; evidently the product, namely, $(a+b+c)(bc+ca+ab)$, cannot be altered by the interchange of any pair of its letters, say b and c, because neither $a+b+c$ nor $bc+ca+ab$ is altered by such interchange. Similarly, the quotient of two symmetrical functions is also symmetrical.

The product of two alternating functions is a symmetrical function. For instance, let L and M denote two alternating functions, each consisting of the three letters a, b, c ; if any two of these three letters be interchanged, L and M will respectively become $-\text{L}$ and $-M$, and hence their product will still be LM as before. Similarly, the quotient of two alternating functions is also a symmetrical function.

Note. The properties of symmetrical and alternating functions proved above should be well remembered as they are of great use in proving a certain class of identities.

8. IDENTITIES.

Example 1. Show that $(x+y+z)^3-(y+z-x)^3-(z+x-y)^3-(x+y-z)^3=24xyz$.

Let P denote the given expression.

Then P may be regarded as a rational and integral function of x, and it evidently vanishes when $x=0$. Hence, by Cor. Art. 2, $x-0$, *i. e.*, x is one of its factors.

Similarly, y and z also are severally factors of P.

Hence xyz is a factor of P ; and therefore, as P is a function of the third degree, it cannot have any other factor involving the variables.

Hence we have $P = kxyz$ ... ... (α)

where k is a mere number which remains the same for all values of x, y, z.

Evidently therefore we shall find k, if in (α) we give to x, y, z any particular numerical values we like. For instance, putting x, y, z each equal to 1, (α) becomes $24 = k$, and thus k is found.

$$\text{Hence } P = 24xyz.$$

Example 2. Prove that $a^3(b^2-c^2)+b^3(c^2-a^2)+c^3(a^2-b^2) = (a-b)(a-c)(b-c)(ab+bc+ca)$.

Let P denote the left-hand expression. Then P may be regarded as a rational and integral function of a, and if we substitute b for a it becomes $b^3\{(b^2-c^2)+(c^2-b^2)\}$ or zero. Hence by Cor., Art. 2, $a-b$ is a factor of P ; and similarly $a-c$ also is a factor.

Again, since P may also be regarded as a rational and integral function of b and as it vanishes when $b = c$, we must have $b-c$ for one of its factors.

We thus see that P is divisible by $(a-b)(a-c)(b-c)$.

Now this expression as well as P are both alternating functions of a, b, c, and they are respectively of the third and fifth degrees ; the quotient therefore must be a symmetrical function of those letters, and of the *second* degree.

Thus it is clear that besides the factors already found, P has another factor which is of the second degree and symmetrical in a, b, c. This factor must therefore be of the form $l(a^2+b^2+c^2)+m(ab+bc+ca)$, where l and m are mere numbers and remain the same for all values of a, b, c.

Hence we have $a^3(b^2-c^2)+b^3(c^2-a^2)+c^3(a^2-b^2)$

$$= (a-b)(a-c)(b-c)\{l(a^2+b^2+c^2)+m(ab+bc+ca)\} \quad \ldots \quad (\beta)$$

Now, putting $a = 0$, $b = 1$, $c = 2$, in (β),

$$\text{we have} \quad -4 = -2(5l+2m)$$

$$\text{or,} \quad 5l+2m = 2 \, ; \quad \ldots \quad (1)$$

also putting $a = 0$, $b = 1$, $c = 3$, we have

$$-18 = -6(10l+3m)$$

or, $$10l+3m = 3 \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Hence, from (1) and (2), $l = 0$ and $m = 1$.

Thus we have

$P = (a-b)(a-c)(b-c)(ab+bc+ca)$.

Example 3. Shew that $8(a+b+c)^3-(b+c)^3-(c+a)^3-(a+b)^3 = 3(2a+b+c)(a+2b+c)(a+b+2c)$.

Putting x for $b+c$, y for $c+a$ and z for $a+b$, the two sides of the above identity respectively become $(x+y+z)^3-x^3-y^3-z^3$ and $3(y+z)(z+x)(x+y)$, and therefore all that we have to do is to prove the identity of these two expressions.

Let P stand for $(x+y+z)^3-x^3-y^3-z^3$. Then P may be regarded as a rational and integral function of y, and it evidently vanishes when $y = -z$. Hence $y+z$ is one of its factors.

Similarly $z+x$ and $x+y$ also are its factors.

Hence $(y+z)(z+x)(x+y)$ is a factor of P; and therefore, as P is a function of the third degree, it cannot have any other factor involving x, y, z.

Hence we have

$$(x+y+z)^3-x^3-y^3-z^3 = k(y+z)(z+x)(x+y) \quad \ldots \quad (B)$$

where k is a mere number which is the same for all values of x, y, z.

Now putting $x = 0$, $y = 1$, $z = 1$, in (B), we have

$$6 = 2k, \text{ and } \therefore k = 3.$$

Hence, $P = 3(y+z)(z+x)(x+y)$.

Example 4. Shew that $(ca-b^2)(ab-c^2)+(ab-c^2)(bc-a^2)+(bc-a^2)(ca-b^2) = (bc+ca+ab)(bc+ca+ab-a^2-b^2-c^2)$.

We have

$$(ca-b^2)(ab-c^2) = a^2bc-a(b^3+c^3)+b^2c^2,$$
$$(ab-c^2)(bc-a^2) = b^2ca-b(c^3+a^3)+c^2a^2,$$
$$(bc-a^2)(ca-b^2) = c^2ab-c(a^3+b^3)+a^2b^2.$$

Hence, the left-hand expression

$$= (b^2c^2+c^2a^2+a^2b^2)+(a^2bc+b^2ca+c^2ab)$$
$$-\{bc(b^2+c^2)+ca(c^2+a^2)+ab(a^2+b^2)\}$$

$$
\begin{aligned}
&= (b^2c^2 + c^2a^2 + a^2b^2) + 2(a^2bc + b^2ca + c^2ab) \\
&\quad - \{bc(b^2 + c^2 + a^2) + ca(c^2 + a^2 + b^2) + ab(a^2 + b^2 + c^2)\} \\
&= (bc + ca + ab)^2 - (bc + ca + ab)(a^2 + b^2 + c^2) \\
&= (bc + ca + ab)(bc + ca + ab - a^2 - b^2 - c^2).
\end{aligned}
$$

Example 5. Prove that, if $3s = 2(a + b + c)$, $(s - a)^3 + (s - b)^3 + (s - c)^3 - 3(s - a)(s - b)(s - c) = a^3 + b^3 + c^3 - 3abc$.

Since $x^3 + y^3 + z^3 - 3xyz$

$$
\begin{aligned}
&= (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx) \\
&= \tfrac{1}{2}(x + y + z)\{(x - y)^2 + (y - z)^2 + (z - x)^2\},
\end{aligned}
$$

we must have

$$
\begin{aligned}
&(s - a)^3 + (s - b)^3 + (s - c)^3 - 3(s - a)(s - b)(s - c) \\
&= \tfrac{1}{2}\{(s - a) + (s - b) + (s - c)\}[\{(s - a) - (s - b)\}^2 \\
&\quad + \{(s - b) - (s - c)\}^2 + \{(s - c) - (s - a)\}^2] \\
&= \tfrac{1}{2}\{3s - (a + b + c)\}[(b - a)^2 + (c - b)^2 + (a - c)^2] \\
&= \tfrac{1}{2}(a + b + c)\{(a - b)^2 + (b - c)^2 + (c - a)^2\} \\
&= a^3 + b^3 + c^3 - 3abc.
\end{aligned}
$$

Example 6. Prove that

$$
\begin{aligned}
&(x^2 + 2yz)^3 + (y^2 + 2zx)^3 + (z^2 + 2xy)^3 - \\
&3(x^2 + 2yz)(y^2 + 2zx)(z^2 + 2xy) = (x^3 + y^3 + z^3 - 3xyz)^2.
\end{aligned}
$$

If 1, ω, ω^2 be the three cube roots of unity we have

$$
\begin{aligned}
a^3 + b^3 + c^3 - 3abc &= (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca) \\
&= (a + b + c)(a + \omega b + \omega^2 c)(a + \omega^2 b + \omega c).
\end{aligned}
$$

[See example 2, page 221]

Hence, the given expression must be the product of three factors of which the first

$$
\begin{aligned}
&= (x^2 + 2yz) + (y^2 + 2zx) + (z^2 + 2xy) \\
&= (x + y + z)^2,
\end{aligned}
$$

the second

$$
\begin{aligned}
&= (x^2 + 2yz) + \omega(y^2 + 2zx) + \omega^2(z^2 + 2xy) \\
&= x^2 + \omega^4 y^2 + \omega^2 z^2 + 2\omega^2 xy + 2\omega^3 yz + 2\omega zx \\
&= (x + \omega^2 y + \omega z)^2,
\end{aligned}
$$

the third

$$
\begin{aligned}
&= (x^2 + 2yz) + \omega^2(y^2 + 2zx) + \omega(z^2 + 2x)y \\
&= x^2 + \omega^2 y^2 + \omega^4 z^2 + 2\omega xy + 2\omega^3 yz + 2\omega^2 zx \\
&= (x + \omega y + \omega^2 z)^2.
\end{aligned}
$$

Hence the given expression

$$= \{(x+y+z)(x+\omega^2 y+\omega z)(x+\omega y+\omega^2 z)\}^2$$
$$= (x^3+y^3+z^3-3xyz)^2.$$

Example 7. If $(x^2+xy+y^2)(a^2+ab+b^2)$
$= X^2+XY+Y^2$, find the values of X and Y.

If 1, ω, ω^2 be the three cube roots of unity we know that $1+\omega+\omega^2 = 0$.

Hence we have

$$\begin{aligned}(x-\omega y)(x-\omega^2 y) &= x^2-(\omega+\omega^2)xy+\omega^3 y^2\\ &= x^2+xy+y^2\ ;\end{aligned}$$

and similarly,

$$(a-\omega b)(a-\omega^2 b) = a^2+ab+b^2. \qquad \ldots \qquad \ldots \qquad (1)$$

Hence, $(x^2+xy+y^2)(a^2+ab+b^2)$

$$\begin{aligned}&= (x-\omega y)(x-\omega^2 y)(a-\omega b)(a-\omega^2 b)\\ &= \{(x-\omega y)(a-\omega b)\}\{(x-\omega^2 y)(a-\omega^2 b)\}\\ &= \{ax+\omega^2 by-\omega(ay+bx)\}\\ &\qquad \times\{ax+\omega by-\omega^2(ay+bx)\}\\ &= \{ax+(\omega^2+\omega)by-\omega(ay+bx+by)\}\\ &\qquad \times\{ax+(\omega+\omega^2)by-\omega^2(ay+bx+by)\}\\ &= \{(ax-by)-\omega(ay+bx+by)\}\\ &\qquad \times\{(ax-by)-\omega^2(ay+bx+by)\}\end{aligned}$$

and $\therefore$, by (1), $= (ax-by)^2+(ax-by)(ay+bx+by)$
$+ (ay+bx+by)^2.$

Thus we see that $\left.\begin{aligned}X &= ax-by\\ Y &= ay+bx+by\end{aligned}\right\}.$

Example 8. If $a+b+c = 0$, shew that

$$\left(\frac{b-c}{a}+\frac{c-a}{b}+\frac{a-b}{c}\right)\left(\frac{a}{b-c}+\frac{b}{c-a}+\frac{c}{a-b}\right) = 9.$$

We have $\dfrac{b-c}{a}+\dfrac{c-a}{b}+\dfrac{a-b}{c} = \dfrac{bc(b-c)+ca(c-a)+ab(a-b)}{abc}$

$$= -\frac{(b-c)(c-a)(a-b)}{abc},$$

because the numerator

$$\begin{aligned} &= a^2(b-c) - a(b^2-c^2) + bc(b-c) \\ &= (b-c)\{a^2 - a(b+c) + bc\} \\ &= (b-c)(a-c)(a-b). \end{aligned}$$

Also, $\dfrac{a}{b-c} + \dfrac{b}{c-a} + \dfrac{c}{a-b}$

$$= \frac{a(c-a)(a-b) + b(a-b)(b-c) + c(b-c)(c-a)}{(b-c)(c-a)(a-b)}$$

of which the numerator

$$\begin{aligned} &= -a(a-c)(a-b) - b(b-a)(b-c) - c(c-b)(c-a) \\ &= -a\{a^2 - a(b+c) + bc\} - b\{b^2 - b(c+a) + ca\} \\ &\qquad - c\{c^2 - c(a+b) + ab\} \\ &= -a(2a^2+bc) - b(2b^2+ca) - c(2c^2+ab) \end{aligned}$$

[$\because$ by hypothesis $b+c = -a$, $c+a = -b$, $a+b = -c$]

$$\begin{aligned} &= -2(a^3+b^3+c^3) - 3abc \\ &= -2(a^3+b^3+c^3-3abc) - 9abc \\ &= -9abc, \end{aligned}$$

[because $a+b+c$, which is zero, is a factor of the first term];

$$\therefore \quad \frac{a}{b-c} + \frac{b}{c-a} + \frac{c}{a-b} = -\frac{9abc}{(b-c)(c-a)(a-b)}.$$

Hence, the given expression

$$\begin{aligned} &= \left[-\frac{(b-c)(c-a)(a-b)}{abc}\right] \times \left[-\frac{9abc}{(b-c)(c-a)(a-b)}\right] \\ &= 9. \end{aligned}$$

Example 9. If $a+b+c+d = 0$, show that

$$\begin{aligned} (a^3+b^3+c^3+d^3)^2 &= 9(bcd+cda+dab+abc)^2 \\ &= 9(bc-ad)(ca-bd)(ab-cd). \end{aligned}$$

Since $a+b+c+d = 0$, we have $a+b = -(c+d)$,

and $\therefore \quad (a+b)^3 = -(c+d)^3$,

or, $a^3+b^3+3ab(a+b) = -\{c^3+d^3+3cd(c+d)\}$.

Hence, by transposition,

$$\begin{aligned} a^3+b^3+c^3+d^3 &= -3cd(c+d) - 3ab(a+b) \\ &= 3cd(b+a) + 3ab(d+c) \\ &= 3(bcd+cda+dab+abc), \end{aligned}$$

$$\therefore \quad (a^3+b^3+c^3+d^3)^2 = 9(bcd+cda+dab+abc)^2.$$

Again, since $(a+b+c+d)^2$

$$= (a+b)^2 + (c+d)^2 + 2(a+b)(c+d)$$
$$= a^2+b^2+c^2+d^2+2\{ab+cd+(a+b)(c+d)\},$$

we must have this expression equal to zero,

and $\therefore$ $a^2+b^2+c^2+d^2 = -2\{ab+cd+(a+b)(c+d)\}$.

Hence, $(bc-ad)(ca-bd)(ab-cd)$

$$= \{ab(c^2+d^2)-cd(a^2+b^2)\}(ab-cd)$$
$$= a^2b^2(c^2+d^2)+c^2d^2(a^2+b^2)-abcd(a^2+b^2+c^2+d^2)$$
$$= a^2b^2(c^2+d^2)+c^2d^2(a^2+b^2)+2abcd\{ab+cd+(a+b)(c+d)\}$$
$$= a^2b^2(c+d)^2+c^2d^2(a+b)^2+2abcd(a+b)(c+d)$$
$$= \{ab(c+d)+cd(a+b)\}^2$$
$$= (bcd+cda+dab+abc)^2.$$

Thus the required relations are established.

Example 10. If $a+b+c+d = 0$, prove that

$$\frac{a^5+b^5+c^5+d^5}{5} = \frac{a^3+b^3+c^3+d^3}{3} \cdot \frac{a^2+b^2+c^2+d^2}{2}.$$

For all values of a, b, c, d, we have

$$(1-ax)(1-bx)(1-cx)(1-dx)$$
$$= 1-(a+b+c+d)x+(ab+ac+ad+bc+bd+cd)x^2$$
$$-(abc+abd+acd+bcd)x^3+abcdx^4.$$

Hence, when their values are restricted by the given condition, we must have

$$(1-ax)(1-bx)(1-cx)(1-dx)$$

identical with $1-px^2-qx^3-rx^4$,

where $-p$, q, $-r$ respectively stand for $ab+ac+ad+bc+bd+cd$, $abc+abd+acd+bcd$ and $abcd$.

Hence we must have

$$\log(1-ax)+\log(1-bx)+\log(1-cx)+\log(1-dx)$$
$$= \log\{1-(px^2+qx^3+rx^4)\}$$

or, $(ax+\frac{1}{2}a^2x^2+\frac{1}{3}a^3x^3+\text{\&c.})$

$$+(bx+\tfrac{1}{2}b^2x^2+\tfrac{1}{3}b^3x^3+\text{\&c.})$$
$$+(cx+\tfrac{1}{2}c^2x^2+\tfrac{1}{3}c^3x^3+\text{\&c.})$$
$$+(dx+\tfrac{1}{2}d^2x^2+\tfrac{1}{3}d^3x^3+\text{\&c.})$$
$$= (px^2+qx^3+rx^4)+\tfrac{1}{2}(px^2+qx^3+rx^4)^2+\text{\&c.}$$

Therefore, equating the co-efficients of x^2, x^3 and x^5 on both sides, we have

$$\frac{a^2+b^2+c^2+d^2}{2} = p,$$

$$\frac{a^3+b^3+c^3+d^3}{3} = q,$$

$$\frac{a^5+b^5+c^5+d^5}{5} = pq;$$

whence $$\frac{a^5+b^5+c^5+d^5}{5} = \frac{a^3+b^3+c^3+d^3}{3}\cdot\frac{a^2+b^2+c^2+d^2}{2}.$$

Exercise (93).

Prove the following identities :—

1. $a(b+c-a)^2+b(c+a-b)^2+c(a+b-c)^2$
$+(b+c-a)(c+a-b)(a+b-c) = 4abc.$

2. $(a+b+c+d)^4+(a+b-c-d)^4+(a+c-d-b)^4$
$+(a+d-b-c)^4-(a+b+c-d)^4-(b+c+d-a)^4$
$-(c+d+a-b)^4-(d+a+b-c)^4 = 192abcd.$

3. $(b-c)^3+(c-a)^3+(a-b)^3$
$= 3(b-c)(c-a)(a-b).$

4. $a^2(b-c)+b^2(c-a)+c^2(a-b)$
$= (a-b)(a-c)(b-c).$

5. $a(b+c)^2+b(c+a)^2+c(a+b)^2-4abc$
$= (b+c)(c+a)(a+b).$

6. $(yz+zx+xy)^3-y^3z^3-z^3x^3-x^3y^3$
$= 3xyz(y+z)(z+x)(x+y).$

7. $x(y-z)(y+z-x)^4+y(z-x)(z+x-y)^4$
$+z(x-y)(x+y-z)^4 = 16xyz(x-y)(x-z)(y-z).$

8. $(x+y+z)^4-(y+z)^4-(z+x)^4-(x+y)^4+x^4+y^4+z^4$
$= 12xyz(x+y+z).$

9. $a^3(b-c)+b^3(c-a)+c^3(a-b)$
$= (a-b)(a-c)(b-c)(a+b+c).$

10. $a(b-c)^3+b(c-a)^3+c(a-b)^3$
$= (b-c)(c-a)(a-b)(a+b+c).$

11. $(b-c)(b+c)^3+(c-a)(c+a)^3+(a-b)(a+b)^3$

$$= 2(a-b)(a-c)(b-c)(a+b+c).$$

12. $(a+b)^5-a^5-b^5 = 5ab(a+b)(a^2+ab+b^2).$

13. $a^4(b-c)+b^4(c-a)+c^4(a-b)$

$$= (a-b)(a-c)(b-c)(a^2+b^2+c^2+ab+bc+ca).$$

14. $(y-z)^5+(z-x)^5+(x-y)^5$

$$= 5(y-z)(z-x)(x-y)(x^2+y^2+z^2-yz-zx-xy).$$

15. $(x+y+z)^5-(y+z-x)^5-(z+x-y)^5$

$$-(x+y-z)^5 = 80xyz(x^2+y^2+z^2).$$

16. $(x+y+z)^5-x^5-y^5-z^5$

$$= 5(y+z)(z+x)(x+y)(x^2+y^2+z^2+yz+zx+xy).$$

17. $a^5(b-c)+b^5(c-a)+c^5(a-b)$

$$= (a-b)(a-c)(b-c)(a^3+b^3+c^3+a^2b+ab^2$$
$$+a^2c+ac^2+b^2c+bc^2+abc).$$

18. $a(b-c)^5+b(c-a)^5+c(a-b)^5$

$$= (b-c)(c-a)(a-b)(a^3+b^3+c^3+b^2c+bc^2$$
$$+c^2a+ca^2+a^2b+ab^2-9abc).$$

19. $2b^2c^2+2c^2a^2+2a^2b^2-a^4-b^4-c^4$

$$= (a+b+c)(b+c-a)(c+a-b)(a+b-c).$$

20. If $s = a+b+c$, prove that

$$s(s-2a)(s-2b)+s(s-2b)(s-2c)+s(s-2c)(s-2a)$$
$$-(s-2a)(s-2b)(s-2c) = 8abc.$$

21. Show that $27(a+b+c)^3-(a+2b)^3-(b+2c)^3$

$$-(c+2a)^3 = 3(a+3b+2c)(b+3c+2a)(c+3a+2b).$$

22. Show that

$$\frac{(x^2-y^2)^3+(y^2-z^2)^3+(z^2-x^2)^3}{(x-y)^3+(y-z)^3+(z-x)^3} = (x+y)(y+z)(z+x).$$

23. Prove that $(x+y-2z)^3+(y+z-2x)^3+(z+x-2y)^3$

$$= 3(x+y-2z)(y+z-2x)(z+x-2y).$$

24. Show that $a(b-c)(1+ab)(1+ac)$

$$+b(c-a)(1+bc)(1+ba)+c(a-b)(1+ca)(1+cb)$$
$$= abc(a-b)(a-c)(b-c).$$

25. Show that $(b-c)(1+a^2b)(1+a^2c)$

$$+(c-a)(1+b^2c)(1+b^2a)+(a-b)(1+c^2a)(1+c^2b)$$
$$= abc(a-b)(a-c)(b-c)(a+b+c).$$

26. Show that

$$\frac{x^3(y+z)}{(x-y)(x-z)}+\frac{y^3(z+x)}{(y-z)(y-x)}+\frac{z^3(x+y)}{(z-x)(z-y)}$$
$$= xy+yz+zx.$$

27. Prove that

$$a^2.\frac{(a+b)(a+c)}{(a-b)(a-c)}+b^2.\frac{(b+c)(b+a)}{(b-c)(b-a)}+c^2.\frac{(c+a)(c+b)}{(c-a)(c-b)}$$
$$= (a+b+c)^2.$$

28. Show that $(b+c)^3+(c+a)^3+(a+b)^3$

$$-3(b+c)(c+a)(a+b) = 2(a^3+b^3+c^3-3abc).$$

29. If $2s = a+b+c$, prove that

$$(s-a)^3+(s-b)^3+(s-c)^3-3(s-a)(s-b)(s-c)$$
$$=\tfrac{1}{2}(a^3+b^3+c^3-3abc).$$

30. Show that $(x^2-yz)^3+(y^2-zx)^3+(z^2-xy)^3$

$$-3(x^2-yz)(y^2-zx)(z^2-xy) = (x^3+y^3+z^3-3xyz)^2.$$

31. If $a+b+c+d = 0$, prove that

$$a^3+b^3+c^3+d^3+3(a+b)(b+c)(c+a) = 0$$

and $(a+b)(a+c)(a+d) = (b+c)(b+d)(b+a)$

$$= (c+d)(c+a)(c+b) = (d+a)(d+b)(d+c).$$

32. Prove that $a^2(b+c)^2+b^2(c+a)^2+c^2(a+b)^2$

$$+2abc(a+b+c) = 2(bc+ca+ab)^2.$$

33. Show that $(bcd+cda+dab+abc)^2$

$$-abcd(a+b+c+d)^2 = (bc-ad)(ca-bd)(ab-cd).$$

34. If $a+b+c = 0$, show that

$$(a^2+b^2+c^2)^2 = 2(a^4+b^4+c^4).$$

35. Show that $\{(a-b)^2+(b-c)^2+(c-a)^2\}^2$

$$= 2\{(a-b)^4+(b-c)^4+(c-a)^4\}.$$

36. Prove that $\{(a+b-2c)^2+(b+c-2a)^2+(c+a-2b)^2\}^2$

$$= 2\{(a+b-2c)^4+(b+c-2a)^4+(c+a-2b)^4\}.$$

37. If $a+b+c = 0$, show that

$$4(b^2c^2+c^2a^2+a^2b^2) = (a^2+b^2+c^2)^2\ ;$$

and hence prove that

$$(y-z)^2(z-x)^2+(z-x)^2(x-y)^2+(x-y)^2(y-z)^2$$
$$=(x^2+y^2+z^2-xy-yz-zx)^2.$$

38. Show that $(a^2+b^2+c^2)^3+2(bc+ca+ab)^3$
$$-3(a^2+b^2+c^2)(bc+ca+ab)^2=(a^3+b^3+c^3-3abc)^2.$$

39. Show that $3(x+y+z)^3-(x+y)^3-(y+z)^3$
$$-(z+x)^3-x^3-y^3-z^3=6(xy+yz+zx)(x+y+z).$$

40. If $2S=a^2+b^2+c^2$, and $2s=a+b+c$, prove that
$$(S-b^2)(S-c^2)+(S-c^2)(S-a^2)+(S-a^2)(S-b^2)$$
$$=4s(s-a)(s-b)(s-c).$$

41. Show that $(a+b+c)^3$
$$=a^3+b^3+c^3+3\{a(b^2+c^2)+b(c^2+a^2)$$
$$+c(a^2+b^2)\}+6abc.$$

42. Prove that $(a+b+c)^3-27abc$
$$=\tfrac{1}{2}\{(a+b+7c)(a-b)^2+(b+c+7a)(b-c)^2$$
$$+(c+a+7b)(c-a)^2\}.$$

43. Prove that $9(a^3+b^3+c^3)-(a+b+c)^3$
$$=(4a+4b+c)(a-b)^2+(4b+4c+a)(b-c)^2$$
$$+(4c+4a+b)(c-a)^2.$$

44. Show that $x(1-y^2)(1-z^2)+y(1-z^2)(1-x^2)$
$$+z(1-x^2)(1-y^2)-4xyz$$
$$=(x+y+z-xyz)(1-xy-yz-zx).$$

45. Prove that
$$8a^2b^2c^2+(b^2+c^2-a^2)(c^2+a^2-b^2)(a^2+b^2-c^2)$$
$$=(a^2+b^2+c^2)(a+b+c)(b+c-a)(c+a-b)(a+b-c).$$

46. If $xy+yz+zx=1$, prove that
$$\frac{x}{1-x^2}+\frac{y}{1-y^2}+\frac{z}{1-z^2}=\frac{4xyz}{(1-x^2)(1-y^2)(1-z^2)}.$$

47. If $xy+yz+zx=1$, prove that
$$\frac{x+y}{1-xy}+\frac{y+z}{1-yz}+\frac{z+x}{1-zx}=\frac{x+y}{1-xy}\cdot\frac{y+z}{1-yz}\cdot\frac{z+x}{1-zx}.$$

48. If $xy+yz+zx=-1$, prove that
$$\frac{x+y}{1+xy}+\frac{y+z}{1+yz}+\frac{z+x}{1+zx}+\frac{x+y}{1+xy}\cdot\frac{y+z}{1+yz}\cdot\frac{z+x}{1+zx}=0.$$

49. If $a^3+b^3+c^3 = (a+b+c)^3$, prove that
$$a^{2n+1}+b^{2n+1}+c^{2n+1} = (a+b+c)^{2n+1},$$
where n is any positive integer.

50. If $\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c} = \dfrac{1}{a+b+c}$, show that
$$\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^{2n+1} = \frac{1}{a^{2n+1}+b^{2n+1}+c^{2n+1}},$$
where n is any positive integer.

51. Show that $(x+y)^7-x^7-y^7$ is divisible by x^2+xy+y^2.

52. Prove that $(3a-b-c)^3+(3b-c-a)^3+(3c-a-b)^3-3(3a-b-c)(3b-c-a)(3c-a-b)=16(a^3+b^3+c^3-3abc)$.

53. Prove that
$$(x^3+y^3+z^3-3xyz)(a^3+b^3+c^3-3abc)$$
$$= (ax+by+cz)^3+(bx+cy+az)^3+(cx+ay+bz)^3$$
$$-3(ax+by+cz)(bx+cy+az)(cx+ay+bz);$$
and hence shew that the product of any number of factors of the form $x^3+y^3+z^3-3xyz$ can be put into the form $X^3+Y^3+Z^3-3XYZ$.

54. If $a+b+c = 0$, prove that
$$\frac{a^5+b^5+c^5}{5} = \frac{a^3+b^3+c^3}{3}\cdot\frac{a^2+b^2+c^2}{2}.$$

55. If $a+b+c = 0$, prove that
$$\frac{a^7+b^7+c^7}{7} = \frac{a^5+b^5+c^5}{5}\cdot\frac{a^2+b^2+c^2}{2}$$
$$= \frac{a^3+b^3+c^3}{3}\cdot\frac{a^4+b^4+c^4}{4}.$$

56. Show that
$$25\{(b-c)^7+(c-a)^7+(a-b)^7\}\{(b-c)^3+(c-a)^3+(a-b)^3\}$$
$$= 21\{(b-c)^5+(c-a)^5+(a-b)^5\}^2.$$

9 ELIMINATION.

Example 1. Eliminate x from the equations
$a_1x^2+b_1x+c_1 = 0$, and $a_2x^2+b_2x+c_2 = 0$.

Let α be the value of x which satisfies both the equations.

Then we must have

$$\left.\begin{aligned} a_1a^2+b_1a+c_1 &= 0 \\ a_2a^2+b_2a+c_2 &= 0 \end{aligned}\right\}$$

Hence, by cross multiplication,

$$\frac{a^2}{b_1c_2-b_2c_1} = \frac{a}{c_1a_2-c_2a_1} = \frac{1}{a_1b_2-a_2b_1};$$

$$\therefore \quad \frac{a^2}{b_1c_2-b_2c_1}\cdot\frac{1}{a_1b_2-a_2b_1} = \left(\frac{a}{c_1a_2-c_2a_1}\right)^2,$$

whence $(b_1c_2-b_2c_1)(a_1b_2-a_2b_1) = (c_1a_2-c_2a_1)^2$, which is the required Eliminant.

Example 2. Eliminate m between the equations

$$\left.\begin{aligned} y = mx+\frac{a}{m} &= m'x+\frac{a}{m'} \\ \text{and } mm' &= -1 \end{aligned}\right\},$$

and prove that $x+a = 0$.

(Calcutta University F. A. Paper, 1888.)

Since $y = mx+\dfrac{a}{m}$

and also $y = m'x+\dfrac{a}{m'}$,

$$\begin{aligned} \therefore \quad 0 &= (m-m')x+a\left(\frac{1}{m}-\frac{1}{m'}\right) \\ &= (m-m')x-\frac{m-m'}{mm'}.a \\ &= (m-m')\left(x-\frac{a}{mm'}\right) \\ &= (m-m')(x+a). \quad \text{[By the 3rd equation.]} \end{aligned}$$

Hence, since $m-m'$ is *not* zero, we have $x+a = 0$, which is the required Eliminant.

Example 3. Eliminate l, m, a', b' from the equations

$$\left.\begin{aligned} lx+my &= \sqrt{a^2l^2+b^2m^2} \\ mx-ly &= \sqrt{a'^2m^2+b'^2l^2} \\ \text{and} \quad a^2-a'^2 &= b^2-b'^2 = k^2 \end{aligned}\right\}$$

(Calcutta University F. A. Paper, 1891.)

From the 1st equation we have

$$l^2x^2 + m^2y^2 + 2lmxy = a^2l^2 + b^2m^2 \qquad \ldots\ldots \text{(A)}$$

Also, since $a'^2 = a^2 - k^2$ and $b'^2 = b^2 - k^2$, from the 2nd equation we have

$$m^2x^2 + l^2y^2 - 2lmxy = (a^2 - k^2)m^2 + (b^2 - k^2)l^2 \quad \ldots \text{(B)}$$

Now, from (A) and (B) by addition, we have

$$(x^2 + y^2)(l^2 + m^2) = (a^2 + b^2 - k^2)(l^2 + m^2).$$

Hence, dividing both sides by $l^2 + m^2$ which is *not* zero, we must have

$$x^2 + y^2 = a^2 + b^2 - k^2,$$

which is the required Eliminant.

Example 4. Eliminate x, y, z from the equations

$$\frac{ax}{by + cz} = \frac{by}{cz + ax} = \frac{z}{x + y} = \frac{1}{2}.$$

Since $\dfrac{ax}{by + cz} = \dfrac{1}{2}$,

$$\therefore\ 2ax = by + cz, \text{ or } 2ax - by - cz = 0 \quad \ldots \quad (1)$$

Also, $\dfrac{by}{cz + ax} = \dfrac{1}{2}$,

$$\therefore\ 2by = cz + ax, \text{ or } ax - 2by + cz = 0 \quad \ldots \quad (2)$$

From (1) and (2), by corss multiplication, we have

$$\frac{x}{-bc - 2bc} = \frac{y}{-ca - 2ca} = \frac{z}{-4ab + ab}$$

$$\text{or, } \frac{x}{-3bc} = \frac{y}{-3ca} = \frac{z}{-3ab},$$

$$\text{or, } \frac{x}{bc} = \frac{y}{ca} = \frac{z}{ab};$$

and supposing each of the above ratios $= k$, we have $x = k.bc$, $y = k.ca$, $z = k.ab$.

Now since $\dfrac{z}{x + y} = \dfrac{1}{2}$, $\therefore\ \dfrac{kab}{k(bc + ca)} = \dfrac{1}{2}$,

whence $2ab = bc + ca$

and $\therefore\ \dfrac{2}{c} = \dfrac{1}{a} + \dfrac{1}{b}$,

which is the required Eliminant.

Example 5. Eliminate x and y between the equations

$$\left.\begin{array}{l} ax^3+bx^2y+cxy^2+dy^3 = 0 \\ a'x^3+b'x^2y+c'xy^2+d'y^3 = 0 \end{array}\right\}.$$

Dividing the equations by y^3, and putting z for $\frac{x}{y}$, we have

$$az^3+bz^2+cz+d = 0 \quad \ldots \ldots \ldots \ldots \ldots \quad (1)$$

$$a'z^3+b'z^2+c'z+d' = 0 \quad \ldots \ldots \ldots \ldots \ldots \quad (2)$$

Multiplying (1) by a' and (2) by a, we have by subtraction

$$(a'b-ab')z^2+(a'c-ac')z+(a'd-ad') = 0 \quad \ldots \ldots \quad (3)$$

Again, multiplying (1) by d' and (2) by d, we have by subtraction

$$(ad'-a'd)z^3+(bd'-b'd)z^2+(cd'-c'd)z = 0$$

and therefore, since z is *not* zero,

$$(ad'-a'd)z^2+(bd'-b'd)z+(cd'-c'd) = 0 \quad \ldots \quad (4)$$

Now all that we have to do is to eliminate z from (3) and (4).

Hence putting p, q, r for the respective co-efficients in (3), and p', q', r' for those in (4), we have, as in Example 1,

$$(qr'-q'r)(pq'-p'q) = (rp'-r'p)^2$$

for the required Eliminant.

Example 6. Eliminate x, y, z for the equations

$$\left.\begin{array}{ll} \frac{1}{x}+\frac{1}{y}+\frac{1}{z} = \frac{1}{a} & \ldots \quad \ldots \quad (1) \\ x+y+z = b & \ldots \quad \ldots \quad (2) \\ x^3+y^3+z^3 = c^3 & \ldots \quad \ldots \quad (3) \\ xyz = d^3 & \ldots \quad \ldots \quad (4) \end{array}\right\}$$

From (1), $\frac{yz+zx+xy}{xyz} = \frac{1}{a}$,

and $\therefore$ by (4), $yz+zx+xy = \frac{d^3}{a} \ldots \quad \ldots \quad (5)$

From (2), $b^2 = (x^2+y^2+z^2)+2(yz+zx+xy)$

$\therefore$ by (5) $= x^2+y^2+z^2+\frac{2d^3}{a}$.

Hence, $x^2+y^2+z^2 = b^2-\frac{2d^3}{a}$. $\ldots$ (6)

Now from (3) and (4), we have

$$\begin{aligned} c^3 - 3d^3 &= x^3 + y^3 + z^3 - 3xyz \\ &= (x+y+z)\{(x^2+y^2+z^2) - (yz+zx+xy)\} \end{aligned}$$

and $\therefore$ $$= b\left\{\left(b^2 - \frac{2d^3}{a}\right) - \frac{d^3}{a}\right\}. \qquad \text{[By (2), (6) and (5).]}$$

Thus we have

$$c^3 - 3d^3 = b^3 - \frac{3bd^3}{a},$$

whence, $a(b^3 - c^3) = 3d^3(b-a)$,

which is the required Eliminant.

Example 7. Eliminate x, y, z from the equations

$$\left.\begin{aligned} ax + yz &= bc \\ by + zx &= ca \\ cz + xy &= ab \\ xyz &= abc \end{aligned}\right\}$$

and show that $b^3c^3 + c^3a^3 + a^3b^3 = 5a^2b^2c^2$.

Multiplying together the first three equations, we have

$$\begin{aligned} abcxyz + bcy^2z^2 + caz^2x^2 + abx^2y^2 + ax^3yz + bxy^3z \\ + cxyz^3 + x^2y^2z^2 = a^2b^2c^2. \end{aligned}$$

Hence, since $xyz = abc$, we have

$$a^2b^2c^2 + bcy^2z^2 + caz^2x^2 + abx^2y^2 + abc(ax^2 + by^2 + cz^2) = 0,$$

or, $$\begin{aligned} a^2b^2c^2 + bc(a^2x^2 + y^2z^2) + ca(b^2y^2 + z^2x^2) \\ + ab(c^2z^2 + x^2y^2) = 0. \quad \ldots \quad \ldots \quad \ldots \quad (1) \end{aligned}$$

But we have $$\begin{aligned} a^2x^2 + y^2z^2 &= (ax+yz)^2 - 2axyz \\ &= b^2c^2 - 2a^2bc\,; \end{aligned}$$

and similarly, $b^2y^2 + z^2x^2 = c^2a^2 - 2ab^2c$,

and $c^2z^2 + x^2y^2 = a^2b^2 - 2abc^2$.

Hence, from (1),

$$\begin{aligned} a^2b^2c^2 + bc(b^2c^2 - 2a^2bc) + ca(c^2a^2 - 2ab^2c) \\ + ab(a^2b^2 - 2abc^2) = 0, \end{aligned}$$

whence, $b^3c^3 + c^3a^3 + a^3b^3 = 5a^2b^2c^2$,

which is the required Eliminant.

Example 8. Eliminate x, y, z from the equations

$$\left.\begin{aligned} x^2+yz &= a \\ y^2+zx &= b \\ z^2+xy &= c \\ x+y+z &= 0 \end{aligned}\right\} \quad \begin{aligned} &\text{and show that} \\ &\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}=0. \end{aligned}$$

We have
$$\begin{aligned} b+c &= y^2+z^2+x(y+z) \\ &= y^2+z^2-(y+z)^2, \quad [\because x=-(y+z)] \\ &\quad -2yz\,; \end{aligned}$$

$$\left.\begin{aligned} \therefore\quad (b+c)x &= -2xyz. \\ \text{Similarly,}\quad (c+a)y &= -2xyz\,; \\ \text{and}\quad (a+b)z &= -2xyz. \end{aligned}\right\}.$$

Hence, putting P for $-2xyz$, we have

$$(b+c)x = (c+a)y = (a+b)z = P\,;$$

$$\therefore\quad x=\frac{P}{b+c},\quad y=\frac{P}{c+a},\quad z=\frac{P}{a+b}.$$

Hence, from the last of the given equations, we have

$$P\left(\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}\right)=0,$$

whence, since P is *not* zero, we have

$$\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}=0,$$

which is the required Eliminant.

Example 9. Eliminate x, y, z from the equations

$$\left.\begin{aligned} x^2-yz &= a \quad \ldots\quad \ldots\quad \ldots\quad (1) \\ y^2-zx &= b \quad \ldots\quad \ldots\quad \ldots\quad (2) \\ z^2-xy &= c \quad \ldots\quad \ldots\quad \ldots\quad (3) \\ ax+by+cz &= 0 \quad \ldots\quad \ldots\quad \ldots\quad (4) \end{aligned}\right\}.$$

Multiplying (1) by y, (2) by z, and (3) by x, we have

$$\left.\begin{aligned} ay &= x^2y-y^2z \\ bz &= y^2z-z^2x \\ cx &= z^2x-x^2y \end{aligned}\right\}.$$

Hence, by addition,

$$cx+ay+bz=0. \quad \ldots\quad \ldots\quad \ldots\quad (5)$$

Again, multiplying (1) by z, (2) by x, and (3) by y, we have

$$\left.\begin{aligned} az &= x^2z - yz^2 \\ bx &= xy^2 - x^2z \\ cy &= yz^2 - xy^2 \end{aligned}\right\}$$

Hence, by addition,

$$bx + cy + az = 0 \quad \ldots \quad \ldots \quad \ldots \quad (6)$$

Now, from (5) and (6), by cross multiplication, we have

$$\frac{x}{a^2 - bc} = \frac{y}{b^2 - ca} = \frac{z}{c^2 - ab}.$$

Therefore, putting each of these ratios $= k$, we have

$$x = k(a^2 - bc), \; y = k(b^2 - ca), \; z = k(c^2 - ab).$$

Hence, from (4), we have

$$k\{a(a^2 - bc) + b(b^2 - ca) + c(c^2 - ab)\} = 0,$$

whence, since k is *not* zero, we must have

$$a^3 + b^3 + c^3 - 3abc = 0,$$

which is the required Eliminant.

Example 10. Eliminate x, y, z from the equations

$$a^2 - \frac{1}{x} = b^2 - \frac{1}{y} = c^2 - \frac{1}{z},$$

$$a^2x^2 + b^2y^2 + c^2z^2 = 0, \; a^2x^3 + b^2y^3 + c^2z^3 = 0;$$

and show that

$$a^{\frac{2}{3}}(b^2 - c^2)^{\frac{2}{3}} + b^{\frac{2}{3}}(c^2 - a^2)^{\frac{2}{3}} + c^{\frac{2}{3}}(a^2 - b^2)^{\frac{2}{3}} = 0.$$

Let $a^2 - \frac{1}{} \quad b^2 - {}^{1} \quad c^2 - {}^{1} \quad k.$

Then we have $a^2 = k + \dfrac{1}{x}$, $b^2 = k + \dfrac{1}{y}$,

and $c^2 = k + {}^{1}$

Hence, since $\left.\begin{aligned} a^2x^2 + b^2y^2 + c^2z^2 &= 0 \\ \text{and } a^2x^3 + b^2y^3 + c^2z^3 &= 0 \end{aligned}\right\}$,

we must have

$$a^4x^3 + b^4y^3 + c^4z^3 = a^2x^3.a^2 + b^2y^3.b^2 + c^2z^3.c^2$$

$$\begin{aligned} &= a^2x^3\left(k+\frac{1}{x}\right)+b^2y^3\left(k+\frac{1}{y}\right) \\ &\qquad\qquad + c^2z^3\left(k+\frac{1}{z}\right) \\ &= k(a^2x^3+b^2y^3+c^2z^3) \\ &\qquad\qquad +(a^2x^2+b^2y^2+c^2z^2) \\ &= 0. \end{aligned}$$

Thus we have

$$\left.\begin{array}{rl} & a^2.(a^2x^3)+b^2.(b^2y^3)+c^2.(c^2z^3) = 0 \\ \text{and also} & (a^2x^3)+\quad(b^2y^3)+\quad(c^2z^3) = 0 \end{array}\right\}$$

Hence, by cross multiplication

$$\frac{a^2x^3}{b^2-c^2} = \frac{b^2y^3}{c^2-a^2} = \frac{c^2z^3}{a^2-b^2},$$

and supposing each of these ratios $= m$, we have

$$\begin{aligned} a^3x^3 &= m.a(b^2-c^2), \text{ and } \therefore\ ax = m^{\frac{1}{3}}.a^{\frac{1}{3}}(b^2-c^2)^{\frac{1}{3}}; \\ b^3y^3 &= m.b(c^2-a^2), \text{ and } \therefore\ by = m^{\frac{1}{3}}.b^{\frac{1}{3}}(c^2-a^2)^{\frac{1}{3}}; \\ c^3z^3 &= m.c(a^2-b^2), \text{ and } \therefore\ cz = m^{\frac{1}{3}}.c^{\frac{1}{3}}(a^2-b^2)^{\frac{1}{3}}. \end{aligned}$$

Hence, since $a^2x^2+b^2y^2+c^2z^2 = 0$, we must have

$$m^{\frac{2}{3}}\left\{a^{\frac{2}{3}}(b^2-c^2)^{\frac{2}{3}}+b^{\frac{2}{3}}(c^2-a^2)^{\frac{2}{3}}+c^{\frac{2}{3}}(a^2-b^2)^{\frac{2}{3}}\right\} = 0,$$

whence, since m is *not* zero, we have

$$a^{\frac{2}{3}}(b^2-c^2)^{\frac{2}{3}}+b^{\frac{2}{3}}(c^2-a^2)^{\frac{2}{3}}+c^{\frac{2}{3}}(a^2-b^2)^{\frac{2}{3}} = 0,$$

which is the required Eliminant.

Exercise (94).

1. Eliminate x, y from the equations

$$\left.\begin{aligned} ax+by &= m \\ bx-ay &= n \\ x^2+y^2 &= 1 \end{aligned}\right\}.$$

2. Eliminate x, y, z from the equations

$$\left.\begin{aligned} a_1x+b_1y+c_1 = 0,\ a_2x+b_2y+c_2 &= 0, \\ a_3x+b_3y+c_3 &= 0. \end{aligned}\right\}$$

3. Eliminate x, y from the equations

$$ax+b = cy,\ a_1y+b_1 = c_1x,\ x^2+y^2 = 1.$$

4. Eliminate a, b, c from the equations

$$bz+cy = a,\quad cx+az = b,\ ay+bx = c.$$

5. Eliminate x, y, z from the equations

$$\frac{x}{y+z} = a,\ \frac{y}{z+x} = b,\ \frac{z}{x+y} = c.$$

6. Eliminate x, y from the equations

$$a_1x^4+b_1x^3+c_1 = 0,\ a_2x^4+b_2x^3+c_2 = 0.$$

7. Eliminate x, y from the equations

$$ax+by = \sqrt{a^2+b^2},\ \frac{x^2}{p^2}+\frac{y^2}{q^2} = \frac{1}{a^2+b^2},\ x^2+y^2 = 1.$$

8. Eliminate x from the equations

$$x^3+\frac{3}{x} = 4(a^3+b^3),\ 3x+\frac{1}{x^3} = 4(a^3-b^3).$$

9. Eliminate x, y from the equations

$$x+y = a,\ x^2+y^2 = b^2,\ x^4+y^4 = c^4.$$

10. Eliminate x, y, z from the equations

$$\left.\begin{aligned} x^3+y^3+z^3-3xyz &= a^3 \\ yz+zx+xy &= b^2 \\ x+y+z &= c \end{aligned}\right\}$$

11. Eliminate x, y, z from the equations

$$\left.\begin{aligned} x+y+z &= a \\ 2(yz+zx+xy) &= b^2 \\ x^3+y^3+z^3 &= c^3 \\ 3xyz &= d^4 \end{aligned}\right\}$$

12. Eliminate x, y, z from the equations

$$\left.\begin{aligned} x+y+z &= a \\ x^2+y^2+z^2 &= b^2 \\ x^3+y^3+z^3 &= c^3 \\ x^4+y^4+z^4 &= d^4 \end{aligned}\right\}$$

13. Eliminate x, y from the equations

$$ax+by = x+y+xy = x^2+y^2-1 = 0,$$

and show that $\dfrac{1}{a^2}+\dfrac{1}{b^2} = \dfrac{1}{(a-b)^2}$.

14. Eliminate x, y from the equations

$$p^2-y^2 = (x-a)^2,\ q^2-y^2 = (x-b)^2,\ r^2-y^2 = (x-c)^2,$$

and show that

$$p^2(b-c)+q^2(c-a)+r^2(a-b)+(b-c)(c-a)(a-b) = 0$$

15. Eliminate x, y, z from the equations

$$\frac{y}{z}+\frac{z}{y} = a,\quad \frac{z}{x}+\frac{x}{z} = b,\quad \frac{x}{y}+\frac{y}{x}$$

16. Eliminate x, y, z from the equations

$$\frac{x}{y}+\frac{y}{z}+\frac{z}{x} = \alpha,\quad \frac{x}{z}+\frac{y}{x}+\frac{z}{y} = \beta,$$

$$\left(\frac{x}{y}+\frac{y}{z}\right)\left(\frac{y}{z}+\frac{z}{x}\right)\left(\frac{z}{x}+\frac{x}{y}\right) = \gamma.$$

17. Eliminate x, y, z from the equations

$$bx+\frac{c}{x} = 2f,\quad cy+\frac{a}{y} = 2g,\quad az+\frac{b}{z} = 2h,$$

$$xyz = 1.$$

18. Eliminate x, y from the equations

$$x-\frac{1}{x} = 2a,\quad y-\frac{1}{y} = 2b,\quad \frac{x}{y}+\frac{y}{x} \quad 2c.$$

19. Eliminate x, y, z from the equations

$$x^2(y+z) = a^3,\ y^2(z+x) = b^3,\ z^2(x+y) = c^3,$$

$$xyz = abc.$$

20. Eliminate l, m, n from the equations

$$a^2l^3+b^2m^3+c^2n^3 = a'^2l+b'^2m+c'^2n\ ;$$

$$al = bm = cn\ ;\ l^2+m^2+n^2 = 1.$$

21. Eliminate x, y, z from the equations

$$\frac{x^2}{a^3} = \frac{y^2}{b^3} = \frac{z^2}{c^3} = \frac{1}{x}+\frac{1}{y}+\frac{1}{z}\ ;$$

$$\frac{b^2}{y^3}+\frac{c^2}{z^3} = \frac{1}{a}+\frac{1}{b}+\frac{1}{c}$$

22. Eliminate x, y from the equations

$$(ax+by)^2 = a+b,$$
$$ax^2+by^2-1 = \frac{2ab}{a+b}\left(x^2+y^2\right),$$
$$\frac{a}{x}+\frac{b}{y} = 1.$$

23. Eliminate x, y, z from the equations

$$x^3-a^3 = y^3-b^3 = z^3-c^3 = xyz\ ;$$
$$\frac{a^3}{x}+\frac{b^3}{y}+\frac{c^3}{z} = \frac{d^3}{x+y+z}.$$

24. Eliminate x, y, z from the equations

$$\frac{x}{a}+\frac{z}{c} = \frac{2y}{b},$$
$$acxz = b^2y^2,$$
$$\frac{yz}{a^2}+\frac{xy}{c^2} = \frac{2xz}{b^2}.$$

25. Eliminate x, y from the equations

$$y^2-x^2 = \alpha y-\beta x,\quad 4xy = \alpha x+\beta y,\quad x^2+y^2 = 1\ ;$$

and show that

$$(\alpha+\beta)^{\frac{2}{3}}+(\alpha-\beta)^{\frac{2}{3}} = 2.$$

26. Eliminate x, y, z, u from the equations

$$x = by+cz+du,\ y = ax+cz+du,$$
$$z = ax+by+du,\ u = ax+by+cz\ ;$$

and show that

$$\frac{a}{1+a}+\frac{b}{1+b}+\frac{c}{1+c}+\frac{d}{1+d} = 1.$$

27. Eliminate x, y, z from the equations

$$ax^2+by^2+cz^2 = ax+by+cz$$
$$= yz+zx+xy = 0.$$

28. Eliminate x, y, z from the equations

$$\frac{x^2-xy-xz}{a} = \frac{y^2-yz-yx}{b} = \frac{z^2-zx-zy}{c},$$
$$ax+by+cz = 0.$$

10. Miscellaneous Artifices.

Example 1. Shew that if a, b, c, d are all real and $(a+b)^2 + (b+c)^2 + (c+d)^2 = 4(ab+bc+cd)$, then will $a = b = c = d$.

We have $(a+b)^2+(b+c)^2+(c+d)^2 = 4(ab+bc+cd)$;

$\therefore$ by transposition,

$$\{(a+b)^2-4ab\}+\{(b+c)^2-4bc\}+\{(c+d)^2-4cd\} = 0,$$

$$\text{or,} \quad (a-b)^2+(b-c)^2+(c-d)^2 = 0.$$

Now since a, b, c, d are all real, *no term* on the left-hand side is negative; hence the above relation will hold only when each term is equal to zero;

$\therefore a-b = 0,\ b-c = 0,\ c-d = 0$; whence $a = b = c = d$.

Example 2. Show that if the sum of any two positive quantities be constant, their product is a *maximum* when the quantities are equal.

Let x, y be any two quantities whose sum $= k$, a constant.

$$\text{Now since } 4xy = (x+y)^2-(x-y)^2$$
$$= k^2-(x-y)^2,$$

it is evident that $4xy$ is *always* less than k^2 *except when* $x-y=0$.

Thus $4xy$ is a maximum when $x=y$; *i.e.*, xy is a maximum when $x=y=\frac{k}{2}$, and the maximum value $= \frac{k^2}{4}$.

Note. The above can be easily verified by a numerical example. For instance, $12 = 11+1 = 10+2 = 9+3 = 8+4 = 7+5 = 6+6$; but of the products 11×1, 10×2, 9×3, 8×4, 7×5, 6×6, the last is the greatest.

Example 3. Shew that if the product of any two positive quantities be constant, their sum is a *minimum* when the quantities are equal.

Let x, y be any two positive quantities whose product $= k^2$.

$$\text{Now since } (x+y)^2 = 4xy+(x-y)^2$$
$$= 4k^2+(x-y)^2,$$

it is evident that $(x+y)^2$ is *always* greater than $4k^2$, *except when* $x-y=0$.

Hence, $x+y$ is a minimum when $x = y = k$, and the minimum value $= 2k$.

Note. Let us verify the above by a numerical example. For instance, $36=36\times1=18\times2=12\times3=9\times4=6\times6$; but of the sums $36+1$, $18+2$, $12+3$, $9+4$, $6+6$, the last is the least.

Example 4. Show that $x^n - a^n$ is divisible by $x - a$, for all positive integral values of n.

By division $\dfrac{x^n - a^n}{x - a} = x^{n-1} + \dfrac{a(x^{n-1} - a^{n-1})}{x - a}$.

Hence, if $x - a$ divides $x^{n-1} - a^{n-1}$ it will also divide $x^n - a^n$.

But we know that $x - a$ divides $x^3 - a^3$; it will therefore also divide $x^4 - a^4$. And, since $x - a$ divides $x^4 - a^4$ it will also divide $x^5 - a^5$; and so on.

Thus the proposition is established.

Example 5. If the two expressions $x^3 + px^2 + qx + r$ and $x^3 + p'x + q'x + r'$ have the same quadratic factor, then

$$\frac{r - r'}{p - p'} = \frac{p'r - pr'}{q - q'} = \frac{q'r - qr'}{r - r'}.$$

Let $x^2 + ax + b$ denote the common quadratic factor; and suppose that

$$x^3 + px^2 + qx + r = (x^2 + ax + b)(x + c)$$

$$\text{and } x^3 + p'x^2 + q'x + r' = (x^2 + ax + b)(x + c').$$

By equating co-efficients of like powers of x, we have

$$\left.\begin{aligned} p &= a + c \\ q &= ac + b \\ r &= bc \end{aligned}\right\} \qquad \left.\begin{aligned} p &= a + c' \\ q' &= ac' + b \\ r' &= bc' \end{aligned}\right\}$$

$$\text{Hence, } \frac{r - r'}{p - p'} = \frac{b(c - c')}{c - c'} = b,$$

$$\frac{p'r - pr'}{q - q'} = \frac{ab(c - c')}{a(c - c')} = b,$$

$$\frac{q'r - qr'}{r - r'} = \frac{b^2(c - c')}{b(c - c')} = b;$$

$$\therefore \frac{r - r'}{p - p'} = \frac{p'r - pr'}{q - q'} = \frac{q'r - qr'}{r - r'}.$$

Example 6. If the equation $ax + by = 1$ and $cx^2 + dy^2 = 1$ have only one solution, prove that $\dfrac{a^2}{c} + \dfrac{b^2}{d} = 1$ and that $x = \dfrac{a}{c}$ and $y = \dfrac{b}{d}$.

From the given equations, we have

$$d.\left(\frac{1-ax}{b}\right)^2 = 1-cx^2,$$

$$\text{or,}\quad d(1-2ax+a^2x^2) = b^2(1-cx^2),$$

$$\text{or,}\quad (a^2d+b^2c)x^2-2adx+(d-b^2) = 0;$$

$$\therefore\quad x = \frac{ad\pm\sqrt{a^2d^2-(d-b^2)(a^2d+b^2c)}}{(a^2d+b^2c)}.$$

Hence, *if there be only one value of x* the expression under the radical sign must vanish; that is, we must have

$$\begin{aligned} a^2d^2 &= (d-b^2)(a^2d+b^2c) \\ &= a^2d^2+b^2dc-b^2(a^2d+b^2c); \end{aligned}$$

$$\therefore\quad a^2d+b^2c = dc,$$

$$\therefore\quad \frac{a^2}{c}+\frac{b^2}{d} = 1.$$

Hence, also $x = \dfrac{ad}{a^2d+b^2c} = \dfrac{ad}{dc} = \dfrac{a}{c}$,

and $\therefore\ y = \dfrac{1}{b}\cdot(1-ax) = \dfrac{1}{b}\left(1-\dfrac{a^2}{c}\right) = \dfrac{1}{b}\cdot\dfrac{b^2}{d} = \dfrac{b}{d}$

Example 7. Find the condition that the roots of $ax^2+2bx+c = 0$ may be formed from those of $a'x^2+2b'x+c' = 0$ by adding the same quantity to each root.

Let α and β be the roots of $a'x^2+2b'x+c' = 0$, then $\alpha+d$, and $\beta+d$ must be the roots of $ax^2+2bx+c = 0$.

$$\text{Hence,}\quad \alpha+\beta = -\frac{2b'}{a'} \qquad \ldots \qquad \ldots \qquad \ldots \quad (1)$$

$$\alpha\beta = \frac{c'}{a'} \qquad \ldots \qquad \ldots \qquad \ldots \quad (2)$$

$$\alpha+\beta+2d = -\frac{2b}{a} \qquad \ldots \qquad \ldots \qquad \ldots \quad (3)$$

$$(\alpha+d)(\beta+d) = \frac{c}{a} \qquad \ldots \qquad \ldots \qquad \ldots \quad (4)$$

From (1) and (3), we have

$$2d = -\frac{2b}{a} + \frac{2b'}{a'},$$

$$\therefore d = -\frac{b}{a} + \frac{b'}{a'}.$$

Hence, from (4),

$$\begin{aligned}\frac{c}{a} &= \alpha\beta + (\alpha + \beta)d + d^2 \\ &= \frac{c'}{a'} - \frac{2b'}{a'}\left(\frac{b'}{a'} - \frac{b}{a}\right) + \left(\frac{b'}{a'} - \frac{b}{a}\right)^2 \\ &= \frac{c'}{a'} - \frac{b'^2}{a'^2} + \frac{b^2}{a^2},\end{aligned}$$

or, $\dfrac{b^2}{a^2} - \dfrac{c}{a} = \dfrac{b'^2}{a'^2} - \dfrac{c'}{a'}$, which is the required condition.

Example 8. If $a + \dfrac{y^2 - z^2}{b - c} = b + \dfrac{z^2 - x^2}{c - a}$, then will each member of the equation be equal to $c + \dfrac{x^2 - y^2}{a - b}$.

Let each member of the given equation $= k$.

Then $a(b - c) + y^2 - z^2 = k(b - c)$,

and $b(c - a) + z^2 - x^2 = k(c - a)$;

$\therefore$ by addition,

$$c(b - a) + y^2 - x^2 = k(b - a).$$

$$\text{Hence, } k = c + \frac{x^2 - y^2}{a - b}.$$

Example 9. If $\dfrac{x^2 - yz}{x(1 - yz)} = \dfrac{y^2 - zx}{y(1 - zx)}$ and x, y, z be unequal, then each member of this equation will be equal to $\dfrac{z^2 - xy}{z(1 - xy)}$, to $x + y + z$, and to $\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z}$.

Let each member of the given equation $= k$.

Then $x^2 - yz = k(x - xyz)$

and $y^2 - zx = k(y - xyz)$.

Hence, by subtraction,

$$x^2 - y^2 + z(x - y) = k(x - y),$$

or, $$(x - y)(x + y + z) = k(x - y),$$

$$\therefore \quad k = x + y + z, \qquad \ldots \quad (1)$$

since $x - y$ is *not zero*.

Again, $$x^2 - yz = k(x - xyz)$$
$$= (x + y + z)(x - xyz)$$
$$= x^2 + x(y + z) - xyz(x + y + z),$$

or, $$xyz(x + y + z) = xy + xz + yz;$$

$$\therefore \quad x + y + z = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}. \qquad \ldots \quad (2)$$

Also, since $$xyz(x + y + z) = xy + xz + yz$$
$$= xy + z(x + y)$$
$$= xy - z^2 + z(x + y + z),$$

therefore $$z^2 - xy = (x + y + z)(z - xyz),$$

$$\therefore \quad x + y + z = \frac{z^2 - xy}{z(1 - xy)}. \qquad \ldots \quad \ldots \quad (3)$$

Hence, from (1), (2) & (3), we have

$$k = x + y + z = \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{z^2 - xy}{z(1 - xy)}.$$

Example 10. Prove that

$$\{(b - c)^2 + (c - a)^2 + (a - b)^2\}\{a^2(b - c)^2 + b^2(c - a)^2 + c^2(a - b)^2\}$$
$$= 3(b - c)^2(c - a)^2(a - b)^2 + \{a(b - c)^2 + b(c - a)^2 + c(a - b)^2\}^2.$$

Let $b - c = x$, $c - a = y$, $a - b = z$.

Then, $$\{(b - c)^2 + (c - a)^2 + (a - b)^2\}$$
$$\times \{a^2(b - c)^2 + b^2(c - a)^2 + c^2(a - b)^2\}$$
$$= (x^2 + y^2 + z^2)(a^2x^2 + b^2y^2 + c^2z^2)$$
$$= a^2x^4 + a^2x^2(y^2 + z^2) + b^2y^4 + b^2y^2(z^2 + x^2)$$
$$+ c^2z^4 + c^2z^2(x^2 + y^2)$$
$$= a^2x^4 + b^2y^4 + c^2z^4 + x^2y^2(a^2 + b^2) + y^2z^2(b^2 + c^2)$$
$$+ z^2x^2(c^2 + a^2).$$

But $$x^2y^2(a^2 + b^2) + y^2z^2(b^2 + c^2) + z^2x^2(c^2 + a^2)$$
$$= x^2y^2\{(a - b)^2 + 2ab\} + y^2z^2\{(b - c)^2 + 2bc\}$$
$$+ z^2x^2\{(c - a)^2 + 2ca\}$$

$$\begin{aligned}
&= x^2y^2(z^2+2ab)+y^2z^2(x^2+2bc)+z^2x^2(y^2+2ca)\\
&= 3x^2y^2z^2+2abx^2y^2+2bcy^2z^2+2caz^2x^2\ ;\\
&\therefore\quad \{(b-c)^2+(c-a)^2+(a-b)^2\}\\
&\qquad\qquad \times\{a^2(b-c)^2+b^2(c-a)^2+c^2(a-b)^2\}\\
&= a^2x^4+b^2y^4+c^2z^4+3x^2y^2z^2+2abx^2y^2\\
&\qquad\qquad\qquad\qquad +2bcy^2z^2+2caz^2x^2\ ;\\
&= 3x^2y^2z^2+(ax^2+by^2+cz^2)^2\\
&= 3(b-c)^2.(c-a)^2.(a-b)^2+\{a(b-c)^2+b(c-a)^2+c(a-b)^2\}^2.
\end{aligned}$$

Example 11. Shew that if $a+b+c=0$, the following expression is also zero :—

$$\frac{a^2}{2a^2+bc}+\frac{b^2}{2b^2+ca}+\frac{c^2}{2c^2+ab}-1.$$

We have

$$\begin{aligned}
2a^2+bc &= a^2+a.a+bc\\
&= a^2-a(b+c)+bc \qquad [\because\ a=-(b+c)]\\
&= (a-b)(a-c).
\end{aligned}$$

Similarly,

$$\begin{aligned}
2b^2+ca &= b^2-b(a+c)+ca\\
&= (b-c)(b-a)\ ;
\end{aligned}$$

and

$$\begin{aligned}
2c^2+ab &= c^2-c(a+b)+ab\\
&= (c-a)(c-b).
\end{aligned}$$

Hence, the given expression

$$\begin{aligned}
&= \frac{a^2}{(a-b)(a-c)}+\frac{b^2}{(b-c)(b-a)}+\frac{c^2}{(c-a)(c-b)}-1\\
&= \frac{a^2(b-c)-b^2(a-c)+c^2(a-b)}{(a-b)(b-c)(a-c)}-1\\
&= \frac{a^2(b-c)-a(b^2-c^2)+bc(b-c)}{(a-b)(b-c)(a-c)}-1\\
&= \frac{(b-c)\{a^2-a(b+c)+bc\}}{(a-b)(b-c)(a-c)}-1\\
&= \frac{(b-c)(a-c)(a-b)}{(a-b)(b-c)(a-c)}-1\\
&= 0.
\end{aligned}$$

Example 12. If n be a positive integer, prove that

$$1-2n+\frac{2n(2n-1)}{\underline{|2}}-\&c.+(-1)^{n-1}.\frac{2n(2n-1)\dots(n+2)}{\underline{|n-1}}$$

$$=(-1)^{n-1}.\frac{\underline{|2n}}{2(\underline{|n})^2}.$$

We have

$$(1-x)^{2n}=1-2n\,x+\frac{2n(2n-1)}{\underline{|2}}x^2-\&c.$$

$$+(-1)^{n-1}.\frac{2n(2n-1)\dots(n+2)}{\underline{|n-1}}x^{n-1}+\&c\,;$$

$$(1-x)^{-1}=1+x+x^2+\dots+x^{n-1}+x^n+\&c\,;$$

Hence, multiplying, we see that

$$1-2n+\frac{2n(2n-1)}{\underline{|2}}-\dots+(-1)^{n-1}.\frac{2n(2n-1)\dots(n+2)}{\underline{|n-1}}$$

$=$ co-efficient of x^{n-1} in the expansion of $(1-x)^{2n}.(1-x)^{-1}$

$=$ co-efficient of x^{n-1} in the expansion of $(1-x)^{2n-1}$

$$={}^{2n-1}C_{n-1}\times(-1)^{n-1}$$

$$=(-1)^{n-1}.\frac{\underline{|2n-1}}{\underline{|n-1}\,\underline{|n}}$$

$$=(-1)^{n-1}.\frac{2n\underline{|2n-1}}{2n\underline{|n-1}\,\underline{|n}}$$

$$=(-1)^{n-1}.\frac{\underline{|2n}}{2\underline{|n}\,\underline{|n}}.$$

Example 13. Prove that

$$\frac{1.3.5..(2r-1)}{\underline{|r}}+\frac{1.3.5....(2r-3)}{\underline{|r-1}}.\frac{3}{\underline{|1}}$$

$$+\frac{1.3.5...(2r-5)}{\underline{|r-2}}.\frac{3.5}{\underline{|2}}.+\&c.=2^r(1+r).$$

We have

$$(1-x)^{-\frac{1}{2}}=1+\frac{1}{2}.x+\frac{1.3}{2^2}.\frac{x^2}{\underline{|2}}+\frac{1.3.5}{2^3}.\frac{x^3}{\underline{|3}}+$$

$$\dots+\frac{1.3.5....(2r-1)}{2^r}.\frac{x^r}{\underline{|r}}+\&c.\,;$$

$$(1-x)^{-\frac{3}{2}} = 1 + \frac{3}{2}.x + \frac{3.5}{2^2}.\frac{x^2}{\underline{|2}} + \frac{3.5.7}{2^3}.\frac{x^3}{\underline{|3}} + \&c$$

Hence, multiplying, we see that

$$\frac{1.3.5....(2r-1)}{2^r\underline{|r}} + \frac{1.3.5....(2r-3)}{2^{r-1}\underline{|r-1}}.\frac{3}{2}$$

$$+ \frac{1.3.5....(2r-5)}{2^{r-2}\underline{|r-2}}.\frac{3.5}{2^2\underline{|2}} + \&c.$$

i.e., $$\frac{1.3.5....(2r-1)}{2^r\underline{|r}} + \frac{1.3.5.....(2r-3)}{2^r\underline{|r-1}}.\frac{3}{\underline{|1}}$$

$$+ \frac{1.3.5.....(2r-5)}{2^r\underline{|r-2}}.\frac{3.5}{\underline{|2}} + \&c.$$

$=$ co-efficient of x^r in the expansion of $(1-x)^{-\frac{1}{2}}(1-x)^{-\frac{3}{2}}$

i.e., of $(1-x)^{-2}$

$= r+1.$

$$\therefore \quad \frac{1.3.5.....(2r-1)}{\underline{|r}} + \frac{1.3.5....(2r-3)}{\underline{|r-1}}.\frac{3}{\underline{|1}}$$

$$+ \frac{1.3.5....(2r-5)}{\underline{|r-2}}.\frac{3.5}{\underline{|2}} + \&c.$$

$= 2^r(1+r).$

Example 14. If $c_0, c_1, c_2, \ldots c_n$ denote the co-efficients in the expansion of $(1+x)^n$, prove that

$$c_1 - \frac{c_2}{2} + \frac{c_3}{3} - \&c. + (-1)^{n-1}.\frac{c_n}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \&c. + \frac{1}{n}.$$

Let S_n denote the given expression.

Then $S_n = {}^nC_1 - \frac{1}{2}.{}^nC_2 + \frac{1}{3}.{}^nC_3 - \frac{1}{4}.{}^nC_4 + \&c.$

$$+ (-1)^{n-1}.\frac{1}{n}.{}^nC_n.$$

Hence $S_{n+1} = {}^{n+1}C_1 - \frac{1}{2}.{}^{n+1}C_2 + \frac{1}{3}.{}^{n+1}C_3 - \&c.$

$$+ (-1)^{n-1}.\frac{1}{n}.{}^{n+1}C_n + (-1)^n.\frac{1}{n+1}.{}^{n+1}C_{n+1}.$$

Therefore, by subtraction,

$$\begin{aligned}
S_{n+1} - S_n &= \left({}^{n+1}C_1 - {}^{n}C_1\right) - \tfrac{1}{2}.\left({}^{n+1}C_2 - {}^{n}C_2\right) \\
&\qquad + \tfrac{1}{3}\left({}^{n+1}C_3 - {}^{n}C_3\right) - \&c. \\
&\qquad + (-1)^{n-1}.\frac{1}{n}.\left({}^{n+1}C_n - {}^{n}C_n\right) + (-1)^{n}.\frac{1}{n+1}.{}^{n+1}C_{n+1} \\
&= {}^{n}C_0 - \tfrac{1}{2}.{}^{n}C_1 + \tfrac{1}{3}.{}^{n}C_2 - \&c. + (-1)^{n-1}.\frac{1}{n}.{}^{n}C_{n-1} \\
&\qquad + (-1)^{n}.\frac{1}{n+1}.{}^{n+1}C_{n+1} \\
&\qquad \left[\because \quad {}^{n+1}C_r - {}^{n}C_r = {}^{n}C_{r-1}\right] \\
&= \frac{1}{n+1}.\Big[(n+1).{}^{n}C_0 - \tfrac{1}{2}.(n+1).{}^{n}C_1 + \tfrac{1}{3}.(n+1).{}^{n}C_2 - \&c. \\
&\qquad + (-1)^{n-1}.\frac{1}{n}.(n+1).{}^{n}C_{n-1} + (-1)^{n}.{}^{n+1}C_{n+1}\Big] \\
&= \frac{1}{n+1}.\Big[{}^{n+1}C_1 - {}^{n+1}C_2 + {}^{n+1}C_3 - \&c. \\
&\qquad + (-1)^{n-1}.{}^{n+1}C_n + (-1)^{n}.{}^{n+1}C_{n+1}\Big] \\
&= \frac{1}{n+1}.\Big[1 - \Big\{1 - {}^{n+1}C_1 + {}^{n+1}C_2 - {}^{n+1}C_3 + \&c. \\
&\qquad + (-1)^{n}.{}^{n+1}C_n + (-1)^{n+1}.{}^{n+1}C_{n+1} \\
&= \frac{1}{n+1}.\left\{1 - (1-1)^{n+1}\right\} \\
&= \frac{1}{n+1}.
\end{aligned}$$

Therefore $S_{n+1} = S_n + \dfrac{1}{n+1}$.

Hence, since $S_1 = 1$, $\therefore\ S_2 = 1 + \frac{1}{2}$; and since $S_3 = S_2 + \frac{1}{3}$,
$\therefore\ S_3 = 1 + \frac{1}{2} + \frac{1}{3}$; and so on.

Thus, $S_n = 1 + \dfrac{1}{2} + \dfrac{1}{3} + \dfrac{1}{4} + \&c. \quad + \dfrac{1}{n}$.

Example 15. Prove that if n and r are positive integers,

$$n^r - n(n-1)^r + \frac{n(n-1)}{\underline{|2}}(n-2)^r - \frac{n(n-1)(n-2)}{\underline{|3}}(n-3)^r + \text{\&c.}$$

is equal to o if r be less than n, and to $\underline{|n}$ if $r = n$.

We have

$$(e^x - 1)^n = \left(x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \frac{x^4}{\underline{|4}} + \text{\&c.}\right)^n$$

$$= x^n + \text{terms containing higher powers of } x. \quad \ldots \quad \ldots \quad (1)$$

Again, by the Binomial Theorem,

$$(e^x - 1) = e^{nx} - ne^{(n-1)x} + \frac{n(n-1)}{\underline{|2}}.e^{(n-2)x} - \text{\&c.} \ldots \quad \ldots \quad (2)$$

By expanding each of the terms e^{nx}, $e^{(n-1)x}$, &c., we find that the co-efficient of x^r in (2) is

$$\frac{n^r}{\underline{|r}} - n\frac{(n-1)^r}{\underline{|r}} + \frac{n(n-1)}{\underline{|2}}\cdot\frac{(n-2)^r}{\underline{|r}}$$
$$- \frac{n(n-1)(n-2)}{\underline{|3}}\cdot\frac{(n-3)^r}{\underline{|r}} + \text{\&c.}$$

and the co-efficient of x^r in (1) is 0 or 1 according as r is less than n or equal to n.

Hence,

$$n^r - n(n-1)^r + \frac{n(n-1)}{\underline{|2}}.(n-2)^r$$
$$- \frac{n(n-1)(n-2)}{\underline{|3}}.(n-3)^r + \text{\&c. is}$$

$= \underline{|n}$, if $r = n$, and is $= 0$, if r be less than n.

Example 16. If $a + \dfrac{bc - a^2}{a^2 + b^2 + c^2}$ be not altered in value by interchanging a pair of the letters a, b, c not equal to each other, it will not be altered by interchanging any other pair; and it will vanish if $a + b + c = 1$.

Suppose the given expression is not altered in value when a and b are interchanged; *i.e.*, suppose

$$a + \frac{bc - a^2}{a^2 + b^2 + c^2} = b + \frac{ac - b^2}{b^2 + a^2 + c^2}$$

Then we must have

$$\left\{a+\frac{bc-a^2}{a^2+b^2+c^2}\right\}-\left\{b+\frac{ac-b^2}{b^2+a^2+c^2}\right\}=0$$

$$\text{or,}\qquad (a-b)+\frac{c(b-a)+(b^2-a^2)}{a^2+b^2+c^2}$$

$$\text{or,}\qquad (a-b)+\frac{(b-a)\{c+(b+a)\}}{a^2+b^2+c^2}=0$$

$$\text{or,}\qquad (a-b)\left\{1-\frac{a+b+c}{a^2+b^2+c^2}\right\}=0.$$

Therefore, since $a-b$ is *not* zero, we have

$$1-\frac{a+b+c}{a^2+b^2+c^2}=0 \quad \dots \quad \dots \quad (1)$$

$$\text{Hence,}\quad \left\{a+\frac{bc-a^2}{a^2+b^2+c^2}\right\}-\left\{c+\frac{ba-c^2}{c^2+b^2+a^2}\right\}$$

$$=(a-c)+\frac{b(c-a)+(c^2-a^2)}{a^2+b^2+c^2}$$

$$=(a-c)+\frac{(c-a)\{b+(c+a)\}}{a^2+b^2+c^2}$$

$$=(a-c)\left\{1-\frac{a+b+c}{a^2+b^2+c^2}\right\}$$

$$=(a-c)\times 0$$

$$=0;$$

$$\text{and}\ \therefore\ a+\frac{bc-a^2}{a^2+b^2+c^2}=c+\frac{ba-c^2}{c^2+b^2+a^2}.$$

Thus it is proved that *if* the given expression is not altered when a and b are interchanged, it is also not altered when a and c are interchanged.

Now to prove that the given expression vanishes if we have the additional condition

$$a+b+c=1 \quad \dots \quad \dots \quad \dots \quad (2)$$

From (1) and (2), we have

$$a^2+b^2+c^2 = 1.$$

Hence,
$$a+\frac{bc-a^2}{a^2+b^2+c^2} = a+bc-a^2$$
$$= \frac{1}{2}(2a+2bc-2a^2)$$
$$= \frac{1}{2}\left\{2a+2bc-a^2-(1-b^2-c^2)\right\}$$
$$= \frac{1}{2}\{(b^2+c^2+2bc)-(1+a^2-2a)\}$$
$$= \frac{1}{2}\{(b+c)^2-(1-a)^2\}$$
$$= \tfrac{1}{2}\{(b+c)^2-(b+c)^2\}, \quad [\text{by } (2)]$$
$$= 0.$$

Example 17. If
$$\left.\begin{aligned} xy+\tfrac{1}{2}(x+y)(a+b)+ab &= 0\\ xy+\tfrac{1}{2}(x+y)(c+d)+cd &= 0\end{aligned}\right\}$$
prove that
$$\frac{x-y}{2} = \frac{\sqrt{(a-c)(a-d)(b-c)(b-d)}}{a+b-c-d}.$$

From the given equations, by cross multiplication, we have
$$\frac{xy}{cd(a+b)-ab(c+d)} = \frac{\tfrac{1}{2}(x+y)}{ab-cd} = \frac{1}{(c+d)-(a+b)};$$
$$\therefore\ xy = \frac{ab(c+d)-cd(a+b)}{a+b-c-d}, \text{ and } \tfrac{1}{2}(x+y) = \frac{cd-ab}{a+b-c-d}.$$

Hence,
$$\left(\frac{x-y}{2}\right)^2 = \left(\frac{c+y}{2}\right)^2 - xy$$
$$= \frac{(cd-ab)^2-(a+b-c-d)\{ab(c+d)-cd(a+b)\}}{(a+b-c-d)^2}.$$

Now, if c is put for a, the numerator on the right-hand side becomes
$$c^2(d-b)^2-(b-d)\{c^2(b-d)\}, \text{ or zero};$$
hence, $a-c$ is one of its factors.

Similarly we find that $a-d$, $b-c$ and $b-d$ also are its factors.

Hence, the numerator which is a rational and integral function of the fourth degree, must be equal to

$$k(a-c)(a-d)(b-c)(b-d),$$

where k is some numerical constant.

Putting $a=0$, $b=0$, $c=1$, $d=1$ in the above relation, we have $1=k\times 1$, and $\therefore k=1$.

Hence, we have

$$\left(\frac{x-y}{2}\right)^2=\frac{(a-c)(a-d)(b-c)(b-d)}{(a+b-c-d)^2}$$

and $\therefore$ $$\frac{x-y}{2}=\frac{\sqrt{(a-c)(a-d)(b-c)(b-d)}}{a+b-c-d}.$$

Example 18. Show that the co-efficient of x^n in the expansion of $\dfrac{x}{(x-a)(x-b)}$ in ascending powers of x is $\dfrac{a^n-b^n}{a-b}\cdot\dfrac{1}{a^nb^n}$.

Assume $$\frac{x}{(x-a)(x-b)}=\frac{A}{x-a}+\frac{B}{x-b},$$

where A and B are quantities independent of x and whose values have to be determined.

Then we must have

$$\begin{aligned} x &= A(x-b)+B(x-a) \\ &= (A+B)x-(Ab+Ba). \end{aligned}$$

Hence, equating co-efficients of like powers of x, we have

$$A+B=1, \quad \text{and } Ab+Ba=0;$$

whence, $$A=\frac{a}{a-b} \quad\text{and}\quad B=\frac{-b}{a-b}.$$

Thus we have the given expression

$$\begin{aligned} &= \frac{1}{a-b}\left(\frac{a}{x-a}-\frac{b}{x-b}\right) \\ &= \frac{1}{a-b}\left\{-\frac{1}{1-\frac{x}{a}}+\frac{1}{1-\frac{x}{b}}\right\} \\ &= \frac{1}{a-b}\left\{-\left(1-\frac{x}{a}\right)^{-1}+\left(1-\frac{x}{b}\right)^{-1}\right\} \end{aligned}$$

$$= \frac{1}{a-b}\left\{-\left(1+\frac{x}{a}+\frac{x^2}{a^2}+\frac{x^3}{a^3}+\dots\dots\right)\right.$$
$$\left.+\left(1+\frac{x}{b}+\frac{x^2}{b^2}+\frac{x^3}{b^3}+\dots\dots\right)\right\}$$

Hence, the required co-efficient

$$= \frac{1}{a-b}\left(\frac{1}{b^n}-\frac{1}{a^n}\right)$$
$$= \frac{a^n-b^n}{a-b}\cdot\frac{1}{a^nb^n}.$$

Miscellaneous Exercises.

I.

1. Solve $9x - 4x^2 + \sqrt{4x^2 - 9x + 11} = 5$.

2. The difference between the hypotenuse and base of a right-angled triangle is = 6, and the difference between the hypotenuse and the perpendicular is = 3. What are the sides ?

3. $\frac{5x-4y}{6a+7b} = \frac{5x-4z}{6b+7c} = \frac{5z-4x}{6c+7a}$; prove that
$(53a+45b+58c)(x+y+z) = 13(a+b+c)(8y+9z-5x)$.

4. Determine the limits between which $\frac{x^2-2x-3}{2x^2+2x+1}$ lies for all real values of x.

5. Resolve $2x^2 - 21xy - 11y^2 - x + 34y - 3$ into its factors.

6. What number must be taken from each of the numbers 13, 15, 19 that the remainders may form a harmonic series ?

7. In how many ways can an eleven be selected from 25 boys, 6 of them being always included and 5 always excluded ?

8. Find the value of—

$$1+\frac{1}{6}+\frac{1.3}{6.12}+\frac{1.3.5.}{6.12.18}+\text{\&c. to infinity.}$$

II.

1. If $x(b-c)+y(c-a)+z(a-b) = 0$, then will
$\frac{bz-cy}{b-c} = \frac{cx-az}{c-a} = \frac{ay-bx}{a-b}$.

2. Solve
$$\left.\begin{aligned} x+y+z &= 7 \\ xy+xz &= yz-2 \\ x^2+y^2+z^2 &= 21 \end{aligned}\right\}$$

3. Prove that $x^4+px^3+qx^2+rx+s$ is a perfect square, if $p^2s = r^2$ and $q = \frac{p^2}{4}+2s^{\frac{1}{2}}$.

4. If $s = a+b+c$, prove that—
$$(as+bc)(bs+ca)(cs+ab) = (b+c)^2.(c+a)^2.(a+b)^2.$$

5. Divide 25 into 5 parts, which are in A. P., such that the sum of the squares of the least and greatest of them is less by 1 than the sum of the squares of the other three.

6. Sum the series :—
$$2^3+4^3+6^3+\&c.+(2n)^3.$$

7. How many different sides can be formed in a croquet party consisting of 5 ladies and 3 gentlemen, the gentlemen never being all on the same side ?

8. Find the co-efficient of x^n in the expansion of
$$(1+3x+6x^2+10x^3+\&c.)^3.$$

III.

1. Solve
$$\left.\begin{aligned} x^2+2xy-y^2 &= ax+by \\ x^2-2xy-y^2 &= bx-ay \end{aligned}\right\}$$

2. Express $4(x^4+x^3+x^2+x+1)$ as the difference between two squares.

3. If a, b, e be in *A. P.*; b, c, d in *G. P.*; and c, d, e in *H. P.*, prove that a, c, e are in *G. P.*

4. Apply the principle of Variation to solve the following question :—If 7 men can dig a trench 27 yards long in 36 days working 5 hours a day, how many men can dig a trench 81 yards long in 15 days working 9 hours a day ?

5. If bc, ca, ab are in *H. P.*, then will
$$a\left(\frac{1}{b}+\frac{1}{c}\right),\ b\left(\frac{1}{a}+\frac{1}{c}\right) \text{ and } c\left(\frac{1}{a}+\frac{1}{b}\right) \text{ be in } A.\ P.$$

6. Eight boats start in a race one after another. Find the number of ways in which they can be arranged so that (i) a particular boat may always have a given boat just behind it, (ii) that it may always be between two given boats.

7. Find the co-efficient of x^n in the expansion of

$$\frac{1+x-2x^2}{(1-x)^3}.$$

8. If A_n is the co-efficient of x^n in the expansion of $\frac{e^x}{1-x}$,

$$\text{then } A_n - A_{n-1} = \frac{1}{\underline{|n}}.$$

IV.

1. Prove that the same values of x and y will satisfy the equations $a_1x+b_1y = c_1$, $a_2x+b_2y = c_2$, $a_3x+b_3y = c_3$, if $(a_2b_3-a_3b_2)c_1+(a_3b_1-a_1b_3)c_2+(a_1b_2-a_2b_1)c_3 = 0$.

2. If α and β be the roots of the equation $x^2+4x+3 = 0$, shew that the equation whose roots are $\frac{\alpha+\beta}{\alpha}$ and $\frac{\alpha+\beta}{\beta}$ is $3x^2-16x+16 = 0$.

3. Show that the least value of $x+\frac{1}{x}$ is 2 and the least value of $x+\frac{9}{x}$ is 6, x being real and positive.

4. Shew that $\left(\frac{-1+i\sqrt{3}}{2}\right)^n+\left(\frac{-1-i\sqrt{3}}{2}\right)^n = -1$, if n be any integer which is not a multiple of 3.

5. If the carriages in a railway train be all of the same class and always just full; if the expense of running a train be proportional to the square of the number of carriages; and if a train of 36 carriages just pay the expense of working it; prove that it will be as profitable to the railway company to run trains of 16 carriages as trains of 20 carriages.

6. How many different rectangular parallelepipeds are there satisfying the condition that each edge of each parallelepiped shall be equal to some one of n given lines all of different lengths?

7. Find a, b so that the co-efficient of x^n in $\frac{a+bx}{(1-x)^2}$ may be $4n+3$.

8. If c_0, c_1, c_2, &c., be the co-efficients in the expansion of $(1+x)^n$, prove that $c_1-2c_2+3c_3-\text{\&c.}+(-1)^{n-1}.nc_n = 0$.

V.

1. Two roots of $a^7 - 23x^5 + 22x^4 + 55x^3 - 32x^2 - 33x + 10 = 0$ are 2 and -5; find all the other roots.

2. If $a^2 + b^2 + c^2 = 1$ and $a_1{}^2 + b_1{}^2 + c_1{}^2 = 1$, prove that $aa_1 + bb_1 + cc_1$ is never greater than 1.

3. If $\frac{1}{x} + \frac{1}{y}$ varies inversely as $x + y$, show that $x^2 + y^2$ varies as xy.

4. If $1, \omega, \omega^2$ are the three cube roots of unity, prove that $(1 - \omega + \omega^2)(1 - \omega^2 + \omega^4)(1 - \omega^4 + \omega^8)$ &c., to $2n$ factors $= 2^{2n}$.

5. Sum the series

(i) $1.3 + 2.4 + 3.5 +$ &c., $+ n(n + 2)$.

(ii) $1.3^2 + 2.4^2 + 3.5^2 +$ &c. to n terms.

6. There are 8 white billiard balls exactly alike, and 4 red ones also alike; in how many ways can an arrangement be made containing 3 of each colour?

7. Find the term independent of x in the expansion of

$$\left(3x^2 - \frac{a}{3x^3}\right)^{10}.$$

8. If x is so small that its square and higher powers may be neglected, prove that

$$\frac{\sqrt[5]{1 + 2x} + \sqrt[3]{1 - x}}{\sqrt[4]{1 + 2x}} = 2 - \frac{14}{15}x.$$

VI.

1. Solve $ax + cy + bz = cx + by + az = bx + ay + cz$
$= a^3 + b^3 + c^3 - 3abc.$

2. Simplify $(a + b + c)(x + y + z) + (a + b - c)(x + y - z) + (b + c - a)(y + z - x) + (c + a - b)(z + x - y)$.

3. Find the value of $(2 + 3\sqrt{-5})^{\frac{1}{2}} + (2 - 3\sqrt{-5})^{\frac{1}{2}}$.

4. Sum to n terms

$$1^3 + 3^3 + 5^3 + 7^3 + \text{\&c.}$$

5. Shew that the sum of all the harmonic means which can be inserted between all the pairs of positive integers the sum of which is n, is $\frac{1}{3}(n^2-1)$.

6. Shew that in ${}^{2n}C_n$ the number of combinations in which a particular thing occurs is equal to the number in which it does not occur.

7. If c_0, c_1, c_2, &c., be the co-efficients in the expansion of $(1+x)^n$, prove that

$$c_0-2c_1+3c_2-\&c. \quad +(-1)^n(n+1)c_n=0.$$

8. Investigate the relations between the co-efficients that the equation $ax^3+bx^2+cx+d=0$ may be put under one of the forms

$$(1) \quad x^4=(x^2+px+q)^2\;;$$
$$(2) \quad q^2=(x^2+px+q)^2.$$

Solve in this way the equation $2x^3-x^2-2x+1=0$.

VII.

1. Solve the equations :—

(i) $\sqrt{x^2-8x+15}+\sqrt{x^2+2x-15}=\sqrt{4x^2-18x+18}$.

(ii) $\dfrac{ax+b}{cx+b}+\dfrac{bx+a}{cx+a}=\dfrac{(a+b)(x+2)}{cx+a+b}$.

2. If $ax+by+cz=0$, $a^2x+b^2y+c^2z=0$, and

$\dfrac{x}{b+c-a}+\dfrac{y}{a+c-b}+\dfrac{z}{a+b-c}=0$, prove that

$$bc(b-c)^3+ac(c-a)^3+ab(a-b)^3=0.$$

3. The hands of a clock are of equal length, and one of them is between the figures 2 and 3 on the dial, and the other between 6 and 7 ; find their position when there is ambiguity as to the time and shew that this position of ambiguity will recur at intervals of $\dfrac{720}{143}$ minutes.

4. If a, b, c be in arithmetic and a, b, d in harmonic progression, prove that

$$\frac{c}{d}=1-\frac{2(a-b)^2}{ab}.$$

5. If $\sqrt[3]{x+yi} = X+Yi$, show that

$$4(X^2-Y^2) = \frac{x}{X}+\frac{y}{Y}.$$

6. There are 3 capitals, 6 consonants and 4 vowels; in how many ways can a word be made beginning with one of the capitals, and containing 3 consonants and 2 vowels?

7. If $\dfrac{a}{b+c}+\dfrac{c}{a+b} = \dfrac{2b}{c+a}$, prove that $a^2+c^2 = 2b^2$,

or, $a+b+c = 0$.

8. Find the co-efficient of x^4 in the expansion of

$$(1-2x+3x^2-4x^3+\&c.)^{-\frac{3}{2}}.$$

VIII.

1. Solve $\sqrt{x^2+2x-1}+\sqrt{x^2+x+1} = \sqrt{2}+\sqrt{3}$.

2. If x be a real quantity, determine the limits of value between which the expression $\dfrac{2x^2+6x+3}{2x+1}$ must lie.

3. Shew that $2^{\frac{1}{4}}.4^{\frac{1}{8}}.8^{\frac{1}{16}}.16^{\frac{1}{32}}.\&c. = 2$.

4. Find the square root of $-79-8\sqrt{-5}$.

5. Shew that in ${}^{3n}C_n$ the number of the combinations in which a particular thing occurs is one-third of the whole number of the combinations.

6. In the expansion of $\left(x^2-\dfrac{1}{x}\right)^6$ write down the co-efficients of x^3 and x^{-3}; also find the term independent of x.

7. Find the 5th root of 3120 to four places of decimals.

8. If $A = (b-c)(a-d)$, $B = (c-a)(b-d)$, $C = (a-b)(c-d)$, find the simplest value of $A^3+B^3+C^3-3ABC$.

IX.

1. Solve

$$\left.\begin{aligned} 2x+3y-8z+35 &= 0 \\ 7x-4y+z-8 &= 0 \\ 12x-5y-3z+10 &= 0 \end{aligned}\right\}$$

2. If x and y are two real quantities connected by the equation $9x^2+2xy+y^2-92x-20y+244=0$, then will x lie between 3 and 6, and y between 1 and 10.

3. Sum to n terms

$$1^2+(1^2+2^2)+(1^2+2^2+3^2)+\text{\&c.}$$

4. The sum of n terms of two arithmetic series are as $3n+31 : 5n-3$; shew that their ninth terms are the same.

5. If ax^3+bx^2+cx+d is a perfect cube, prove that $bc = 9ad$ and $ac^3 = b^3d$

6. Find the number of words beginning and ending with a consonant which can be formed out of the word *equation.*

7. Show that the co-efficient of x^n in the expansion of

$$\frac{(1-2x)^2}{(1-x)^4} \text{ is } \frac{1}{6}(n-6)(n^2-1).$$

8. Sum the series

$$\frac{1}{1.2}+\frac{1.3}{1.2.3.4}+\frac{1.3.5}{1.2.3.4.5.6}+\text{\&c. to infinity.}$$

X.

1. If $a(y+z) = b(z+x) = c(x+y)$, prove that

$$\frac{y-z}{a(b-c)} = \frac{z-x}{b(c-a)} = \frac{x-y}{c(a-b)}.$$

2. Find the value of x for which $3x^2+5x+3$ has its least possible value, and show that the least value is $\frac{11}{12}$.

3. Sum the series

$$1.2+2^2.3+3^2.4+\text{\&c. to } n \text{ terms.}$$

4. If a, b, c are in *H. P.*, show that

$$\left(\frac{1}{a}+\frac{1}{b}-\frac{1}{c}\right)\left(\frac{1}{b}+\frac{1}{c}-\frac{1}{a}\right) = \frac{4}{ac}-\frac{3}{b^2}.$$

5. If the number of permutations of n things taken r together be denoted by the symbol ${}^nP_r$; shew that the number of such permutations in which p particular things occur, will be

$${}^rP_p.{}^{n-p}P_{r-p}.$$

6. Find the seventh root of 126 to 4 places of decimals.

7. Find the co-efficients of x^{14} and x^{-10} in the expansion of $\left(x^4 + \frac{1}{x^2}\right)^8$

8. Shew that, if $x^2 + y^2 + z^2 = yz + zx + xy$, then $x = y = z$.

XI.

1. Solve
$$\begin{aligned} lx + my + nz &= mn + nl + lm, \\ x + y + z &= l + m + n, \\ (m-n)x + (n-l)y + (l-m)z &= 0. \end{aligned}$$

2. A train travelling at the rate of 37½ miles an hour passes a person walking on a road parallel to the railway in 6 seconds; it also meets another person walking at the same rate as the other but in the opposite direction, and passes him in 4 seconds. Find the length of the train.

3. Shew that if x, y, z be in *H. P.*, then will
$$\frac{x}{y+z-x},\ \frac{y}{z+x-y},\ \frac{z}{x+y-z},$$ be also in *H. P.*

4. The series of natural numbers are divided into groups 1; 2, 3, 4,; 5, 6, 7, 8, 9; and so on: prove that the sum of the numbers in the nth group is $n^3 + (n-1)^3$.

5. Given $\dfrac{3x + 2y\sqrt{-1}}{5\sqrt{-1} - 2} = \dfrac{15}{8x + 3y\sqrt{-1}}$, find x and y.

6. Out of $2n$ men who have to sit down, half on each side of a long table, p particular men desire to sit on one side and q on the other; find the number of ways in which this may be done.

7. Shew that if t_r denote the middle term of $(1+x)^{2r}$, then will $t_0 + t_1 + t_2 +$ &c. $= (1-4x)^{-\frac{1}{2}}$.

8. Shew that $4e = 1.3 + \dfrac{2.4}{1.2} + \dfrac{3.5}{1.2.3} + \dfrac{4.6}{1.2.3.4} +$ &c.

XII.

1. Shew that
$$a^3 + b^3 + c^3 + 24abc$$
$$= (a+b+c)^3 + 3\{a(b-c)^2 + b(c-a)^2 + c(a-b)^2\}.$$

2. If $\dfrac{ad-bc}{a-b-c+d} = \dfrac{ac-bd}{a-b-d+c}$, then each of them

$$= \frac{a+b+c+d}{4}.$$

3. If a, b, c, x are all real quantities, and $(a^2+b^2)x^2 - 2b(a+c)x + (b^2+c^2) = 0$, then a, b, c are in *G. P.*, and x is their common ratio.

4. Having given that $\dfrac{a}{b-c}, \dfrac{b}{c-a}, \dfrac{c}{a-b}$ are in *A. P.*; prove that

$$\frac{a^3+c^3-2b^3}{a^2+c^2-2b^2} = \frac{a+b+c}{2}.$$

5. Sum to n terms

$$1^3 - 3^3 + 5^3 - 7^3 + \&c.$$

6. The number of combinations of $2n$ things taken n together when n of the things and no more are alike, is 2^n; and the number of combinations of $3n$ things, n together, when n of the things and no more are alike is $2^{2n-1} + \dfrac{\underline{|2n}}{2(\underline{|n})^2}$.

7. If c_0, c_1, c_2, c_3, &c. be the co-efficients in the expansion of $(1+x)^n$, prove that

$$c_1^2 + 2c_2^2 + 3c_3^2 + \quad . \quad . \quad . \quad . \quad . \quad + nc_n^2 = \frac{\underline{|2n-1}}{\underline{|n-1}\,\underline{|n-1}}.$$

8. Shew that the sum to infinity of the series whose $(r+1)$th term is $\dfrac{(r+1)^3}{\underline{|r}}$ is $15e$.

XIII.

1. If $a(by+cz-ax) = b(cz+ax-by) = c(ax+by-cz)$ and if $a+b+c = 0$, then will $x+y+z = 0$.

2. If x^2+ax+b and $x^2+a'x+b'$ have a common measure of the first degree, prove that their L. C. M. is

$$x^3 + \frac{ab-a'b'}{b-b'}.x^2 + \frac{a'b^2-ab'^2}{a'b-ab'}.x + \frac{bb'(a-a')}{b-b'}.$$

3. Solve
$$\left.\begin{aligned} (x-2)^2+(y-3)^2+(z-1)^2 &= 24 \\ xy+yz+zx &= 63 \\ 2x+3y+z &= 30 \end{aligned}\right\}$$

4. If a, b, c be in *A. P.*, α, β, γ in *H. P.*, $a\alpha, b\beta, c\gamma$ in *G. P.*, then will $a : b : c = \frac{1}{\gamma} : \frac{1}{\beta} : \frac{1}{\alpha}$.

5. If one geometrical mean G and two arithmetical means p and q be inserted between two given quantities, shew that

$$G^2 = (2p-q)(2q-p).$$

6. Shew that

$$\left(\frac{a+x}{a-x}\right)^{\frac{1}{2}} = 1 + \frac{x}{a+x} + \frac{3}{2}\left(\frac{x}{a+x}\right)^2 + \frac{5}{2}\left(\frac{x}{a+x}\right)^3 + \&c.$$

7. If c_0, c_1, c_2, &c. be the co-efficients in the expansion of $(1+x)^n$, prove that

$$c_0^2 + 2c_1^2 + 3c_2^2 + \&c. \quad + (n+1)c_n^2 = \frac{(n+2)\,\underline{|2n-1}}{\underline{|n-1}\,\underline{|n}}.$$

8. Shew that if $3(a^2+b^2+c^2) = (a+b+c)^2$, then $a = b = c$.

XIV.

1. If $(x^2+y^2+z^2)(a^2+b^2+c^2) = (ax+by+cz)^2$, shew that $x : a = y : b = z : c$.

2. Solve

$$\left.\begin{aligned} 2x + 4y - 3z &= 22 \\ 4x - 2y + 5z &= 18 \\ 5x + y - 2z &= 14 \end{aligned}\right\}$$

3. If $\frac{bz+cy}{b-c} = \frac{cx+az}{c-a} = \frac{ay+bx}{a-b}$, shew that

$$(a+b+c)(x+y+z) = ax+by+cz.$$

4. Show that

$$1 + 3\left(\frac{2n+1}{2n-1}\right) + 5\left(\frac{2n+1}{2n-1}\right)^2 + \&c. + (2n-1)\left(\frac{2n+1}{2n-1}\right)^{n-1} = n(2n-1).$$

5. If from a series of n terms in *A. P.*, another series be formed such that each term is an arithmetic mean between the consecutive terms of the first series, and if S and S' be the sums of these series respectively, prove that $S : S' = n : n-1$.

6. Shew that, if x, y, z be in *H. P.*, $(y+z-x)^2$, $(z+x-y)^2$, $(x+y-z)^2$ will be in *A. P.*

7. In the expansion of $\left(2x^2+\frac{1}{x}\right)^n$ find the co-efficients of x and of x^{-2}.

8. There are three candidates for a Professorship, and one is to be elected by the votes of 5 men; in how many ways can the votes be given?

XV.

1. Shew that

$$\begin{aligned}&(b-c)^7+(c-a)^7+(a-b)^7\\&=7(b-c)(c-a)(a-b)(a^2+b^2+c^2-bc-ca-ab)^2.\end{aligned}$$

2. Having given $\frac{x}{1-x^2}=\frac{y+z}{m+nyz}$, $\frac{y}{1-y^2}=\frac{z+x}{m+nxz}$, prove that if x, y be unequal, $\frac{z}{1-z^2}=\frac{x+y}{m+nxy}$.

3. Shew that $y^{mn}-1$ is divisible by either of the quantities y^m-1 or y^n-1 without remainder, m and n being positive integers.

4. Solve $\frac{x}{y+z+1}=\frac{y}{z+x}=\frac{z}{x+y-1}=x+y+z$.

5. Prove that $a_2a_3-a_1a_4$ is positive, zero or negative according as a_1, a_2, a_3, a_4 are in arithmetical, geometrical or harmonical progression

6. In a plane there are m straight lines which all pass through a given point, n others which all pass through another given point, and p others which all pass through a third given point; supposing no other three to intersect at any point, find the number of triangles formed by the intersection of the straight lines.

7. Shew that

$$\left(\frac{10}{7}\right)^{\frac{1}{3}}=1+\frac{1}{10}+\frac{1.4}{10.20}+\frac{1.4.7}{10.20.30}+\&c.$$

8. In the expansion of $(a+b)^{2n}$, prove that the sum of the squares of the first, third, &c., co-efficients differs from the sum of the squares of the second, fourth, &c. by $\frac{2n(2n-1)\dots(n+1)}{1.2.3\dots\dots n}$.

XVI.

1. Solve $\sqrt{\frac{a}{b}\left(bx-a^2\right)}-\sqrt{\frac{b}{a}\left(ax-b^2\right)} = a-b.$

2. If $l(my+nz-lx) = m(nz+lx-my)$ $= n(lx+my-nz)$, show that

$$\frac{l}{y+z-x} = \frac{m}{z+x-y} = \frac{n}{x+y-z}.$$

3. From the equations $x^2+2yz = a,\ y^2+2zx = b,\ z^2+2xy = c$, obtain the result $3(yz+zx+xy) = a+b+c-\sqrt{a^2+b^2+c^2-bc-ca-ab}.$

4. Prove that

$$\frac{(ab-cd)(a^2-b^2+c^2-d^2)+(ac-bd)(a^2+b^2-c^2-d^2)}{(a^2-b^2+c^2-d^2)(a^2+b^2-c^2-d^2)+4(ab-cd)(ac-bd)}$$
$$= \frac{(b+c)(a+d)}{(b+c)^2+(a+d)^2}.$$

5. If x, y, z are such that their sum is constant, and if $(z+x-2y)(x+y-2z)$ varies as yz, prove that $2(y+z)-x$ varies as yz.

6. Prove that

$$2^n = 1+\frac{(n+1)n}{\underline{|2}}+\frac{(n+1)n(n-1)(n-2)}{\underline{|4}}+\text{\&c.}$$

7. Show that

$$\frac{1}{2}+\frac{1.3}{2.4}\cdot\frac{1}{4}+\frac{1.3.5}{2.4.6}\cdot\frac{1}{4^2}+\frac{1.3.5.7}{2.4.6.8}\cdot\frac{1}{4^3}+\text{\&c.}$$
$$= \frac{4}{3}\left(2-\sqrt{3}\right)\sqrt{3}.$$

8. If $yz+\frac{x^3}{y+z} = zx+\frac{y^3}{z+x} = xy+\frac{z^3}{x+y}$, x, y and z being supposed unequal, prove that each of these quantities is equal to $xy+yz+zx$; and that

$$x+y+z = 0.$$

ANSWERS.

Exercises.

1 [Pages 25, 26.]

1. $7ax^2 - 3b^2$. **2.** $a - b + c$. **3.** $2a - b - 3c$.
4. $a^2 + 2b^2 - 3c^2$. **5.** $x + \frac{a}{3} - \frac{b}{2}$. **6.** $x - 2 - \frac{1}{x}$.
7. $x^2 + 1 + \frac{1}{x^2}$. **8** $\frac{x}{y} - \frac{1}{\sqrt{2}} + \frac{y}{x}$. **9.** $a - b + c - d$.
10. $a^2 + b^2$. **11.** $a^2 - b^2 + c^2 - d^2$. **12.** $2(ab + ac + bc)$.

2 [Pages 29, 30.]

1. $x^2 - 2x + 3$. **2.** $x^3 - x + 1$. **3.** $2x^2 + 2ax + 4b^2$.
4. $7x^2 - \frac{x}{5} + 3$. **5.** $\frac{x^2}{2} - 2x + \frac{a}{3}$. **6.** $\frac{x}{y} - \frac{1}{2} - \frac{y}{x}$.
7. $\frac{2x}{7y} - 5 + \frac{3y}{4x}$. **8.** $x^{\frac{5}{6}} - 2x^{\frac{1}{2}} - x^{\frac{1}{6}}$. **9.** $ax^{-1} + 1 + a^{-1}x$.
10. $x^{\frac{3}{4}} - x^{\frac{1}{2}}y^{-\frac{1}{4}} + y^{\frac{1}{2}}$. **11.** $\frac{3x^{\frac{3}{2}}}{2} - \frac{5xy^{\frac{1}{2}}}{3} + \frac{2x^{\frac{1}{2}}y}{5}$.
12. $x^2 + xy + xz + y^2 + z^2$.

3 [Page 36.]

1. 193. **2.** 908. **3.** 1679. **4.** 4698. **5.** 9876.
6. 23456. **7.** 5·37. **8.** ·027. **9.** 15·367. **10.** ·0374.
11. 4·0305. **12.** 1·4142. **13.** ·0252. **14.** 2·2360.
15. 4·4721.

4 [Page 42.]

1. $x + 9$. **2.** $3x - 8$. **3.** $4a - 3b$. **4.** $x^3 - 3x + 2$.
5. $2x^2 + x - 3$. **6.** $1 - 3x^2 + 2x^4$. **7.** $2x^2 - 3cx + c^2$.

5 [Page 46.]

1. 25. **2.** 48. **3.** 98. **4.** 203. **5.** 307.
6. 457. **7.** 579. **8.** 607. **9.** 945. **10.** 8765.
11. 3·19. **12.** 30·02. **13.** 3·546. **14.** 1·442.
15. 8·320.

6 [Page 52.]

1. $\sqrt[7]{a^5}$. **2.** $\frac{1}{\sqrt{x^3}}$. **3.** $3\sqrt[5]{x^4}$. **4.** $\frac{3}{\sqrt[5]{x^2}\times\sqrt{a}}$.
5. $\frac{8}{\sqrt[3]{m^8}}$. **6.** $\frac{\sqrt[4]{a^5}}{3\sqrt[5]{x^4}}$. **7.** $\frac{1}{2\sqrt[6]{x}}$. **8.** $\sqrt[5]{x^{a+2}}$.
9. $\sqrt[2m]{a^{11}}$. **10.** $\sqrt[a]{x^4}$. **11.** $x^{\frac{7}{3}}$. **12.** $\frac{1}{a^{\frac{3}{2}}}$. **13.** $x^{\frac{2}{3}}$.
14. $a^{\frac{2}{5}}$. **15.** $x^{\frac{3}{2}}$. **16.** $a^{\frac{3}{4}}$. **17.** $\frac{1}{2}$. **18.** 4.
19. 27. **20.** 32. **21.** $\frac{1}{27}$. **22.** 36. **23.** $\frac{1}{25}$.
24. 81. **25.** 36. **26.** x^{-m}.

7 [Pages 55, 56.]

1. a^{-6}. **2.** $a^{-\frac{1}{2}}b^{\frac{5}{8}}$. **3.** ab^6. **4.** $a^{-8}b^{-\frac{5}{3}}$.
5. a^8b^6. **6.** $x^{-\frac{9}{2}}y^4$. **7.** $x^{\frac{5}{32}}$. **8.** a^{-1}.
9. y. **10.** $\frac{4}{9}x^2a^2$. **11.** $\frac{9}{16}x^{-2}a^{-2}$. **12.** $a^{\frac{1}{2}}b^{\frac{1}{3}}c^{\frac{2}{3}}$.
13. $a^{-1}b^{\frac{2}{3}}c^{\frac{1}{3}}$. **14.** $a^{\frac{3}{2}}b^{-\frac{1}{3}}c^{\frac{1}{2}}$. **15.** a^4b^2.

8 [Pages 59—61.]

1. $x-2x^{\frac{1}{2}}+1$. **2.** $a-27b$. **3.** $1+a^2b^{-2}+a^4b^{-4}$.
4. $x^2+6xz^{\frac{1}{3}}-4y+9z^{\frac{2}{3}}$. **5.** $x^{-2}+x^{-1}y^{-1}+y^{-2}$.
6. $a+x^{\frac{1}{3}}-1+a^{-\frac{1}{3}}+a^{-1}$. **7.** $x+y+z-3x^{\frac{1}{3}}y^{\frac{1}{3}}z^{\frac{1}{3}}$.
8. $a^{2m}-9b^{2n}+12b^nc^p-4c^{2p}$. **9.** a^3-64b^2.
10. $a+a^{\frac{3}{4}}x^{-\frac{1}{4}}-a^{\frac{1}{4}}x^{-\frac{3}{4}}-x^{-1}$. **11.** $x-x^{\frac{1}{2}}$.

12. $2x^{-2}+4x^{-1}+2$. **13.** $y+x^{\frac{1}{2}}y^{\frac{1}{2}}+x$. **14.** $a+a^{\frac{1}{2}}b^{\frac{1}{2}}-b$.
15. $x^{2n}-1+x^{-2n}$. **16.** $4x-2x^{\frac{1}{2}}y^{-\frac{1}{2}}+2x^{\frac{1}{2}}z^{\frac{1}{3}}+y^{-1}+$
$y^{-\frac{1}{2}}z^{\frac{1}{3}}+z^{\frac{2}{3}}$. **18.** $x^{2^n}-a^{2^n}$. **19.** $x^{2^{n-1}}-y^{2^{n-1}}$.
20. a^{m-1}. **21.** $x^{\frac{3}{4}}+3x^{\frac{1}{2}}-1$. **22.** $x^{\frac{3}{2}}+xy^{-\frac{1}{2}}-2x^{\frac{5}{4}}y^{-\frac{1}{4}}$.
$2x^{\frac{1}{2}}y^{\frac{1}{4}}+2x^{\frac{3}{4}}y^{\frac{1}{2}}+y$. **23.** $x^n+x^{\frac{n}{2}}a^{\frac{n}{2}}+a^n$. **24.** $x^{\frac{2}{3}}-4x^{\frac{5}{6}}$
$+4x+2x^{\frac{7}{6}}-4x^{\frac{4}{3}}+x^{\frac{5}{3}}$. **25.** $a^{\frac{2}{3}}x^{-\frac{2}{3}}+a^{\frac{1}{3}}x^{-\frac{1}{3}}+a^{-\frac{1}{3}}x^{\frac{1}{3}}+a^{-\frac{2}{3}}x^{\frac{2}{3}}$.
26. $\dfrac{1}{\sqrt{a}}+\dfrac{1}{\sqrt{b}}$. **27.** $\dfrac{2x+36x^{\frac{1}{3}}y^{\frac{2}{3}}}{x-27y}$. **28.** $\dfrac{x+a}{x^2+3xa+a^2}$.
29. 1. **30.** $x^{\frac{1}{4}}y^{\frac{1}{4}}-x^{-\frac{1}{4}}y^{\frac{3}{4}}$. **31.** 1. **32.** 3. **33.** 5.
34. 2. **35.** 4. **36.** $x=1, y=2$. **37.** $x=2, y=3$.
38. $x=1, y=3$. **39.** $x=1, y=2, z=3$. **40.** $x=2$,
$y=4, z=6$.

9 [Page 62.]

1. $\sqrt{45}$. **2.** $\sqrt[3]{24}$. **3.** $\sqrt[4]{96}$. **4.** $\sqrt[4]{1280}$.
5. $\sqrt[n]{a^n b}$. **6.** $\sqrt[a]{x^{3a}y}$. **7.** $\sqrt[5]{a^{20}b^2}$.

10 [Page 63.]

1. $3\sqrt{2}$. **2.** $4\sqrt{5}$. **3.** $5\sqrt[3]{2}$. **4.** $2\sqrt[5]{4}$. **5.** $3\sqrt[4]{5}$.
6. $7\sqrt[3]{4}$. **7.** $5\sqrt[4]{3}$. **8.** $a^2\sqrt[3]{b}$. **9.** $x^4\sqrt[n]{a}$. **10.** $-8\sqrt[3]{5}$.
11. $-4ab\sqrt[3]{3b}$. **12.** $5a^2x\sqrt[3]{4ax}$.

11 [Page 64.]

1. $7\sqrt{3}$. **2.** $7\sqrt{2}$. **3.** $8\sqrt{5}$. **4.** $2\sqrt{2}$. **5.** $\sqrt[3]{2}$.
6. $5\sqrt[4]{5}$. **7.** $\sqrt[4]{3}$. **8.** $3\sqrt{3}$. **9.** $6\sqrt{5}$. **10.** 0. **11.** 0.
12. $17\sqrt[3]{2}$. **13.** $(7x+y)\sqrt{5x}$. **14.** $(x^2-2y^2+3z^2)\sqrt[3]{a}$.
15. $4a\sqrt[4]{2x}$.

12 [Page 65.]

1. $\sqrt[6]{27}$ and $\sqrt[6]{4}$. **2.** $\sqrt[12]{256}$ and $\sqrt[12]{125}$. **3.** $\sqrt[15]{8}$ and $\sqrt[15]{243}$. **4.** $\sqrt[12]{27}$ and $\sqrt[12]{25}$. **5.** $\sqrt[24]{256}$ and $\sqrt[24]{216}$.
6. The latter. **7.** The former. **8.** The former. **9.** $\sqrt[3]{4}$, $\sqrt[4]{6}$, $\sqrt{2}$. **10.** $\sqrt[8]{10}$, $\sqrt[4]{3}$, $\sqrt[12]{25}$.

13 [Page 67.]

1. $5\sqrt{2}$. **2.** $4\sqrt{3}$. **3.** 9. **4.** $3\sqrt{10}$. **5.** 30. **6.** 5.
7. $3ax\sqrt[3]{6x}$. **8.** $\sqrt[6]{864}$. **9.** $\sqrt[6]{288}$. **10.** $4\sqrt[6]{2}$.
11. $9\sqrt[6]{3}$. **12.** $\sqrt[18]{72}$. **13.** $\sqrt[8]{27}$. **14.** $\sqrt[18]{32}$.
15. $\sqrt[12]{1024}$. **16.** $40\sqrt{3}$. **17.** $288\sqrt{2}$. **18.** $480\sqrt[3]{3}$.
19. $210abx\sqrt[3]{x}$. **20.** $2\sqrt{\frac{2}{3}}$. **21.** $\frac{1}{3}$. **22.** $\sqrt[3]{\frac{3}{4}}$.
23. $\sqrt[6]{\frac{2}{3}}$. **24.** ·577. **25.** 1·341. **26.** 3·535.
27. 26·832.

14 [Page 68.]

1. $a\sqrt{b}+b\sqrt{a}$. **2.** $a-b$. **3.** $6a-10\sqrt{a}$. **4.** $16x-9y$.
5. $6x-54$. **6.** $6+\sqrt{10}$. **7.** $7+4\sqrt{6}$. **8.** $6-6\sqrt{5}$.
9. $2+6\sqrt{2}$. **10.** $5+3\sqrt[3]{18}+3\sqrt[3]{12}$. **11.** $2x-2\sqrt{x^2-a^2}$.
12. $182+80\sqrt{3}$. **13.** $83+12\sqrt{35}$. **14.** $2a^2-2\sqrt{a^4-4b^4}$.
15. $29x^2-21y^2+20\sqrt{x^4-y^4}$.

15 [Pages 71, 72.]

1. $\frac{23-3\sqrt{21}}{10}$. **2.** $5+2\sqrt{6}$. **3.** $24+17\sqrt{2}$.
4. $9+2\sqrt{15}$. **5.** $\frac{a+\sqrt{a^4-x^2}}{x}$. **6.** $x^2-\sqrt{x^4-1}$.
7. $\frac{1}{4}(2+\sqrt{2}-\sqrt{6})$. **8.** $1+\sqrt{5}+\sqrt{10}-\sqrt{2}$.
9. $\frac{\sqrt{2}+\sqrt{3}+\sqrt{5}}{2}$. **10.** $\frac{1}{2}(\sqrt{21}+\sqrt{10}-\sqrt{14}-\sqrt{15})$.
11. 5·828. **12.** 6·464. **13.** 5·414. **14.** 3·650.
15. 6·854. **16.** ·504. **17.** $2x$. **18.** $\sqrt{5}(1+\sqrt{2})$.
19. $2+\sqrt{3}$. **20.** 198. **21.** $4x\sqrt{x^2-1}$. **22.** $2x^2$.

16 [Page 74.]

1. $3^{\frac{5}{2}}+3^2.4^{\frac{1}{3}}+3^{\frac{3}{2}}.4^{\frac{2}{3}}+3.4+3^{\frac{1}{2}}.4^{\frac{4}{3}}+4^{\frac{5}{3}}$.

2. $6^{\frac{3}{2}}-6.3^{\frac{1}{4}}+6^{\frac{1}{2}}.3^{\frac{1}{2}}-3^{\frac{3}{4}}$. **3.** $3^4-3^3.4^{\frac{1}{5}}+3^2.4^{\frac{2}{5}}-3.4^{\frac{3}{5}}+4^{\frac{4}{5}}$. **4.** $2^{\frac{35}{4}}-2^{\frac{15}{2}}.3^{\frac{3}{8}}+2^{\frac{25}{4}}.3^{\frac{3}{4}}-2^5.3^{\frac{9}{8}}+2^{\frac{15}{4}}.3^{\frac{3}{2}}-2^{\frac{5}{2}}.3^{\frac{15}{8}}+2^{\frac{5}{4}}.3^{\frac{9}{4}}-3^{\frac{21}{8}}$. **5.** $3^{\frac{20}{3}}-3^{\frac{16}{3}}.4^{\frac{5}{6}}+3^4.4^{\frac{5}{3}}-3^{\frac{8}{3}}.4^{\frac{5}{2}}+3^{\frac{4}{3}}.4^{\frac{10}{3}}-4^{\frac{25}{6}}$.

6. $4^6 + 4^5.5^{\frac{1}{7}} + 4^4.5^{\frac{2}{7}} + 4^3.5^{\frac{3}{7}} + 4^2.5^{\frac{4}{7}} + 4.5^{\frac{5}{7}} + 5^{\frac{6}{7}}$.

7. $\frac{1}{3}(1 - 2.4^{\frac{1}{3}} + 4^{\frac{2}{3}})$.

8. $\dfrac{31 + 2(3^{\frac{5}{2}}.2^{\frac{1}{3}} + 3^2.2^{\frac{2}{3}} + 3^{\frac{3}{2}}.2 + 3.2^{\frac{4}{3}} + 3^{\frac{1}{2}}.2^{\frac{5}{3}})}{23}$.

9 $3 - 2\,2^{\frac{3}{4}} + 2.2^{\frac{1}{2}} - 2.2^{\frac{1}{4}}$. 10. $\dfrac{13 - 3.3^{\frac{1}{5}} - 9.3^{\frac{2}{5}} + 3^{\frac{3}{5}} + 3.3^{\frac{4}{5}}}{13}$.

17 [Page 77.]

1. $\sqrt{5} + \sqrt{3}$. 2. $3 - \sqrt{5}$. 3. $3 + 2\sqrt{2}$. 4. $5 -$
$2\sqrt{3}$. 5. $2\sqrt{7} + \sqrt{3}$. 6. $3\sqrt{5} - 2\sqrt{7}$. 7. $2\sqrt{11} + \sqrt{3}$.
8. $\sqrt{\frac{7}{2}} - \sqrt{\frac{5}{2}}$. 9. $\sqrt[4]{2}(\sqrt{2} - 1)$. 10. $\sqrt[4]{2}(\sqrt{3} - 1)$.
11. $\sqrt[4]{3}(1 + \sqrt{2})$. 12. $\sqrt[4]{5}(\sqrt{3} + \sqrt{2})$. 13. $x + \sqrt{a^4 - x^2}$.
14. $\sqrt{a + b} + \sqrt{a - b}$. 15. $\sqrt{a + \frac{1}{2}x} + \sqrt{\frac{1}{2}x}$.
16. $\sqrt{x + 2} + \sqrt{x - 3}$. 17. $\sqrt{x + y} + \sqrt{z}$.

18 [Page 79.]

1. $1 + \sqrt{6}$. 2. $2 - \sqrt{3}$. 3. $\sqrt{5} + \sqrt{2}$. 4. $3\sqrt{2} - \sqrt{5}$.
5. $4\sqrt{3} + \sqrt{6}$.

19 [Pages 84, 85.]

1. $1 + \sqrt{2} + \sqrt{5}$. 2. $\sqrt{2} + \sqrt{3} + \sqrt{5}$. 3. $\sqrt{3} +$
$\sqrt{2} + \sqrt{6}$. 4. $2\sqrt{3} + 2 - \sqrt{5}$. 5. $\sqrt{2} + \sqrt{6} - \frac{4}{3}\sqrt{3}$.
6. ·44721. 7. $4 - \sqrt{10} + \sqrt{2}$ 8. 1. 9. $\dfrac{1}{\sqrt{3}}$.
10 b. 11. $a + b$. 12 $\sqrt[3]{3} + 1$. 13 $\frac{3}{4}\sqrt{5}$. 14. 125.
15. 1692.

20 [Pages 89, 90.]

1. The latter. 2. The latter. 3. The former.
4. The former. 5. The latter. 6. $a : d$. 7. 1 : 4.
8. 1 : 1. 9. 75 : 8. 10. 28 : 27. 11. 5 : 7.
12. 3 : 4. 13. 63 and 72. 14. 85 and 51. 15. 28 and 35.
16. 42 and 54. 17. −15. 18. 13. 19. 17.
20. $\dfrac{ad - bc}{c - d}$. 23. 76:75. 24. 1772 : 1771. 25. B.

21 [Page 91.]

1. 4. **2.** 18. **3.** $37\frac{1}{2}$. **4.** 36. **5.** 20. **6.** 60. **7.** 20. **8.** 6. **9.** 14. **10.** 18.

22 [Pages 96, 97.]

1. $x = 9, y = 6$. **2.** $x = 25, y = 9$. **3.** $x = 56, y = 30$. **4.** $\frac{1}{3}a$. **5.** $\frac{5}{8}$. **6.** $\frac{3}{4}$. **7.** $\frac{4}{81}$. **8.** $\frac{1}{5}$. **9.** $2\frac{7}{9}$. **10.** $\sqrt{2ab - b^2}$. **11.** $a\left\{1 - \frac{16b^2}{(1+b)^4}\right\}$. **14.** $\frac{29}{61}$. **16.** 2.

24 [Pages 102—104.]

21. 0.

25 [Pages 113—116.]

1. $3y = x$. **2.** 14. **3.** $2\frac{1}{2}$. **4.** 1. **5.** $27x^2 = 4y^3$. **6.** $y = 2x + \frac{2}{x}$. **7.** $xy = \frac{12}{25}\left(x^2 + y^2\right)$. **8.** $y = 2x + \frac{4}{x^2}$. **9.** $y = 3 + 2x - x^2$. **10.** $y = \frac{b}{a^2}\sqrt{c}$

12. 45 inches. **13.** £26. 5*s*. **15.** 45 sq. ft. **16.** $346\frac{1}{2}$ sq. ft. **17.** 960 cubic ins. **18.** $1\frac{5}{9}$ ft. **19.** 10. **20.** 1·2426 ins. nearly. **21.** ·01875 in. **22.** 1610 feet ; 305·9 feet. **23.** 3 days, 6 hours. **24.** $224\frac{1}{2}$ days nearly. **25.** 9 : 4.

26. Value of diamond $= \frac{mcn^2}{(m+1)a^2}$, value of ruby $= \frac{cn^{\frac{3}{2}}}{(m+1)b^{\frac{3}{2}}}$. **29.** $x = \frac{22}{15}z + \frac{2}{15z}$. **31.** The cost is least when the rate is 12 miles an hour ; and the cost per mile is £$\frac{3}{32}$, and for the journey is £9. 7*s*. 6*d*.

26 [Pages 118, 119.]

1. $\pm 2\frac{1}{3}$. 2. ± 5. 3. ± 2. 4. ± 9. 5. ± 2. 6. $\pm \frac{\sqrt{3}}{2}$.

7. $\pm \frac{a}{2}\left(\frac{a^2-4}{a^2-1}\right)^{\frac{1}{2}}$. 8. $\pm \sqrt{mab}$. 9. $\pm \frac{2}{b\left(4a-b^2\right)^{\frac{1}{2}}}$.

10. $\pm \frac{2a}{\sqrt{5}}$. 11. $\pm \sqrt{\frac{3}{5}}$. 12. $\pm \sqrt{\frac{13}{2}}$.

27 [Pages 119—122.]

3. $\frac{3}{4}$, $7\frac{1}{4}$. 4. $3\frac{3}{5}$, $2\frac{2}{5}$. 5. $2\frac{1}{3}$, $2\frac{1}{4}$. 6. $2 \pm \frac{1}{5}\sqrt{3}$.
8. 9, 8. 9. $\frac{3}{5}$, $\frac{4}{7}$. 10. $\frac{2}{3}$, $\frac{3}{10}$. 11. 29, -10. 12. 10, -29.
15. $\frac{4}{3}$, 0. 16. 10, $-\frac{2}{3}$. 17. 3. 18. 6, $\frac{40}{13}$. 19. 1, $-\frac{40}{17}$.
20. $\frac{-11 \pm \sqrt{13}}{6}a$.

28 [Page 124.]

1. 3, $2\frac{2}{\text{[illegible]}}$. 2. -4, -5. 3. $\frac{4}{3}$, $-\frac{5}{2}$. 4. $\frac{5}{3}$, $-\frac{7}{3}$.
5. $2\frac{1}{2}$, $-\frac{3}{4}$. 6. 5, $\frac{5}{3}$. 7. -1, $\frac{5}{4}$.

29 [Page 124.]

1. $\frac{3}{2}$, -6. 2. $1\frac{2}{5}$, $-1\frac{1}{3}$. 3. $\frac{5}{13}$, -8. 4. $\frac{8}{3}$, -34.
5. $9\frac{15}{17}$, -11. 6. $\frac{a}{2}$, $\frac{b}{c}$. 7. ab, $-\frac{a}{3}$.

30 [Page 127.]

1. 2, 4. 2. 5, -4. 3. 1, -4. 4. 4, -13
5. 3, $-\frac{4}{5}$. 6. $\frac{1}{2}$, $-\frac{4}{3}$. 7. 7, $-\frac{1}{3}$. 8. 2, $\frac{1}{2}$.
9. 2, $-\frac{2}{15}$. 10. a, b. 11. 1, $\frac{2b}{a-b}$. 12. a, $-b$.
13. $\frac{a}{2}$, $\frac{3a}{4}$. 14. 4, 8. 15. 0, $\frac{2ab-ac-bc}{a+b-2c}$.
16. a, $\frac{a}{5}$.

31 [Page 128.]

1. 2, 3. **2.** ± 2, $\pm\sqrt{10}$. **3.** $\frac{1}{4}$, $\frac{1}{9}$. **4.** $\pm\frac{1}{3}$, ± 1.

5. $\frac{1}{4}$, $-\frac{1}{2}$. **6.** $\frac{1}{a^2}$, $\frac{1}{4a^2}$. **7.** -8, $-\frac{1}{8}$. **8.** 3^n, 5^n.

9. 2^{2n}, 2^{4n}. **10.** 4, $\sqrt[3]{49}$. **11.** $(\frac{1}{32})^{\frac{1}{3}}$, $(-\frac{4}{243})^{\frac{1}{3}}$.

12. 4, $\frac{1}{6}$. **13.** $2^{\frac{3}{2n}}$, $(-\frac{8}{3})^{\frac{3}{2n}}$.

32 [Pages 130, 131.]

1. 0, $\frac{1}{2}$, $\frac{1}{3}$. **2.** -1, 2, -2. **3.** $-\frac{2}{5}$, $\frac{1\pm\sqrt{10}}{3}$.

4. 2, $-1\pm\sqrt{-10}$. **5.** 1, $\frac{1}{4}(-1\pm\sqrt{-7})$.

6. -3, $\frac{1}{2}(3\pm\sqrt{-3})$. **7.** ± 1, $\pm\left(\frac{-11\pm\sqrt{85}}{6}\right)^{\frac{1}{2}}$.

8. $-m$, $m\pm\sqrt{m^2+2}$. **9.** 1, 2, 3, 4. **10.** 4, $1\pm\sqrt{-1}$.

11. 3, -1, $-1\pm\sqrt{6}$. **12.** $1-p$, $\frac{1}{2}\left(-1\pm\sqrt{\frac{p-5}{p-1}}\right)$.

13. $\frac{1}{1-p}$, $\frac{1}{2(p-1)^2}\left\{-1\pm\sqrt{1-4(p-1)^3}\right\}$.

33 [Pages 131—136.]

3. 0, 2. **4.** $\pm\frac{1}{2}$. **5.** -4. **6.** ± 3. **7.** 0, -1.

8. 0, 2. **11.** 2, -8, $-3\pm 3\sqrt{5}$. **12.** 1. **13.** 4, -9.

14. 0, -3, 12, -15. **15.** 1, $\frac{1}{2}$. **16.** 7, $-\frac{14}{3}$, $\frac{7\pm\sqrt{37}}{6}$.

17. 2, $-\frac{1}{2}$. **18.** 1, -2.

19. 2, $\frac{1}{4}$, $\frac{1}{8}(9\pm\sqrt{-31})$. **20.** 1, 2, $\frac{1}{2}\{3\pm\sqrt{-1}\}$.

21. 4, -6, $-1\pm 4\sqrt{2}$. **22.** 0, 3, $\frac{3}{2}\pm\frac{\sqrt{-23}}{2}$.

25. $\frac{1}{2}$, $-\frac{9}{2}$, $-2\pm\sqrt{-6}$. **26.** 9, -7, $1\pm\sqrt{-24}$.

27. 2, -4, $-1\pm\sqrt{71}$. **28.** 9, -7, $1\pm\sqrt{-6}$.

29. 3, $-\frac{3}{2}$, $\frac{3\pm\sqrt{-47}}{4}$. **30.** $-\frac{3}{2}$, $\frac{-3\pm\sqrt{10}}{2}$.

31. 4, -3, $\frac{1}{2}(1\pm\sqrt{-43})$. **32.** 5, -1, $2\pm\sqrt{5}$.

34 [Pages 136, 137.]

1. $2, -\frac{9}{2}$. **2.** $2, -\frac{13}{3}$. **3.** $4, -\frac{14}{5}$. **4.** $5, -\frac{13}{4}$. **5.** $\frac{-7 \pm \sqrt{17}}{8}$.

35 [Page 140.]

1. $2 \pm \sqrt{3}, \frac{-1 \pm \sqrt{-3}}{2}$. **2.** $\frac{1}{4}(3 \pm \sqrt{-7}), \frac{1}{4}(-1 \pm \sqrt{-15})$.

3. $\pm 1, \frac{1}{2}(1 \pm \sqrt{-3}), \pm \sqrt{-1}$. **4.** $1, \frac{5 \pm \sqrt{5}}{2}$.

5. $2, \frac{1}{2}, \frac{1}{4}(3 \pm \sqrt{-7})$. **6.** $2, \frac{1}{2}, \frac{1}{4}(9 \pm \sqrt{65})$. **7.** $3, -1, \frac{3 \pm \sqrt{21}}{2}$.

8. $2, -\frac{1}{2}, \frac{5 \pm \sqrt{41}}{4}$. **9.** $2, -\frac{1}{2}, 3, -\frac{1}{3}$. **10.** $3, \frac{1}{3}, \frac{-8 \pm \sqrt{55}}{3}$.

11. $x + \frac{c}{ax} = \frac{1}{2a}\left\{-a^2 \pm \sqrt{a^4 - 4a^2b + 8ac}\right\}$.

12. $(1 + \sqrt{3}) \pm \sqrt{3 + 2\sqrt{3}}, \quad (1 - \sqrt{3}) \pm \sqrt{3 - 2\sqrt{3}}$.

36 [Pages 140—143.]

1. $3, -1$. **2.** $\pm \frac{2a}{5}, -\frac{a}{2}$. **3** ± 1. **4.** $1, 9, -\frac{18}{5}$.

5. $a, \frac{a}{2}, -\frac{a}{3}$. **6.** $\frac{9}{13}, \frac{4}{13}$. **7.** $0, \frac{63a}{65}$. **8.** $\pm \frac{n^5 + 1}{n^5 - 1} \cdot m$.

9. $5, -4, \frac{1 \pm \sqrt{-75}}{2}$. **10.** $1, \frac{1}{2}(-3 \pm \sqrt{5})$.

11. $\pm 1, \frac{1}{2}(3 \pm \sqrt{5})$. **12.** $\pm 5, \pm 4\sqrt{2}$.

13. $1, 4$. **14.** $-2, -2 \pm \sqrt{3}$. **15.** $\frac{2ac}{2c + b \pm \sqrt{b^2 - 4c^2}}$.

16. $2, -\frac{8}{7}, \frac{-3 \pm \sqrt{93}}{7}$. **17.** $9, 4, \frac{-3 \pm \sqrt{-7}}{2}$.

18. $4, -2, -1 \pm \sqrt{7}$. **19.** $3, -1$. **20.** $\frac{3 \pm \sqrt{17}}{2}, \frac{3 \pm \sqrt{13}}{2}$.

21. $5, -1, 2 \pm \sqrt{5}$. **22.** $\frac{5c}{3}, -\frac{29c}{21}$.

23. $\frac{a}{2}(-1 \pm \sqrt{5})$. **24.** $\pm\sqrt{\frac{n}{n-2}}$, $\pm\sqrt{\frac{n-1}{n+1}}$.

25. $\frac{a}{2}(1 \pm \sqrt{5})$. **26.** 1, $\frac{1}{2}(1 \pm \sqrt{5})$. **27.** 5, $2 \pm \sqrt{-2}$.

28. a, $\frac{b}{2a}\left\{a + b \pm \sqrt{5a^2 + 2ab + b^2}\right\}$. **29.** 6, $\pm\sqrt{-11}$.

30. 5, $\frac{1}{2}(1 \pm \sqrt{-23})$. **31.** $\frac{a}{2} \pm \sqrt{\frac{a^2}{4} - 1}$, $\frac{-1 \pm \sqrt{-3}}{2}$.

32. $\frac{1}{2}(-3 \pm \sqrt{5})$, $\frac{1}{6}(-5 \pm \sqrt{-11})$. **33.** 1, $\frac{-1 \pm \sqrt{-3}}{2}$.

34. $\frac{1 \pm \sqrt{-1}}{\sqrt{2}}$, $\frac{-1 \pm \sqrt{-1}}{\sqrt{2}}$. **35.** $\pm 1\frac{13}{14}$.

36. $\frac{1}{3}\left(6 \pm \sqrt{27 \pm 3\sqrt{57}}\right)$.

37 [Pages 145—147.]

4. (i) Real but not rational; (ii) Imaginary; (iii) Rational; (iv) Real and equal; (v) Real but not rational; (vi) Imaginary.
7. ± 12. **8.** 8. **9.** 0, 3.

38 [Pages 147—153.]

4. (i) $-\frac{1}{16}$; (ii) $1\frac{11}{12}$; (iii) $\frac{7}{8}$. **5.** $\frac{9}{8}$, $\frac{5}{8}$, $3\frac{1}{4}$. **7.** (i) 8; (ii) 7; (iii) 6. **8.** 3, $\frac{1}{2}$, $\frac{1}{2}$. **14.** $+3$ and -3. **22.** x between -3 and $+3$, y between 0 and 4.

39 [Page 155.]

1. $x^2 - 5x + 6 = 0$. **2.** $x^2 + 3x - 28 = 0$. **3.** $x^2 + 5x - 104 = 0$. **4.** $3x^2 - 53x + 34 = 0$. **5.** $5x^2 - 12x - 9 = 0$. **6.** $x^2 - 10x + 23 = 0$. **7.** $x^2 - 8x + 9 = 0$. **8.** $x^2 - 14x + 29 = 0$. **9.** $x^2 + 14x + 4 = 0$. **10.** $(m^2 - n^2)x^2 + 4mnx - (m^2 - n^2) = 0$. **11.** $x^2 - (1 + \sqrt{3})x + 1 = 0$.

40 [Page 155.]

1. 3, 4. **2.** 5, -3. **3.** -2, -4. **4.** $3 + \sqrt{7}$, $3 - \sqrt{7}$.

41 [Page 156.]

1. $7,\ -1$. **2.** $a,\ a-b-c$. **3.** $a,\ 7-a$. **4.** $1,\ \dfrac{47-44\sqrt{6}}{23}$.

42 [Pages 159, 160.]

1. $qx^2-(p^2-2q)x+q=0$. **2.** $qx^2-2px+4=0$.

3. $x^2-p^2x+p^2q=0$. **5.** 36. **7.** (i) $\dfrac{bc^4(3ac-b^2)}{a^7}$;

(ii) $\dfrac{b(3ac-b^2)}{a^2c}$; (iii) $\dfrac{b^4-4b^2ac+2a^2c^2}{a^2c^2}$.

9. 5. **10.** 12. **11.** 8. **12.** 3.

43 [Pages 162, 163.]

1. $4(x-3)(x+\frac{7}{4})$. **2.** $(x+5)(x+17)$. **3.** $(x+20)(x-32)$.

4. $3\left(x+\dfrac{4-\sqrt{19}}{3}\right)\left(x+\dfrac{4+\sqrt{19}}{3}\right)$.

5. $5\left(x+\dfrac{6-\sqrt{26}}{5}\right)\left(x+\dfrac{6+\sqrt{26}}{5}\right)$. **6.** Yes. **7.** No.

8. Yes. **9.** No. **10.** No. **13.** Between $\frac{3}{4}$ and $\frac{3}{2}$.

44 [Pages 169, 170.]

1. ± 10. **2.** ± 7. **10.** $(cc'-aa')^2=(bc'-ab')(b'c-a'b)$.

12. $\dfrac{(ad'-a'd)^2-(ab'-a'b)(cd'-c'd)}{(ab'-a'b)(bd'-b'd)-(ac'-a'c)(ad'-a'd)}$

or, $\dfrac{(ac'-a'c)(cd'-c'd)-(ad'-a'd)(bd'-b'd)}{(ad'-a'd)^2-(ab'-a'b)(cd'-c'd)}$.

45 [Pages 175, 176.]

1. $x=1,\ -\dfrac{53}{88}$; $y=1,\ -\dfrac{25}{22}$. **2.** $x=5,\ \dfrac{333}{28}$; $y=9,\ \dfrac{185}{42}$. **3.** $x=12,\ 18$; $y=18,\ 12$.

4. $x=29,\ -47$; $y=47,\ -29$. **5.** $x=3,\ 2$; $y=2,\ 3$. **6.** $x=11,\ -8$; $y=8,\ -11$.

7. $x = 7, -4$; **8.** $x = 4, 3$; **9.** $x = 4, \frac{16}{3}$;
$y = 4, -7.$ $y = 3, 4.$ $y = 3, \frac{5}{3}.$

10. $x = \frac{1}{a}$; **11.** $x = 7, 3, -7, -3$; **12.** $x = 8, 5, -8, -5$;
$y = \frac{1}{b}.$ $y = 3, 7, -3, -7.$ $y = 5, 8, -5, -8.$

13. $x = 5, 7, -5, -7$; **14.** $x = 5, -\frac{8}{3}$; **15.** $x = 8, 5$;
$y = 7, 5, -7, -5.$ $y = 4, -\frac{15}{2}.$ $y = 5, 8.$

16. $x = 7, -5$; **17.** $x = 7, 2, -7, -2$; **18.** $x = 9, 5, -9, -5$;
$y = 5, -7.$ $y = 2, 7, -2, -7.$ $y = 5, 9, -5, -9.$

19. $x = 17, 1$; **20.** $x = 3, 2, -3, -2$; **21.** $x = 5, 3, -5, -3$;
$y = 1, 17.$ $y = 2, 3, -2, -3.$ $y = 3, 5, -3, -5.$

22. $x = 4, 2$; **23.** $x = 7, -3$; **24.** $x = \frac{6}{5}, \frac{8}{3}$;
$y = 2, 4.$ $y = 3, -7.$ $y = \frac{8}{3}, \frac{6}{5}.$

25. $x = 5, 1$; **26.** $x = 8, 2$; **27.** $x = 9, 4$;
$y = 1, 5.$ $y = 2, 8.$ $y = 4, 9.$

28. $x = 5, 1$; **29.** $x = 2, 8$; **30.** $x = 729, 343$;
$y = 1, 5.$ $y = 8, 2.$ $y = 343, 729.$

31. $x = 1, 64$; **32.** $x = 2, 1$; **33.** $x = \frac{1}{6}, -\frac{1}{6}$; **34.** $x = \frac{1}{3}, -\frac{1}{2}$;
$y = 64, 1.$ $y = 1, 2.$ $y = \frac{1}{6}, -\frac{1}{6}.$ $y = \frac{1}{2}, -\frac{1}{3}.$

46 [Pages 178, 179.]

1. $x = 3, \frac{5}{\sqrt{2}}, -3, -\frac{5}{\sqrt{2}}$; **2.** $x = 2, \sqrt{\frac{2}{5}}, -2, -\sqrt{\frac{2}{5}}$;
$y = 2, \frac{1}{\sqrt{2}}, -2, -\frac{1}{\sqrt{2}}.$ $y = \frac{1}{2}, -2\sqrt{\frac{2}{5}}, -\frac{1}{2}, 2\sqrt{\frac{2}{5}}.$

3. $x = 2, -2$; **4.** $x = \frac{3}{2}, \frac{1}{2}, -\frac{3}{2}, -\frac{1}{2}$; **5.** $x = 2, \frac{4}{3}\sqrt{3}, -2, -\frac{4}{3}\sqrt{3}$;
$y = 3, -3.$ $y = \frac{1}{2}, \frac{3}{2}, -\frac{1}{2}, -\frac{3}{2}.$ $y = 6, \frac{10}{3}\sqrt{3}, -6, -\frac{10}{3}\sqrt{3}.$

6. $x = 5, 3, -5, -3$; **7.** $x = \sqrt{3}, \sqrt{\frac{3}{19}}, -\sqrt{3}, -\sqrt{\frac{3}{19}}$;

$y = 3, 4, -3, -4.$ $y = 0, 6\sqrt{\frac{3}{19}}, 0, -6\sqrt{\frac{3}{19}}.$

8. $x = 1, \frac{13}{21}\sqrt{21}, -1, -\frac{13}{21}\sqrt{21}$;

$y = 3, \frac{2}{21}\sqrt{21}, -3, -\frac{2}{21}\sqrt{21}.$

9. $x = 3, \frac{4}{3}\sqrt{6}, -3, -\frac{4}{3}\sqrt{6}$; **10.** $x = 4, 3\sqrt{3}, -4, -3\sqrt{3}$;

$y = 1, \frac{1}{6}\sqrt{6}, -1, -\frac{1}{6}\sqrt{6}.$ $y = 5, \sqrt{3}, -5, -\sqrt{3}.$

11. $x = 7, \sqrt{3}, -7, -\sqrt{3}$; **12.** $x = 4, 3$; **13.** $x = 7, -6$;

$y = 2, -3\sqrt{3}, -2, 3\sqrt{3}.$ $y = 3, 4.$ $y = 6, -7.$

14. $x = 4, \frac{38}{\sqrt{87}}, -4, -\frac{38}{\sqrt{87}}$; **15.** $x = 2, -2, \pm 5\sqrt{\frac{5}{67}}$;

$y = 1, \frac{3}{\sqrt{87}}, -1, -\frac{3}{\sqrt{87}}.$ $y = 3, -3, \mp \frac{34}{5}\sqrt{\frac{5}{67}}.$

16. $x = 3, -3, 8\sqrt{\frac{37}{127}}, -8\sqrt{\frac{37}{127}}$;

$y = 4, -4, 6\sqrt{\frac{37}{127}}, -6\sqrt{\frac{37}{127}}.$

47 [Page 182.]

1. $x = 3, 1, 2 \pm 5\sqrt{-1}$; **2.** $x = \frac{a}{2}$;

$y = 1, 3, 2 \mp 5\sqrt{-1}.$ $y = \frac{b}{2}.$

3. $x = 4, -2, \pm\sqrt{-15} + 1$; **4.** $x = 4, 1, \frac{1}{2}(5 \pm \sqrt{-159})$;

$y = 2, -4, \pm\sqrt{-15} - 1.$ $y = 1, 4, \frac{1}{2}(5 \mp \sqrt{-159}).$

5. $x = 11, \frac{1}{2}\left\{1 \pm \sqrt{-211}\right\}$; **6.** $x = 4, -2, \pm\sqrt{-11} + 1$;

$y = 2, -4, \pm\sqrt{-11} - 1.$

$y = 3, \frac{1}{2}\left\{-15 \pm \sqrt{-211}\right\}.$

7. $x = 2, 1, \frac{1}{2}\left\{3 \pm \sqrt{-19}\right\}.$

$y = 1, 2, \frac{1}{2}\left\{3 \mp \sqrt{-19}\right\}.$

8. $x = \frac{a}{2}\left\{1 \pm \sqrt{3}\right\}, \frac{a}{2}\left\{1 \pm \frac{1}{\sqrt{3}}\right\};$

$y = \frac{a}{2}\left\{1 \mp \sqrt{3}\right\}, \frac{a}{2}\left\{1 \mp \frac{1}{\sqrt{3}}\right\}.$

9. $x = a, b;$ **10.** $x = 3, 2, \frac{1}{2}\left(5 \pm \sqrt{-\frac{103}{3}}\right);$

$y = b, a.$ $y = 2, 3, \frac{1}{2}\left(5 \mp \sqrt{-\frac{103}{3}}\right).$

48 [Pages 185, 186.]

1. $x = 2, -2;$ $y = 4, -4;$ $z = 6, -6.$

2. $x = 3, y = 4, z = 1.$

3. $$\left.\begin{aligned} x &= 0, \pm a(b^2 - c^2) \\ y &= 0, \pm b(c^2 - a^2) \\ z &= 0, \pm c(a^2 - b^2) \end{aligned}\right\}.$$

4. $x = 5, y = -1, z = 7.$ **5.** $x = \pm 3, y = \pm 1, z = \pm 2.$

6. $x = \pm 3, y = \pm 5, z = \pm 4.$

7. $x = 3, \frac{3}{4}, \frac{9}{10}, -\frac{9}{4};$ $y = 4, -1, -\frac{12}{15}, -3;$ $z = 1, -\frac{1}{4}, \frac{8}{10}, \frac{3}{4}.$

8. $$\left.\begin{aligned} x &= \frac{a(b^2 - c^2)}{2(ab + bc + ca)}.P \\ y &= \frac{b(c^2 - a^2)}{2(ab + bc + ca)}.P \\ z &= \frac{c(a^2 - b^2)}{2(ab + bc + ca)}.P \end{aligned}\right\} \text{ where } P = \frac{a}{b - c} + \frac{b}{c - a} + \frac{c}{a - b}.$$

49 [Pages 201—205.]

1. $x = 0, 2, \pm \sqrt{2};$ $y = 0, 2, 2 \mp \sqrt{2}.$

2. $x = 4, 2;$ $y = 2, 4.$

3. $x = 4, 2, -1 \pm \sqrt{\frac{11}{3}};$ $y = 2, 4, -1 \mp \sqrt{\frac{11}{3}}.$

4. $x = 5, 13;$ $y = 4, 12.$

5. $x = 4, 9;$ $y = 9, 4.$

6. $x = 3, -1\frac{26}{27};$ $y = -4, 8\frac{11}{27}.$

7. $x = 7, -1, \pm \sqrt{-3} + 3;$ $y = 1, -7, \pm \sqrt{-3} - 3.$

8. $x = 3, 2, 5, 1.$ $y = 2, 3, 1, 5;$

9. $x = 5,\ 18\frac{1}{8}$; $y = 4,\ 17\frac{7}{8}$.

10. $x = 7,\ 2$; $y = 2,\ 7$.

11. $x = 2,\ -\frac{551}{82}$; $y = 3,\ \frac{1247}{82}$.

12. $x = 9,\ \frac{13}{3}$; $y = 7,\ 0$.

13. $x = 5$; $y = \pm 4$.

14. $x = 6,\ 2,\ 4,\ 3$; $y = 1,\ 3,\ \frac{3}{2},\ 2$.

15. $x = 0,\ 1,\ \frac{15}{22}$; $y = 0,\ 2,\ \frac{9}{22}$.

16. $x = 4,\ -2,\ \ 1 \pm \sqrt{\frac{3}{11}}$; $y = 2,\ -4,\ -1 \pm \sqrt{\frac{3}{11}}$.

17. $x = 3,\ 1$; $y = 1,\ 3$.

18. $x = \frac{1}{2}\left\{a \pm \sqrt{a^2 - \frac{4a}{b}}\right\}$;

$y = \frac{1}{2}\left\{b \pm \sqrt{b^2 - \frac{4b}{a}}\right\}$.

19. $x = 3,\ 3,\ -\frac{1}{3},\ -\frac{1}{3}$; $y = 3,\ -\frac{1}{3},\ 3,\ -\frac{1}{3}$.

20. $x = 10 \pm 4\sqrt{6},\ 10 \mp \frac{5}{2}\sqrt{15}$; $y = 10 \mp 4\sqrt{6},\ 10 \pm \frac{5}{2}\sqrt{15}$.

21. $x = 1,\ 2,\ \frac{1}{6}(-11 \pm \sqrt{209})$; $y = 2,\ 1,\ \frac{1}{6}(-11 \mp \sqrt{209})$.

22. $x = \pm\frac{1}{3}\sqrt{3(2b-a)},\ \pm a\sqrt{-1}$; $y = \pm\frac{1}{3}\sqrt{3(2a-b)},\ \mp b\sqrt{-1}$.

23. $x = 6,\ 6,\ -6,\ -6$; $y = 3,\ -3,\ 3,\ -3$.

24. $x = \pm 3$; $y = \pm 2$.

25. $x = y = \sqrt[3]{m+n}$.

26. $x = \frac{1 \pm \sqrt{2}}{4}$; $y = \frac{1}{4}$.

27. $x = \frac{ac}{a+b}$; $y = \frac{bc}{a+b}$.

28. $x = a,\ b$; $y = b,\ a$.

29. $x = 5,\ -1$; $y = 1,\ -5$; $z = 2$.

30. $x = 8,\ -8$; $y = 5,\ -5$; $z = 3,\ -3$.

31. $\left.\begin{matrix} x = 1 \\ y = 3,\ -1 \\ z = -1,\ 3 \end{matrix}\right\}$ $\left.\begin{matrix} x = -1 \\ y = 3,\ 1 \\ z = 1,\ 3 \end{matrix}\right\}$ $\left.\begin{matrix} x = 3 \\ y = 1,\ -1 \\ z = -1,\ 1 \end{matrix}\right\}$

32. $$x = 4,\ \frac{355}{42};\quad y = 5,\ \frac{190}{21};\quad z = 6,\ -\frac{5}{2}.$$

33. $$x = 4,\ \frac{60}{7};\quad y = 6,\ \frac{66}{7};\quad z = 2,\ -6.$$

34. $$x = \frac{12}{7};\quad y = \frac{12}{5};\quad z = -12.$$

35. $$x = 9,\ 1;\quad y = 3;\quad z = 1,\ 9.$$

36. $$x = 3;\quad y = 7;\quad z = 11;\quad u = 20.$$

37. $$x = \pm 3;\quad y = \pm 5;\quad z = \pm 7.$$

38. $$\left.\begin{array}{l} x = 6 \\ y = 5,\ 2 \\ z = 2,\ 5 \end{array}\right\}\quad \left.\begin{array}{l} x = 7 \\ y = 3 \pm \sqrt{-1} \\ z = 3 \mp \sqrt{-1} \end{array}\right\}$$

39. $$x = 4;\quad y = 6,\ 3;\quad z = 3,\ 6.$$

40. $$x = 2,\ -2;\quad y = 0,\ \ 0;\quad z = -1,\ 1.$$

41. $$x = \pm 2;\quad y = \pm 3;\quad z = \pm 4.$$

42. $$x = \pm\frac{b+c-a}{2\sqrt{a^2+b^2+c^2}},\quad y = \pm\frac{c+a-b}{2\sqrt{a^2+b^2+c^2}};$$
$$z = \pm\frac{a+b-c}{2\sqrt{a^2+b^2+c^2}}.$$

43. $$\left.\begin{array}{l} x = 1,\ -1,\ \pm\dfrac{7}{\sqrt{3}}; \\ y = 3,\ -3,\ \mp\dfrac{2}{\sqrt{3}}; \\ z = 5,\ -5,\ \mp\dfrac{11}{\sqrt{3}}. \end{array}\right\}$$

44. $$\left.\begin{array}{l} x = 3,\ 3 \\ y = 4,\ 5 \\ z = 5,\ 4 \end{array}\right\}\left.\begin{array}{l} x = 4,\ 4 \\ y = 5,\ 3 \\ z = 3,\ 5 \end{array}\right\}\left.\begin{array}{l} x = 5,\ 5 \\ y = 3,\ 4 \\ z = 4,\ 3 \end{array}\right\}$$

45. $$\left.\begin{array}{l} x = \pm 1 \\ y = \pm 2 \end{array}\right\}\left.\begin{array}{l} x = \pm 2 \\ y = \pm 1 \end{array}\right\}.$$

50 [Pages 213—216.]

1. 16; £5. **2.** 18. **3.** 3 inches. **4.** *A*'s capital = £5, *B*'s capital = £120. **5.** 5 miles per hour. **6.** 12, 5. **7.** 5, 3. **8.** *A*, 120; *B*, 80. **9.** 7, 2. **10.** Rs. 90. **11.** Small wheel

4 feet; large wheel 13 feet. **12.** 4 pence. **13.** 56. **14.** 20, 30 miles an hour. **15.** £60, or £40. **16.** 12, 16, 18. **17.** 26 and 38 feet. **18.** 25, 13, 6. **19.** 40 and 45 miles an hour. **20.** 256 square yards. **21.** 14, 10, 2. **22.** 6400. **23.** *A*, 10 miles an hour; *B*, 12 miles an hour. **24.** $\frac{(a-b)^4}{a^5}$. **25.** The sides were 30 and 19, and the height 4 yards. **26.** 100 shares at £15 each. **27.** 15, 12, 10, 7. **29.** 625. **30.** 324 square feet.

51 [Page 217.]

1. i **2.** -1. **3.** i. **4.** 1. **5.** -1. **6.** $-i$. **7.** i. **8.** $-i$. **9.** $-i$. **10.** -1.

52 [Pages 221—223.]

1. $99+23\sqrt{-3}$. **2.** $-6\sqrt{15}$. **3.** $63+11\sqrt{21}$.

4. $\frac{17}{13}+\frac{6}{13}i$. **5.** $\frac{ac+bd}{c^2+d^2}+\frac{bc-ad}{c^2+d^2}i$. **6.** x^2-x+1.

8. $2+\sqrt{-3}$. **9.** $5-3\sqrt{-2}$. **10.** $2+\sqrt{-5}$.

11. $-1+2i$. **12.** $1+i$. **13.** 100. **14.** 6.

15. $\pm\frac{1}{\sqrt{2}}\left(\sqrt{x^2+x+1}+i\sqrt{x^2-x+1}\right)$.

16. $-1, \frac{1\pm\sqrt{-3}}{2}$. **19.** -27.

53 [Page 224.]

1. 16, 40, $2n-6$. **2.** 29th, 46th, $(3n-10)$th. **3.** 6. **4.** 98.

54 [Pages 225, 226.]

1. 325. **2.** 900. **3.** $52\frac{1}{2}$. **4.** 0. **5.** 25452.

6. $\frac{1}{2}(n-1)$. **7.** $\frac{n}{a+b}\left\{na-\frac{n+1}{2}b\right\}$.

55 [Page 228.]

1. 3. **2.** 9. **3.** 7. **4.** 13 or 7. **5.** Last term 3, or -1; number of terms 10 or 12.

6. 18 or 19. **7.** n^2. **8.** $1, \frac{5}{4}, \frac{3}{2}, \frac{7}{4}, 2$, &c.; 1470.

9. 1, 3, 5, 7, &c.; n^2. **10.** 2. **11.** 4 or 10.

56 [Page 230.]

1. $6\frac{1}{2}$; 8; m; a^2+x^2. **2.** $9\frac{1}{3}, 10\frac{2}{3}$; $\frac{2}{3}, 7\frac{1}{3}$. **3.** 207, 297, 387. **4.** $-2, -6, -10, -14$. **5.** $1, -1\frac{1}{2}$, &c., -39. **6.** 14.

57 [Page 234.]

1. $\frac{n}{2}(6n^2+3n-1)$. **2.** $\frac{n(n+1)(n+2)(3n+5)}{12}$.

3. $\frac{n}{3}(4n^2+6n-1)$. **4.** $n^2(2n^2-1)$.

5. $\frac{n(n+1)(n+2)}{6}$.

58 [Pages 238, 239.]

1. $\frac{ma-nb}{a-b}(2n+1)$. **2.** 9, 13, 17, 21, 25.

3. 13; 6. **4.** 70. **5.** $\frac{n(n+1)(n+2)}{6}$.

6. $\frac{n}{6}(2n^2+3n+7)$. **7.** $\frac{n}{6}(2n^2+9n+1)$.

8. $\frac{n}{3(2n+3)}$. **9.** 8, 12, 16, 20.

10. 3, 5, 7. **11.** 1, 3, 5, 7.

12. 3, 5, 7, 9, 11, 13.

14. $\frac{a}{p}(q-r)+\frac{b}{q}(r-p)+\frac{c}{r}(p-q)=0$.

15. $\frac{P(r-n)+Q(m-r)}{m-n}$. **16.** 16.

19. $\frac{n(n+1)(n+2)}{6}$. **20.** $\frac{1}{3}(n-1).n.(2n-1)$ yards.

59 [Page 240.]

1. 8748. 2. $\frac{4}{9}$. 3. 65536. 4. -243.

5. $\frac{8}{27}$; $\pm\frac{2^{n-3}}{3^{n-3}}$, + or − according as n is even or odd.

6. $-\frac{448}{243}$. 7. $\frac{1}{25}$, $\frac{1}{125}$.

60 [Page 242.]

1. 265720. 2. $60\frac{20}{27}$. 3. -682.

4. $\frac{181}{576}$. 5. $\frac{2}{3}(1-2^{2r})$.

6. $\frac{1}{14}\cdot\frac{5^n \mp 2^n}{5^{n-2}}$, − or + according as n is even or odd.

61 [Page 244.]

1. 1. 2. $\frac{2}{3}$. 3. $\frac{31}{36}$. 4. $\frac{3}{8}$. 5. $10\frac{1}{3}$.

6. $\frac{13}{24}$. 7. $\frac{33}{48}$. 8. $\frac{3\sqrt{3}}{2}$. 9. $\frac{1}{2}(4+3\sqrt{2})$.

10. $\frac{1}{11}$.

62 [Page 246.]

1. 6, 12. 2. $\frac{3}{2}$, 1, $\frac{2}{3}$. 3. -1, $\frac{3}{2}$, $-\frac{9}{4}$,

4. $\frac{16}{3}$, 8, 12, 18, 27.

63 [Pages 249--251.]

1. $\frac{1}{36}$; $1\frac{8}{55}$; $\frac{358}{1665}$; $\frac{1}{7}$. 2. $\frac{1+x}{(1-x)^2}$.

3. $\frac{2x}{(1-2x)^2}$. 4. $\frac{(1+6x).3x}{(1-3x)^2}$.

5. $\frac{a(1-a^n)}{(1-a)^2}-\frac{na^{n+1}}{1-a}$. 6. $\frac{1-x}{(1+x)^2}$.

7. 1. 8. $4-\frac{n+2}{2^{n-1}}$.

9. $2^{n-1}.(2n-1)$; $2^n(2n-3)+3$.

10. $\frac{5^{n+1}-5-4n}{16.5^{n-1}}$. 11. $\frac{40}{81}.(10^n-1)-\frac{4n}{9}$.

12. $\frac{n}{9} - \frac{1}{9^2}\left(1 - \frac{1}{10^n}\right)$. 13. $2^{n+1} - 2 - n$.

14. $2.(2^n - 1 - 4n)$. 15. $\frac{1}{3}(4^n - 1 + 15n)$.

18. 2, 5, 8. 19. 4, 8, 16.

20. $\frac{4}{5}$, 4, 20. 21. $a^{q-r}.b^{r-p}.c^{p-q} = 1$.

22. $\left(\frac{P^{n-q}}{Q^{n-r}}\right)^{\frac{1}{r-q}}$. 24. $n\,2^{n+2} - 2^{n+1} + 2$.

30. $\frac{1}{(1-r)(1-ar)}$.

64 [Pages 253, 254.]

1. $5\frac{1}{3}$. 2. $\frac{4}{11}, \frac{2}{7}$. 3. $1, \frac{1}{7}, \frac{2}{5}, \frac{1}{13}$. 4. (i) $\frac{5}{3}$: (ii) $\frac{3}{2}$

5. $\frac{1}{2}$. 6. $(3, \frac{3}{2}, 1)$; $(\frac{3}{7}, \frac{3}{8}, \frac{1}{3})$ 7. $\frac{3}{2}, 1, \frac{3}{4}$.

8. $\frac{ab}{b + (n-1)(a-b)}$.

9. $\frac{(n+1)xy}{ny+x}, \frac{(n+1)xy}{(n-1)y+2x}, \frac{(n+1)xy}{(n-2)y+3x}$, &c., $\frac{(n+1)xy}{y+nx}$.

10. $\frac{60}{16-n}$.

65 [Pages 257, 258.]

5. 8, 72. 6. $\frac{mn}{m+n}$

66 [Pages 262, 263.]

2. 42. 3. 30. 5. 120. 7. 992.

67 [Pages 267, 268.]

1. 120 ; 24 ; 6. 2. 151200 ; 19958400. 3. 6. 4. 4. 5. 8.
6. 210. 7. 16. 8. 125. 9. 240. 10. 20160 ; 40320.
11. 11880. 12. $n(n-1)(n-2)\ldots(n-m+1)$. 13. 40319.
14. 80640 15. 4320. 16. 864. 17. 1440. 18. 576.
19. 399168000. 20. 332640. 21. 34560. 22. 40320.

68 [Pages 275—278.]

1. 19600 ; 17550 ; 341149446. **2.** 300 ; 351. **3.** 20 ; 8.
4. 190 ; 171. **5.** 120 ; 35. **6.** $\frac{n(n-3)}{2}$.
7. 210 ; 84. **8.** 1330. **9.** 59850.
10. 816000. **11.** 30030 ; 20020. **12.** 792 ; 5940.
13. 344. **15.** 40320. **16.** 282240.
17. 300. **18.** 8436. **19.** 144000.
20. 25920. **21.** 210 **22.** 57.
23. $\frac{1}{6}\left\{n(n-1)(n-2)-m(m-1)(m-2)\right\}$. **24.** 1680
25. 27720.

69 [Pages 280, 281.]

1 4, or 5 ; 21, or 35. **2.** 10 ; 92378. **3.** 70 ; 35.
4. $\frac{11\times 13}{4025}$. **5.** 5511.

70 [Pages 287, 288.]

1. 7560 ; 60. **2.** 138600 **3.** 60 ; 36.
4. 4540535999 ; 605404800 ; 9979200 **5.** 59875200.
6. 4989599. **7.** 2519. **8.** 21599.
9. 1960. **10.** 1680. **11.** 50. **12.** 8400.
13. 27720 ; 280. **14.** 7, or 8.

71 [Pages 289, 290.]

1. 343. **2.** 243. **3.** 16384. **4.** 6144.
5. 12902400. **6.** 729. **7.** m^n.

72 [Page 292.]

1. 56. **2.** 28. **3.** 351. **4.** 84.

73 [Pages 296, 297.]

1. 42. **2.** 113. **3.** 2190. **4.** 6570. **5.** 160.
6. 315. **7.** 464730. **8.** 22100. **9.** 52. **10.** 159.

74 [Pages 306–310.]

1. 720 ; 120. **2.** 658409472000. **3.** 181440.
4. 60. **5.** 13305600. **6.** 72576000.
7. 256. **8.** 160. **9.** 4309200. **10.** 3963960.
11. 10648. **12.** 498015. **13.** 127.
14. 8085. **15.** n^m. **16.** 1023. **22.** 6666600.
24. 76076.
25. $\frac{\underline{|25}}{(\underline{|4})^4(\underline{|3})^3}$. **30.** 254. **31.** 190. **33.** 576.
34. $24 \times \frac{\underline{|36}}{(\underline{|9})^4}$; $576 \times \frac{\underline{|36}}{(\underline{|9})^4}$.

76 [Page 316.]

1. $a^6 - 18a^5x + 135a^4x^2 - 540a^3x^3 + 1215a^2x^4 - 1458ax^5 + 729x^6$.

2. $15625 - 3125x + \frac{3125}{12}x^2 - \frac{625}{54}x^3 + \frac{125}{432}x^4 - \frac{5}{1296}x^5 + \frac{1}{46656}x^6$.

3. $x^{20} - 5x^{18}y^2 + 10x^{16}y^4 - 10x^{14}y^6 + 5x^{12}y^8 - x^{10}y^{10}$.

4. $x^2 - x^{\frac{5}{3}}y^{\frac{1}{3}} + 15x^{\frac{4}{3}}y^{\frac{2}{3}} - 20xy + 15x^{\frac{2}{3}}y^{\frac{4}{3}} - 6x^{\frac{1}{3}}y^{\frac{5}{3}} + y^2$.

5. $32x^5 - 240x^4y + 720x^3y^2 - 1080x^2y^3 + 810xy^4 - 243y^5$.

6. $\frac{64x^6}{729} - \frac{32x^4}{27} + \frac{20x^2}{3} - 20 + \frac{135}{4x^2} - \frac{243}{8x^4} + \frac{729}{64x^6}$.

7. $64a^6 - 96a^4 + 36a^2 - 2$. **8.** $2(365 - 363x + 63x^2 - x^3)$.

9. $2x(64x^6 - 112x^4 + 56x^2 - 7)$.

10. $x^3 + 3x^2 - 5 + \frac{3}{x^2} - \frac{1}{x^3}$.

77 [Pages 319, 320.]

1. $495a^{16}b^8$. **2.** $\frac{55}{2304}a^4b^8$. **3.** $-120x^8y^{12}$. **4.** $760xy^6$.

5. $924a^{6m}x^{6m}$. **6.** $1716a^7x^6$; $1716a^6x^7$. **7.** $-\frac{120x^4}{a^4}$.

8. $84a^3b^6$. 9. $-\frac{429}{16}x^{14}$. 10. 0; $210a^{16}b^6$. 11. $312y^2$.

12. $\frac{|\underline{2n+1}}{|\underline{n-r}\,|\underline{n+r+1}}.(-1)^{n-r}$. 13. $\frac{7}{18}$. 15. $\frac{|\underline{2n}}{\left|\underline{\frac{2n}{3}}\right.\left|\underline{\frac{4n}{3}}\right.}$.

78 [Page 324.]

1. 7*th*. 2. 22*nd*. 3. 20*th*. 4. 5*th* and 6*th*.

5. 56229888. 6. $\frac{7}{144}$. 7. $\frac{1771875}{8}$.

79 [Pages 330, 331.]

1. $2^{20}-1$. 2. 2^{25}. 5. $p\left(1+\frac{P'}{p}\right)-1$.

7. $(-1)^{r-1}.\frac{|\underline{2n}}{|\underline{r-1}\,|\underline{2n-r+1}}.x^{2n-2r+2}$; $(-1)^{r-1}.\frac{|\underline{2n}}{|\underline{r-1}\,|\underline{2n-r+1}}$ $\times x^{2r-2n-2}$; $\frac{(-1)^n|\underline{2n}}{|\underline{n}\,|\underline{n}}$. 8. $(-1)^r.\frac{|\underline{2n+1}}{|\underline{r+1}\,|\underline{2n-r}}.x^{2r-2n+1}$.

11. $\frac{|\underline{2n}}{|\underline{n-1}\,|\underline{n+1}}$. 12. $\frac{|\underline{2n}}{|\underline{n-2}\,|\underline{n+2}}$.

13. $2^{n-1}(n+2)$. 14. $\frac{n(n+1)}{2}$.

80 [Pages 336, 337.]

1. $1+x+x^2+x^3+x^4+x^5+x^6+x^7+x^8+x^9+$ &c.
2. $1+2x+3x^2+4x^3+5x^4+6x^5+7x^6+8x^7+$ &c.
3. $1+3x+6x^2+10x^3+15x^4+21x^5+28x^6+$ &c.
4. $1+\frac{1}{2}x+\frac{3}{8}x^2+\frac{5}{16}x^3+\frac{35}{128}x^4+$ &c.
5. $1+6x+24x^2+80x^3+240x^4+$ &c.
6. $1+x+\frac{x^2}{6}-\frac{x^3}{54}+$ &c.
7. $\frac{1}{27}(1+x+\frac{5}{6}x^2+\frac{35}{54}x^3+$ &c.)
8. $1-\frac{7}{3}x^2+\frac{14}{9}x^4-\frac{14}{81}x^6-\frac{7}{243}x^8-$ &c.
9. $1+\frac{2}{3}x+\frac{5}{9}x^2+\frac{40}{81}x^3+$ &c.

10. $1 + x + \frac{5}{8}x^2 + \frac{7}{16}x^3 + \frac{35}{256}x^4 + \&c.$

11. $\frac{a^4}{81} + \frac{8a^5x}{243} + \frac{40a^6x^2}{729} + \frac{160a^7x^3}{2187} + \frac{560a^8x^4}{6561} + \&c.$

12. $\left(\frac{a^2}{x}\right)^{\frac{1}{3}}\left\{3 + \frac{x}{a} + \frac{2}{3}\left(\frac{x}{a}\right)^2 + \frac{14}{27}\left(\frac{x}{a}\right)^3 + \&c.\right\}$

13. $-945a^{-21}b^{5}$; $-30618a^{-20}b^{5}$.

14. $\frac{77}{256}x^{30}$. 15. $-\frac{(p-1)(2p-1)(3p-1).\&c.\ (\overline{r-1}.p-1)}{\underline{|r}}x^r$.

16. $\frac{1.5.9.\ \&c.\ (4r-3)}{4^r\underline{|r}}x^r$. 17. $\frac{7.9.11.\ \&c.\ (2r+5)}{\underline{|r}}x^r$.

18. $(-1)^{r-1}.r.a^{-2(r+1)}.x^{2r-2}$ 19. $-\frac{5}{1024}.a^{-\frac{35}{2}}.b^{18}$.

20. $-1848x^{13}$

21. $3\left\{1 + \frac{1}{3}\cdot\frac{x^2}{a^3} + \frac{2}{9}\cdot\frac{x^4}{a^6} + \frac{14}{81}\cdot\frac{x^6}{a^9} + \frac{35}{243}\cdot\frac{x^8}{a^{12}} + \&c.\right\}$;

$\frac{1.4.7.\ \&c.\ (3r-2)}{3^{r-1}.\underline{|r}}\left(\frac{x^2}{a^3}\right)^r$.

22. $\frac{(n+1)(2n+1)(3n+1)\ \&c.\ (\overline{r-2}.n+1)}{n^{r-1}\underline{|r-1}}.a^{-\frac{n(r-1)+1}{n}}.x^{r-1}$

23. $\frac{3.5.7.\ .\ .\ .\ (r+1)}{\underline{|r}}.x^r$.

81 [Page 340.]

1. 3rd. 2. 13th. 3. 2nd. 4. 4th and 5th. 5. 5th.

82 [Page 344.]

1. 1·99776. 2. 1·0099. 3. 3·002. 4. 1·9520.

5. 3·14138. 6. 5·01329. 7. 9·99332. 8. $1 - \frac{41}{24}x$.

9. $1 - \frac{989}{192}x$. 10. $1 - \frac{5x}{8}$. 11. $\frac{1}{2} + \frac{643}{3024}x$.

83 [Pages 349—352.]

1. $\dfrac{n(n-1)(n-2).\&c.(n-r+2)}{\underline{|r-1}}.(4x)^{n-r+1}(-3y)^{r-1}$;

$-\dfrac{3.4.11.18.25.\&c.(7r-17)}{7^{r-1}.\underline{|r-1}}(3x^2)^{r-1}$; co-efficient of $x^{11} = -1$'

and that of $x^{12}=0$. **2.** 0. **3.** 25. **4.** $\frac{1}{6}(r+1)(r+2)(r+3)$.

5. $\sqrt[3]{4}$. **6.** $\dfrac{\sqrt{3}}{2}$. **9.** $2\sqrt{2}$ **15.** $\frac{1}{2}\Big\{(n+1)(n+2)ac^n$

$+n(n-1)\times bc^{n-2}\Big\}$.

16. $\dfrac{(-1)^n}{2}(9n^2+3n+2)$. **18.** Co-efficient of $x^{2n}=n^2+2n+1$;

that of $x^{2n+1}=(n+1)(n+2)$.

24. Co-efficient of $x^{2r} = \dfrac{1.3.5\ \&c\ (2r-1)}{2^r.a^{2r}.\underline{|r}}$;

co-efficient of $x^{2r+1} = \dfrac{1.3.5\ \&c.(2r-1)}{2^r.a^{2r+1}.\underline{|r}}$.

84 [Page 355.]

1. 6. **2.** 6. **3.** 9. **4.** -4. **5.** $\frac{5}{3}\log m+\frac{3}{2}\log n$.

6. $\frac{1}{4}(17\log a-19\log b)$. **7.** $\dfrac{41\log 3+\log 2}{12}$.

8. $\dfrac{4\log c-3\log b}{\log a+2\log b+\log c}$. **9.** $\dfrac{3\log 3+4\log 5-3\log 2}{\log 2+\log 5}$.

85 [Page 359.]

1. 5 ; 3 ; -6 ; 0 ; -1. **2.** ·7283375 ; 2·7283375 ; 1 7283375 ; 8 7283375. **3.** 15. **4.** $\bar{3}$·8515933. **5.** 2·88869495. **6.** ·005942.

7. 19. **8.** 7043652 ; 5·09691 ; 1·2730013 ; ·2272439.

9. ·44092388. **10.** 1·206. **11.** $\left.\begin{aligned} x &= \cdot 76028, \\ y &= \cdot 02060. \end{aligned}\right\}$

86 [Page 360.]

1. $\bar{1}$·6777943. **2.** 4·226.

87 [Pages 366—368.]

2. $1+\frac{1}{\underline{|3}}+\frac{1}{\underline{|5}}+\frac{1}{\underline{|7}}+\&c.$ **4.** ·61. **6.** $\frac{7}{12}$. **7.** $\frac{e-e^a}{1-a}$.

8. $2\times\left\{1+\frac{(2x)^2}{\underline{|2}}+\frac{(2x)^4}{\underline{|4}}+\&c.+\frac{(2x)^{2r}}{\underline{|r}}+\&c.\right\}$.

9. $\frac{(-1)^r}{\underline{|r}}\left\{1+ar-r(r-1)\right\}$. **10.** $\frac{(-1)^n}{\underline{|n-2}}\left\{\frac{a}{n(n-1)}-\frac{b}{n-1}+c\right\}$.

11. $1-\frac{x^2}{\underline{|2}}+\frac{x^4}{\underline{|4}}-\frac{x^6}{\underline{|6}}+\&c.$ **12.** $x-\frac{x^3}{\underline{|3}}+\frac{x^5}{\underline{|5}}-\frac{x^7}{\underline{|7}}+\&c.$

19. $12e-5$. **20.** $11e$.

88 [Pages 371—373.]

4. ·434. **5.** $2\log_e 2-\log_e 3$. **12.** $2\log_e 2-1$.

13. $\frac{1}{2}+\log_e 3-\log_e 2$. **14.** $2-\log_e 2$. **16.** $\frac{1}{2n}$ if n be odd, and $-\frac{3}{2n}$ if n be even ; $\frac{1}{2n+1}$.

89 [Pages 375, 376.]

1. $x=2, y=3, z=4$. **2.** $x=3, y=2, z=1$. **3.** $x=1$, $y=-2, z=3$. **4.** $x=3, y=\frac{1}{2}, z=\frac{2}{3}$. **5.** $x=-(a+b+c)$, $y=bc+ca+ab$, $z=-abc$. **6.** $x=\frac{a^3}{(a-c)(a-b)}$, $y=\frac{b^3}{(b-a)(b-c)}$, $z=\frac{c^3}{(c-b)(c-a)}$. **7.** $x=2, y=4, z=6$. **8.** $x=2$, $y=-3, z=4$. **9.** $x=1, y=4, z=27$.

90 [Page 377.]

1. $(x-1)(x+2)(x+3)$. **2.** $(x-2)(x+3)(x+4)$.

3. $(x-3)(x+5)(x+7)$. **4.** $(x-1)(x-2)(x+4)(x+5)$.

5. $(x-2)(x^2+2x-14)$. **6.** $(x-3)(x^2+3x-20)$.

7. 16. **8.** 17. **9.** -114. **10.** -336.

91 [Page 381.]

1. $\frac{1}{2}n(6n^2-3n-1)$. **2.** $\frac{1}{3}n(n+1)(n+2)$. **3.** $\frac{1}{3}n(4n^2-1)$. **4.** $\frac{1}{3}n(n+1)(4n-1)$. **5.** $n^2(2n^2-1)$. **8.** $ac^3=db^3$, $c^2=3db$.

92 [Page 383.]

1. $x^3+7x^2+17x+29$; 65. **2.** $2x^3+15x^2+3x+12$; 113. **3.** $5x^4-2x^3+15x^2-4x-5$; 43. **4.** $4x^4+5x^3+6x^2+7x+8$; 35. **5.** $x^4+3x^3-13x+100$; 5. **6.** $7x^5-3x^4+x^3+8x^2+x+10$; -20. **7.** $x^6+3x^5-2x^4-2x^3-5x^2+60x-20$; 329. **8.** $x^8+2x^7+9x^6+18x^5+36x^4+72x^3+100x^2+200x+407$; 825. **9.** $x^6+4x^5+3x^4+12x^3+13x^2+52x+200$; 823. **10.** $x^6-5x^5+31x^4-155x^3+775x^2+3850x+19250$; -95485. **11.** 800. **12.** 742. **13.** -1525. **14.** 75. **15.** 8.

94 [Pages 402—405.]

1. $a^2+b^2=m^2+n^2$. **2.** $a_3(b_1c_2-b_2c_1)+b_3(c_1a_2-c_2a_1)+c_3(a_1b_2-a_2b_1)=0$. **3.** $(b_1c+a_1b)^2+(bc_1+ab_1)^2=(cc_1-aa_1)^2$. **4.** $x^2+y^2+z^2+2xyz=1$. **5.** $ab+bc+ca+2abc=1$. **6.** $(b_1c_2-b_2c_1)^3(a_1b_2-a_2b_1)=(c_1a_2-c_2a_1)^4$. **7.** $\frac{a^2}{p^2}+\frac{b^2}{q^2}=1$. **8.** $a^2-b^2=1$. **9.** $a^4-2a^2b^2-b^4+2c^4=0$. **10.** $c^3-a^3=3b^2c$. **11.** $2a^3-3ab^2-2c^3+2d^3=0$. **12.** $6d^4-a^4+6a^2b^2-3b^4-8ac^3=0$. **15.** $a^2+b^2+c^2-abc=4$. **16.** $a\beta=1+\gamma$. **17.** $abc+2fgh-af^2-bg^2-ch^2=0$. **18.** $a^2+b^2+1=c(c+2ab)$. **19.** $a^3+b^3+c^3+abc=0$.

20. $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\left(\frac{a'^2}{a}+\frac{b'^2}{b}+\frac{c'^2}{c}\right)\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)$.

21. $\frac{1}{a^{\frac{3}{2}}}\left(\frac{1}{b}+\frac{1}{c}\right)+\frac{1}{b^{\frac{3}{2}}}\left(\frac{1}{c}+\frac{1}{a}\right)+\frac{1}{c^{\frac{3}{2}}}\left(\frac{1}{a}+\frac{1}{b}\right)=0$.

22. $(a-b)^4=a+b$. **23.** $a^3+b^3+c^3=d^3$. **24.** $4a^3c^3(a^3-bc)(ab-c^2)=b^4(a^3-c^3)^2$. **26.** $\frac{a}{1+a}+\frac{b}{1+b}+\frac{c}{1+c}+\frac{d}{1+d}=1$. **27.** $(b+c-a)(c+a-b)(a+b-c)=abc$. **28.** $a^3+b^3+c^3+2abc=(b+c)(c+a)(a+b)$.

Miscellaneous Exercises.

[Pages 419–430]

I.

1. $x = 2, \frac{1}{4}, \frac{9 \pm \sqrt{-31}}{8}$. **2** 15, 12, 9. **4** 1 and -4.

5. $(2x + y - 3)(x - 11y + 1)$. **6.** 7. **7.** 2002. **8.** $\frac{1}{2}\sqrt{6}$.

II.

2. $x = 1, y = 4, z = 2$. **5.** 1, 3, 5, 7, 9. **6.** $2n^2(n + 1)^2$.
7. 60. **8.** 3003.

III.

1. $x = \frac{1}{2}(a + b), y = \frac{1}{2}(a - b)$. **2.** $(2x^2 + x + 2)^2 - (\sqrt{5}.x)^2$.
4. 28. **6.** (*i*) 5040 ; (*ii*) 1440. **7.** $3n + 1$.

IV.

6. $\frac{n(n-1)(n-2)}{1.2.3}$. **7.** $a = 3, b = 1$.

V.

1. $1, -1, -1, 2 \pm \sqrt{3}$. **5.** (i) $\frac{n(n+1)(2n+7)}{6}$;

(ii) $\frac{n(n+1)(3n^2+19n+32)}{12}$. **6.** 20. **7.** $1890a^4$.

VI.

1. $x = y = z = a^2 + b^2 + c^2 - ab - bc - ca$.
2. $4(ax + by + cz)$. **3.** $3\sqrt{2}$. **4.** $n^2(2n^2 - 1)$.
8. (i) $4cbd = 8ad^2 + c^3$; (ii) $b^3 + 8a^2d = 4abc$; (iii) $1, \frac{1}{2}, -1$.

VII.

1. (i) $3, \frac{17}{3}$; (ii) $0, \frac{a^2(c-a) + b^2(c-b)}{ac(a-c) + bc(b-c)}$.

3. The time is either $\frac{60}{143} \times 30$ minutes past 6 o'clock, or $\frac{60}{143} \times 74$ minutes 2 o'clock.

6. 43200. **8.** 0.

VIII.

1. $1, -\frac{53+20\sqrt{6}}{19+8\sqrt{6}}$. 2 It cannot lie between 1 and 3, but can have any other value. 4. $1 \pm 4\sqrt{-5}$. 6. -20; 6; 15. 7. 4·9984. 8. 0.

IX.

1. $x = 3, y = 5, z = 7$. 3. $\frac{1}{12}n(n+1)^2.(n+2)$. 6. 4320. 8. $\sqrt{e}-1$.

X.

2. $-\frac{5}{8}$. 3. $\frac{n(n+1)(n+2)(3n+1)}{12}$. 6. 1·9955. 7. 56 ; 8.

XI.

1. $x = \frac{1}{2}(m+n), y = \frac{1}{2}(n+l), z = \frac{1}{2}(l+m)$. 2. 88 yards. 5. $x = 1, y = 3$. 6. $\frac{2n-p-q}{\overline{n-p}\;\overline{n-q}}$

XII.

5. $(-1)^{n-1}(4n^3-3n)$.

XIII.

3. $x = 6, y = 5, z = 3$; or, $x = \frac{10}{3}, y = \frac{10}{3}, z = \frac{13}{3}$.

XIV.

2. $x = 3, y = 7, z = 4$. 7. 448, 112. 8. 243.

XV.

4. $x = 0, y = 0, z = 0$; or, $x = \frac{1}{2}, y = \frac{1}{6}, z = -\frac{1}{6}$.

6. $\frac{m(m-1)}{2}(n+p)+\frac{n(n-1)}{2}(m+p)+\frac{p(p-1)}{2}(m+n)+mnp$.

XVI.

1. $x = \frac{a^3+b^3}{ab}+4(a+b)$, or $\frac{a^3+b^3}{ab}$.

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1888.

1. Solve the equations :—

(i) $x^2 - 48x + 527 = 0$;

(ii) $\left.\begin{aligned} 2x^2 + 3xy + y^2 &= 15 \\ 5x + 2y &= 12 \end{aligned}\right\}$;

(iii) $x^3 + 4x^2 + 12x - 48 = 0$.

2. Eliminate m and m' between the equations :—

$$\left.\begin{aligned} y = mx + \frac{a}{m} &= m'x + \frac{a}{m'}, \\ \text{and } mm' &= -1 \end{aligned}\right\}$$

(See page 396).

nd prove that $x + a = 0$.

3. Sum the series :—

(α) $5^2 + 7^2 + 9^2 + \&c. + 25^2$;

(β) $\frac{1}{2} + \frac{1}{2} + \frac{3}{8} + \frac{4}{16} + \&c.$ to infinity. **(See page 244.)**

4. Write down the 4th term of $\left(\frac{x}{a} - \frac{a}{x}\right)^{10}$. **(See page 320.)**

If c be a quantity so small that c^3 may be neglected in comparison with l^3, shew that

$\sqrt{\frac{l}{l+c}} + \sqrt{\frac{l}{l-c}}$ is *very nearly* equal to $2 + \frac{3c^2}{4l^2}$ **(See page 342.)**

5. Find the value of $\frac{1}{\sqrt[5]{e}}$ to four places of decimals. **(See page 363.)**

Shew that $\frac{1}{\lfloor 2} + \frac{1+2}{\lfloor 3} + \frac{1+2+3}{\lfloor 4} + \frac{1+2+3+4}{\lfloor 5} + \&c. = \frac{e}{2}$.

(See page 363.)

1889.

1. Solve the following equations—

(i) $\frac{(x-2)^2}{(x^2-4x)} + \frac{2}{(x-2)^2} = 4$; **(See page 143.)**

(ii) $\left.\begin{aligned} x^4 + 2x^3y + x^2y^2 + 2xy^3 + y^4 &= 41 \\ \frac{x}{y} + \frac{y}{x} &= \frac{5}{2} \end{aligned}\right\}$ **(See page 205.)**

2. (a) If α and β be the roots of the equation $ax^2 + bx + c = 0$, prove that $\alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$.

(*b*) If the equations $ax^3+3bx^2+3cx+d=0$ and $ax^2+2bx+c=0$ have a common root, prove that

$$(bc-ad)^2 = 4(ac-b^2)(bd-c^2).$$ **(See page 166.)**

3. Sum the series :—

$n.1+(n-1).2+(n-2).3+(n-3).4 \ldots\ldots +1.n$ **(See page 239.)**

and $1+\frac{2}{5}+\frac{3}{5^2}+\frac{4}{5^3}+$ &c. to n terms. **(See page 249.)**

4. (*a*) Assuming the number of permutations of n things taken r at a time find the number of combinations of n things taken r at a time.

(*b*) Find the number of combinations in the letters of the word *alliteration* taken four at a time. **(See page 296.)**

5. Find the $(r+1)$th term of $(1-2x)^{-\frac{5}{2}}$. **(See page 337.)**

6. (*a*) Expand a^x in a series of ascending powers of x.

(*b*) Shew that $\frac{e-1}{e+1} = \dfrac{\frac{1}{|\underline{2}}+\frac{1}{|\underline{4}}+\frac{1}{|\underline{6}}+\ldots\ldots\ldots}{\frac{1}{|\underline{1}}+\frac{1}{|\underline{3}}+\frac{1}{|\underline{5}}+\ldots\ldots\ldots}$. **(See page 367).**

1890.

1. Solve the equations :—

(i) $x^2-13x-888=0$;

(ii) $\left.\begin{aligned} 4xy-x^2 &= 15 \\ 39y^2-xy &= 150 \end{aligned}\right\}$

Eliminate x between the equations :—

$$\left.\begin{aligned} Ax^2+Bx+C &= 0 \\ A_1x^2+B_1x+C_1 &= 0 \end{aligned}\right\}$$

2. Find the condition that the roots of the equation $Ax^2+Bx+C=0$, should be (i) equal, (ii) real and unequal : a, b, c, m are real quantities and not zero : given that the quadratic in x, *viz.*,

$$x^2(m^2a^2+b^2)+2mca^2x+a^2(c^2-b^2) = 0$$

has equal roots ; show that $c=\sqrt{m^2a^2+b^2}$.

3. Insert four arithmetic and three geometric means between 4 and 324. Sum the series (*a*) $1^3+2^3+3^3+$ &c. to n terms.

(*b*) $1.2.3+2.3.4+3.4.5+$ &c. n terms.

4. Shew that every term in the expansion of $(1-x)^{-\frac{p}{q}}$ is positive.

Sum the infinite series :—

$$1+2.\frac{1}{3^2}+\frac{2.5}{1.2}.\frac{1}{3^4}+\frac{2.5.8}{1.2.3}.\frac{1}{3^6}+\&c.$$

5. (*a*) Show that—

$$\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + (-1)^{n-1}\cdot\frac{x^n}{n} + \dots\dots$$

(*b*) Prove that—

$$\begin{aligned}\log_{10}(x+5) &= \log_{10}(x+3) + \log_{10}(x-3) \\ &+ \log_{10}(x+4) + \log_{10}(x-4) \\ &- \log_{10}(x-5) - 2\log_{10}x \\ &= \cdot86859\left\{\frac{72}{x^4-25x^2+72}\right. \\ &\left.+\frac{1}{3}\left(\frac{72}{x^4-25x^2+72}\right)^3 + \dots\right\}\end{aligned}$$

(See page 373.)

1891.

1. Solve the following equations :—

(*α*) $12x^2 - 61x + 91 = 0$;

(*β*) $$\left.\begin{aligned}\frac{1}{x^3} - \frac{1}{y^3} &= \frac{1}{b}\left(\frac{3}{a^2} + \frac{1}{b^2}\right) \\ \frac{1}{x} - \frac{1}{y} &= \frac{1}{b}.\end{aligned}\right\}$$

2. If α, β are the roots of the equation $ax^2 + 2bx + c = 0$ find the value of $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$.

Eliminate l, m, a', b' from the equations :—

$$\left.\begin{aligned}lx + my &= \sqrt{a'^2l^2 + b'^2m^2} \\ mx - ly &= \sqrt{a'^2m^2 + b'^2l^2}\end{aligned}\right\}$$

and $a^2 - a'^2 = b^2 - b'^2 = k^2$.

(See page 396.)

3. If the first term of a Geometric Progression be a, and the common ratio r, find the sum of the first n terms.

Sum the following series :

(i) $(2+\sqrt{3}) + 1 + (2-\sqrt{3}) + \dots \dots$ to ∞.

(ii) $a + b + (a^2 + 2ab) + (a^3 + 3a^2b) + \dots \dots \dots$ to n terms.

4. If $f(m)$ denote the series

$1 + mx + \frac{m(m-1)}{1.2}x^2 + \frac{m(m-1)(m-2)}{1.2.3}x^3 + \dots\dots\dots$for all values of m, prove that $f(m) \times f(n) = f(m+n)$.

Hence, deduce the Binomial Theorem for any index.

5. Show that $e^x = 1 + x + \frac{x^2}{1.2} + \dots + \frac{x^n}{\lfloor n} + \dots$

Expand $\log(1 + x + x^2 + x^3)$ in powers of x, and find the co-efficients of x^{2n} and x^{2n+1}.

(See page 373.)

1892.

1. Solve the equations :—

(i) $35x^2 + x = 204$;

(ii) $\left.\begin{array}{l} 2x^2 + 3xy + y^2 = 20 \\ 5x^2 + 4y^2 = 41 \end{array}\right\}$

2. Under what circumstances is the expression $ax^2 + 2bx + c$ essentially positive whatever real value may be attributed to x? **(See page 163.)**

Prove that if the equations $x^2 + bx + ca = 0$, $x^2 + cx + ab = 0$ have a common root, their other roots will satisfy the equation $x^2 + ax + bc = 0$.

(See page 166.).

3. What is an imaginary expression?

Show that $\left(\frac{-1+\sqrt{-3}}{2}\right)^n + \left(\frac{-1-\sqrt{-3}}{2}\right)^n$ is equal to 2, if n be a multiple of 3, and equal to -1, if n be any other integer.

4. Find the sum of n terms of an arithmetical progression of which the first term is a, and common difference b.

Sum the series $1^2 + 2^2 + 3^2 + \ldots$ to n terms.

The first term of a geometrical progression exceeds the second term by 2 and the sum to infinity is 50; find the series.

5. Write down the middle term of the expansion of $(1+x)^{2n}$.

Prove that the co-efficients of the $(r+1)^{th}$ term of $(1+x)^{n+1}$ is equal to the sum of the co-efficients of the r^{th} and $(r+1)^{th}$ terms of $(1+x)^n$.

6. Expand $\log(1+x)$ in a series of ascending powers of x.

1893.

1. Solve the equation $ax^2 + 2bx + c = 0$.

If α and β be its roots, determine $\alpha^2 + \beta^2$ and $\alpha^3 + \beta^3$ in terms of a, b and c.

2. If $(a+b+c+d)(a-b+d-c) = (a-b+c-d)(a+b-c-d)$, prove that a, b, c, d are proportionals.

3. Find the middle term in the expansion of $(1+x)^{2n}$.

4. Find the sum of n terms of the series

$$a + (a+b)r + (a+2b)r^2 + (a+3b)r^3 + \&c.$$

5. Expand e^x in a series of ascending powers of x.

Hence calculate the square root of e to three places of decimals correctly.

1894.

1. Solve the equations :—

(i) $10x^2 - 21x = 187$;

(ii) $5\sqrt{\frac{3}{x}} + 7\sqrt{\frac{x}{3}} = 22\frac{2}{3}$.

2. α, β are the roots of the equation $ax^2 + bx + c = 0$. Find the limits of y that the expression $ay^2 + by + c$ may have the opposite sign from a.

Show that the greatest and least values of $\frac{6x^2 - 22x + 21}{5x^2 - 18x + 17}$ for real values of x are $\frac{5}{4}$ and 1, corresponding to the values 1 and 2 respectively of x.

3. Find the sum of the squares of the first n natural numbers.

Show how the value of a recurring decimal can be found by Geometrical Progression, and deduce the rule for expressing a recurring decimal as a proper fraction.

Sum to infinity $432 + 324 + 243 + \&c.$

How many successive odd numbers beginning with 3 must be taken, in order that their sum may amount to 675 ?

4. Write down the $(r+1)^{th}$ term of $(1+x)^n$, where n is a positive integer.

Within what limits will the series be true when n is negative or fractional ?

Show that

$$1 + \frac{3}{4} + \frac{3.5}{4.8} + \frac{3.5.7}{4.8.12} + \&c. \text{ to infinity } = \sqrt{8}.$$

5. Write down the series for the expansion of $\log_e (1+x)$, and deduce a series for $\log_e \left(\frac{1+y}{y}\right)$.

Show that $\log_e 3 = 1 + \frac{1}{3.2^2} + \frac{1}{5.2^4} + \frac{1}{7.2^6} + \&c.$

1895.

1. If α, β be the roots of the quadratic equation $ax^2 + bx + c = 0$, find the quadratic equation whose roots are α^2, β^2.

2. Solve :— (i) $(x+a)(x+b)(x+c) = abc$;

(ii) $\left.\begin{aligned} 3x + 2y &= 2xy \\ 9x + 4y &= 5xy \end{aligned}\right\}$

3. Prove that

(i) $(a+b+c)(a^2+b^2+c^2-bc-ca-ab) = a^3+b^3+c^3-3abc$;

(ii) $(a+b+c+d+e+f)^2$

$$= a^2+2a(b+c+d+e+f)+b^2+2b(c+d+e+f)$$
$$+c^2+2c(d+e+f)+d^2+2d(e+f)+e^2+2ef+f^2.$$

4. Prove that $n^2+2n\{1+2+3+\ldots(n-1)\} = n^2$.

Hence prove that

$$1^3+2^3+3^3+\ldots+n^3 = (1+2+3+\ldots+n)^2.$$

5. Expand by the Binomial Theorem $(1-x)^{\frac{1}{2}}$.

Hence find the sum of the series

$\frac{1}{3}+\frac{1.3}{3.6}+\frac{1.3.5}{3.6.9}+\frac{1.3.5.7}{3.6.9.12}+\ldots$ to infinity. **(See page 350, Ex. 7).**

6. Find the limit of $\left(1+\frac{1}{n}\right)^n$ when n becomes infinite.

Calculate the result to *six* places of decimals.

1896.

1. Solve :—

(i) $(x-2)(x+3)(x+6)(x+1)+56 = 0$;

(ii) $(x+1)^4+(x-3)^4 = 256$.

2. Prove that a quadratic equation cannot have more than two roots.

Find the relation between the co-efficients and the roots.

Form the equation whose roots are 5 and -6.

3. Explain the meaning of Arithmetical Mean, Geometrical Mean, and Harmonical Mean.

If a, b, c be in G. P. and if p be the A. M. between a and b, and q be the A. M. between b and c, then b will be the H. M. between p and q.

4. Write down the expansion of $(1+x)^n$.

Expand $(1+\sqrt{x})^4+(1-\sqrt{x})^4$.

5. Show that the sum of the cubes of the first n natural numbers is equal to the square of their sum.

6. Find the number of permutations of n things r at a time.

1897.

1. If α and β be the roots of the equation $ax^2+2bx+c = 0$, prove that $\alpha+\beta = -\frac{2b}{a}$, and $\alpha\beta = \frac{c}{a}$.

Trace the changes of sign of the expression $ax^2+2bx+c$, as x is made to take, in succession, all values between a very large negative and a very large positive quantity.

2. Solve the equations—

$$\text{(i)}\quad \frac{3}{2(x^2-1)}+\frac{x}{4(x+1)}=\frac{2}{3};$$

$$\text{(ii)}\quad \left.\begin{aligned} x^2+x &= 6y, \\ x^3+1 &= 9y. \end{aligned}\right\}$$

3. Find the product of the two imaginary expressions

$$\alpha+\beta\sqrt{-1} \text{ and } \gamma+\delta\sqrt{-1}.$$

Prove that the cube of $-\frac{1}{2}+\frac{\sqrt{3}}{2}\sqrt{-1}$ is unity; and write down another imaginary quantity whose cube is unity.

4. Find the sum of n terms of an arithmetical progression of which the first term is a and the common difference b.

Show how to insert $2n+1$ arithmetical means between two given quantities a and b, and give the value of the middle mean.

5. It being given that a, b, c are in arithmetical progression, and α, β, γ in harmonical progression;

if $\frac{\alpha}{\gamma}+\frac{\gamma}{\alpha}=\frac{a}{c}+\frac{c}{a}$, prove that $a\alpha$, $b\beta$, $c\gamma$ are in geometrical progression.

6. Prove the binomial theorem for any positive integral exponent.

Expand $(2-3x)^{\frac{1}{2}}$ to 5 terms, and give the co-efficient of x^r in the expansion.

7. Prove the formula

$$\log_{10}(n+1)-\log_{10}n = 2m\left(\frac{1}{2n+1}+\frac{1}{3(2n+1)^3}\right)+\&c., \text{ explaining what}$$

m is.

1898.

1. Solve the equations :—

$$(1)\quad \sqrt{x^2+ax-1}+\sqrt{x^2+bx-1}=\sqrt{a}+\sqrt{b};$$

$$(2)\quad \left.\begin{aligned} xy-2x^2 &= 4 \\ y^2+xy-6x^2 &= 21 \end{aligned}\right\}.$$

2. Explain the difference between a quadratic expression and a quadratic equation.

Prove that ax^2+bx+c may be written in the form

$$a\left\{\left(x+\frac{b}{2a}\right)^2-\frac{b^2-4ac}{4a^2}\right\},$$

and from the latter expression find under what circumstances the expression ax^2+bx+c has necessarily the same sign as a.

3. If s the sum of an arithmetical progression, a the first term, and b the common difference be given, find the number of terms.

If one of the results in the above case be $-n_1$, where n_1 is a whole number, prove that if the series be counted backwards n_1 terms beginning with $a-b$, the sum so obtained will be $-s$.

4. Sum the following series :—

(1) $2+5+10+17+$&c. to n terms ;

(2) $a+(a+b)r+(a+2b)r^2+$&c. to n terms.

5. Find the number of permutations of n things taken r at a time.

6. Write down all the terms of $\left(5-\frac{x}{6}\right)^6$.

Also show that the middle term in the expansion of $(1+x)^{2n}$ is

$$\frac{1.3.5\ldots(2n-1)}{\underline{|n}}2^n x^n.$$

7. Having given that

$$\log_e(x+1) = 2\log_e x - \log_e(x-1)$$
$$-2\left\{\frac{1}{2x^2-1}+\frac{1}{3}\frac{1}{(2x^2-1)^3}+\ldots\right\},$$

also that $\log_{10} 3 = \cdot 47712$, and $\frac{1}{\log_e 10} = \cdot 43429$, find $\log_{10} 11$ correct to the fifth decimal place.

1899.

1. Solve the equations :—

(1) $x^2+2ab = (2a+b)x$;

(2) $\frac{1}{x^2}+\frac{1}{y^2} = 10$; $3xy = 1$.

2. Show when the roots of $ax^2+bx+c = 0$ are imaginary, and write out the equation whose roots are $1\pm\sqrt{-1}$ in its simplest form.

3. (1) Prove that the reciprocals of quantities in Arithmetical Progression are in Harmonic Progression.

(2) If $\cdot 5$ and $\cdot 25$ are the first and second terms of an Arithmetical, Geometrical and Harmonic Progression, find in each of the three cases, by means of decimal fractions, the third term.

4. (1) Find the number of Combinations of n things taken r at a time.

(2) In how many different ways may four Spanish ships fight with three American ships, so that all are engaged, and not more than two Spanish vessels attack any one American vessel at the same time ?

5. (1) State the general term in the expansion of

$$(1-x)^{-\frac{1}{n}}.$$

(2) Find the cube root of $1\cdot 02$ to six places of decimals.

6. What is the exponential theorem, and how is it proved to be true.

1900.

1. If α and β are the roots of $x^2+px+q = 0$, form the equation whose roots are $(\alpha-\beta)^2$ and $(\alpha+\beta)^2$.

2. Solve the equations :—

(1) $$\frac{x+\sqrt{x^2-1}}{x-\sqrt{x^2-1}}+\frac{x-\sqrt{x^2-1}}{x+\sqrt{x^2-1}} = 98 ;$$

(2) $$\left.\begin{aligned} 3x^2-5y^2 &= 7 \\ 3xy-4y^2 &= 2 \end{aligned}\right\} .$$

3. If a, b, c, d be in G. P., then

$$\left.\begin{aligned} (a+b+c+d)^2 &= (a+b)^2+(c+d)^2+2(b+c)^2 \\ (a-d)^2 &= (b-c)^2+(c-a)^2+(d-b)^2 \end{aligned}\right\} .$$

4. Find the sum of the cubes of the first n natural numbers.

5. Find for what value of r the number of combinations of n things r at a time is greatest.

6. Expand $(1-3x)^{-\frac{1}{3}}$ to four terms.

7. Expand $\log_e(1+x)$ in ascending powers of x.

1901.

1. Find the square roots of $a^2-1+2a\sqrt{-1}$.

Find also the cube roots of -1.

2. Solve the equations :—

(1) $$\sqrt{a^2+2ax-3x^2}-\sqrt{a^2+ax-6x^2} = \sqrt{2a^2+3ax-9x^2} ;$$

(2) $$\left.\begin{aligned} \sqrt{\frac{x}{y}}+\sqrt{\frac{y}{x}} &= \frac{10}{3} \\ x+y &= 10 \end{aligned}\right\} .$$

3. Distinguish between the Arithmetical mean, the Geometrical mean and the Harmonic mean of any two quantities.

Prove that the Arithmetic mean of any two positive quantities is greater than their Geometric mean.

4. Distinguish between Permutation and Combination.

Find the number of combinations of n dissimilar things taken r at a time.

5. Prove that in the expansion of $(1+x)^n$ the sum of the co-efficients of the odd terms is equal to the sum of the co-efficients of the even terms.

Find the first three terms in the expansion of

$$\frac{1}{(1+x)^2\sqrt{1+4x}} .$$

1902.

1. Show that a quadratic equation cannot have more than two roots.

If α, β are the roots of $ax^2+bx+c = 0$, form the equation whose roots are $\alpha^2+\beta^2$ and $\alpha^{-2}+\beta^{-2}$.

2. Solve the equations :—

(1) $$\frac{x+\sqrt{12a-x}}{x-\sqrt{12a-x}} = \frac{\sqrt{a}+1}{\sqrt{a}-1};$$

(2) $$\left.\begin{aligned}(x^2+y^2)(x-y) &= 16xy,\\ (x^4-y^4)(x^2-y^2) &= 640x^2y^2.\end{aligned}\right\}$$

3. Find the sum of the series

$x+a$, x^2+2a, x^3+3a...to n terms.

4. Find for what value of r the number of combinations of n things r at a time is greatest.

5. Assuming the binomial theorem for a positive integral index, prove it for a positive fractional index.

Show that the nth co-efficient in the expansion of $(1-x)^{-n}$ is double of the $(n-1)$th.

1903.

1. Find the condition that the roots of the equation

$$ax^2+bx+c = 0$$

should be (1) equal in magnitude and opposite in sign, (2) reciprocals.

(a) Find the equation whose roots are

$$\frac{\sqrt{a}}{\sqrt{a}\pm\sqrt{a-b}}.$$

2. Solve the equations :

(1) $$\sqrt{2x-1}+\sqrt{3x-2} = \sqrt{4x-3}+\sqrt{5x-4};$$

(2) $$\left.\begin{aligned}\frac{x^2+y^2}{xy}+x^2+y^2 &= 13\tfrac{1}{3},\\ \frac{xy}{x^2+y^2}+xy &= 3\tfrac{3}{10}.\end{aligned}\right\}$$

3. Applying the rules of Geometrical Progression, how would you find the value of a recurring decimal ?

Find the sum of n terms of the series whose n^{th} term is $n(n+2)$.

4. Find the number of ways in which n things may be arranged among themselves, taking them all at a time, when p of the things are exactly alike of one kind, q of them exactly alike of another kind, r of them exactly alike of a third kind, and the rest different.

5. Find in its simplest form the general term in the expansion of $(1-x)^{-n}$.

Show that

$$\frac{1}{2}-\frac{1}{2.2^2}+\frac{1}{3.2^3}-\frac{1}{4.2^4}+\frac{1}{5.2^5}-\ldots = \log_e 3-\log_e 2.$$

1904.

1. Find the condition that the roots of the equation $ax^2+bx+c = 0$ should be equal in magnitude but opposite in sign.

(*a*) Show that if the roots of the equation

$$x^2(b^2+b'^2)+2x(ab+a'b')+a^2+a'^2 = 0$$

be real, they will be equal.

2. Solve the equations :—

(1) $(a-1)(1+x+x^2)^2 = (a+1)(1+x^2+x^4)$;

(2) $xy - \dfrac{x}{y} = a$

$xy - \dfrac{y}{x} = \dfrac{1}{a}$.

3. Find the sum of the cubes of the first n natural numbers.

If a, b, c be in A. P. and a^2, b^2, c^2 be in H. P., prove that $-\frac{a}{2}$, b, c are in G. P. or else $a = b = c$.

4. Find the number of ways in which $m+n+p+q$ things can be divided into four groups containing m, n, p, q things severally.

5. For what values of n are the co-efficients of the second, third, and fourth terms of the expansion of $(1+x)^n$ in arithmetical progression ?

In the statement of the Exponential theorem

$$e^x = 1+x+\frac{x^2}{\lfloor 2} + \ldots + \frac{x^r}{\lfloor r} + \ldots,$$

prove that e is an incommensurable number.

1905.

1. Show the relations between the roots and co-efficients of a quadratic equation.

Prove that if x be real, $2(a-x)(x+\sqrt{x^2+b^2})$ cannot exceed a^2+b^2.

2. Solve the equations :—

(1) $(x+a)(x+3a)(x+5a)(x+7a) = 384a^4$;

(2) $\dfrac{x+y}{1-xy} = 3$,

$\dfrac{x-y}{1+xy} = \frac{1}{3}$.

3. Find the sum of any number of terms in Geometrical Progression.

The series of natural numbers is divided into groups as follows ; 1 ; 2, 3 ; 4, 5, 6 ; 7, 8, 9, 10 ; and so on. Prove that the sum of the numbers in the m^{th} group is $\frac{1}{2}m(m^2+1)$.

4. Find the greatest number of combinations of n things (where n has a given value) taken r at a time.

Show in how many different ways $2n$ persons may be seated at two round tables, n persons being seated at each.

5. In the expansion of $(1 \pm x)^n$ by the binomial theorem, find the greatest term.

Show that
$$\left(1+\frac{1}{\underline{|2}}+\frac{1}{\underline{|4}}+\frac{1}{\underline{|6}}+\ldots\right)^2$$
$$=1+\left(1+\frac{1}{\underline{|3}}+\frac{1}{\underline{|5}}+\frac{1}{\underline{|7}}+\ldots\right)^2.$$

1906.

1. If α is a root of the equation $ax^2+bx+c=0$, prove that $x-\alpha$ is a factor of the expression ax^2+bx+c.

2. (1) Solve the equations :—

(i) $x^2+1=x\left(a+\frac{1}{a}\right)$;

(ii) $5x^3-13x^2-13x+21=0$;

(iii) $\frac{1}{x}+\frac{1}{y}=\frac{1}{2}$, $x+y=9$.

(2) A man bought a certain number of articles of equal value for Rs. 375. By selling them at Rs. 9 each he gained as much as 10 of them cost him. How many did he buy.

3. If $S_1 = 1+2+3+\ldots\ldots+n$,
and $S_2 = 1^2+2^2+3^2+\ldots+n^2$,
find S_1 and S_2 in terms of n, and shew that
$$3s_2+3s_1+(n+1)=(n+1)^3.$$

4. Find the number of ways in which a selection of r things can be made out of n things which are all different.

If ${}^nC_r$ denote this number, prove that
$${}^nC_r+{}^nC_{r-1}={}^{n+1}C_r\,.$$

5. Prove the Binomial theorem for a positive integral exponent. Shew that the sum of the co-efficients of the odd powers of x in the expansion of $(1+x)^n$ is 2^{n-1}.

1907.

1. Prove that if $ax^2+bx+c=0$ is satisfied by giving x the values α, β, γ, all different, it is an *identity*.

Verify that
$$(x-2)(x-3)-8(x-3)(x-1)+9(x-1)(x-2)=2x^2$$
is an identity.

2. Solve (1) $\frac{a^2}{(x-a)^2}=\frac{b^2}{(x+b)^2}$;

(2)
$$x(x+y+z)=a^2$$
$$y(x+y+z)=b^2$$
$$z(x+y+z)=c^2$$

3. Show how to sum a Geometric series (a, r) of n terms.

If the sum of n terms of such a series is 728, the common ratio being 3 and the first term 2, find n.

4. Show how to find the number of permutations of n things, r at a time.

Denoting this by P_r^n, prove that

$$P_r^{n+1} = P_r^n + r.\ P_{r-1}^n.$$

5. Write down the expansion of $(1 - \frac{1}{2}x)^{-\frac{1}{2}}$, $x < 1$.

6. Expand a^x in powers of x.

1908.

1. Prove the rule for the conversion of recurring decimals into vulgar fractions.

2. Show how to solve a quadratic equation.

If α, β are the roots of $ax^2 + bx + c = 0$, form the equation whose roots are $\alpha - 1$, $\beta - 1$.

3. Solve :—

(1) $x^3 - 1 = 0$;

(2) $(x^2 + x + 1)(x^2 + x + 2) = 12$;

(3) $\left.\begin{aligned} x + y &= 6, \\ x^3 + y^3 &= 72. \end{aligned}\right\}$

4. Show to insert n geometric means between two given quantities ;

Or,

If the p^{th} term of an arithmetical progression is q, and the q^{th} term is p, find the m^{th} term.

5. If C_r^n is the number of combinations of n things r at a time, show that

$$C_r^n = \frac{n}{r} \cdot C_{r-1}^{n-1}.$$

Hence or otherwise find C_r^n. Find when this is a maximum.

6. Write down the expansion of $(1 - x)^{\frac{1}{3}}$.

. Find the cube root of 999 to three places of decimals.

CALCUTTA UNIVERSITY INTER. PAPER.

1909.

1. Prove that the equation $ax^2 + bx + c = 0$
cannot have three different roots.

Find the condition that it should have two equal roots.

If α and β be the roots of $x^2 - (1 + k^2)x + \frac{1}{2}(1 + k^2 + k^4) = 0$,
show that $\alpha^2 + \beta^2 = k^2$.

2. Solve

(i) $$\frac{a}{x}+\frac{b}{y}=2$$

$$\frac{a^2}{x^2}+\frac{b^2}{y^2}=2$$

(ii) $$4^x=2^y$$

$$27^{xy}=9^{y+1}.$$

3. Find the number of combinations of n different things taken r at a time.

There are sixteen points in a plane, no three of which are in the same straight line. Find the number of straight lines which can be formed by joining them.

4. Find the co-efficient of x^7 in

(i) $(1-5x)^{-\frac{3}{4}}$

(ii) $\dfrac{1-2x}{3+2x-x^2}$.

5. (i) Simplify

$$(2+3\sqrt{-1})^4+(2-3\sqrt{-1})^4.$$

(ii) Find the value of

$$\frac{1+x}{1+\sqrt{1+x}}+\frac{1-x}{1-\sqrt{1-x}}$$

when $x=\dfrac{\sqrt{3}}{2}$.

---

1. Prove that the latus-rectum of a parabola is four times the focal distance of the vertex.

Find a double ordinate of a parabola which shall be double the latus-rectum.

2. The sub-tangent at any point of a parabola is bisected at the vertex.

Given the vertex, a tangent, and its point of contact, construct the curve.

3. The sum of the focal distances of any point on an ellipse is equal to the major axis.

The distance of either extremity of the minor axis from either focus, is equal to the semi-axis-major.

4. The tangents at the ends of a focal chord of an ellipse intersect on the directrix.

Where do the tangents at the extremities of the latus-rectum intersect?

5. Prove that straight lines in space which are parallel to a given straight line, are parallel to one another.

6. If a solid angle be contained by three plane angles, any two of them must be together greater than the third.

MADRAS UNIVERSITY F. A. PAPERS.

1880.

1. Solve the equation—

$$\frac{1}{3x^2+11x+10}+\frac{1}{6x^2+19x+15}=\frac{2x^2+9x+10}{150}.$$

(See page 135.)

2. Given a quadratic expression in x, prove the rule for resolving it into its component factors and shew that its sign will be constant for all real values of x, if the binomial factors are either imaginary or identical.

The expression $\frac{1}{x+1}+\frac{1}{3x+1}-\frac{1}{(x+1)(3x+1)}$

cannot lie between the values of 1 and 4, for any real value of x.

(See page 151.)

3. Point out the artifice by which a series in *G. P.* is summed, and employ it to obtain the sum of the following series :—

(*a*) $a, (a+b)x, (a+2b)x^2, \ldots \{a+(n-1)b\}x^{n-1}$;

(*b*) $x, 2^2x^2, 3^2x^3, \ldots . n^2x^n$.

4. Find the number of permutations of n different things, taken r at a time.

If P_r denote this number, shew that

$$P_1+\frac{P_2}{|\underline{2}}+\frac{P_3}{|\underline{3}}+\ldots+\frac{P_n}{|\underline{n}}=2^n-1.$$

(See page 308.)

1881.

1. Solve the equation—

$(x+3)^2-2(x^2+3)=2x(x+1)^2$.

2. Investigate the conditions under which the roots of the equation $ax^2+bx+c=0$, are real and equal, real and unequal, and imaginary.

3. (*a*) Form the equation of which the roots are $\frac{1}{2}(1+\sqrt{3}+\sqrt{2\sqrt{3}})$ and $\frac{1}{2}(1+\sqrt{3}-\sqrt{2\sqrt{3}})$.

(See page 155.)

(*b*) Shew that the expression

$$\frac{(x^2-4)(x^2+3x+2)(x^2-x-2)+10}{x^2+5x+7}$$

is positive for all real values of x.

(See page 169.)

4. Find the sum to n terms of the series in *G. P.*, whose first term is a, and the common ratio r.

(*a*) The first term of a Geometric series continued to infinity is 1, and any term is equal to the sum of all the succeeding terms. Find the

5. Assuming the formula for permutations, find the number of combinations of n things taken r at a time.

(*a*) 25 passengers arrive at a Railway station, and proceed to the neighbouring village. At the station, there are 2 coaches accommodating 4 each, and 3 carts accommodating 3 each. Find the number of ways in which they can proceed to the village, assuming

(1) that the conveyances are always fully occupied, and

(2) that the conveyances are all distinguishable from each other.

(See page 308.)

6. Find the sum of the co-efficients of the terms in the expansion of $(1+x)^n$, when n is a positive integer.

(*a*) If the co-efficients in order are p_0, p_1, p_2, p_3, &c., deduce

$$p_0+\frac{p_1}{2}+\frac{p_2}{3}+\frac{p_3}{4}+\&c. = \frac{2^{n+1}-1}{n+1}.$$

1882.

1. Solve the following equations :—

(1) $\sqrt[12]{x^5}-\frac{1}{2}\frac{p^2-q^2}{p^4+q^2}\left(\sqrt{x}+\sqrt[3]{x}\right) \qquad 0.$

$$\left.\begin{array}{r}(2)\quad x(9-xy) - y(xy-36) \\ xy(3x+12y-xy) = 108(x+y-3)\end{array}\right\}$$

(See page 202.)

2. Show that a quadratic equation has two, and only two, roots.

(*a*) Give the conditions, in order that the roots of the equation $\frac{x+a}{a}\cdot\frac{x+b}{b}\cdot\frac{x+c}{c} = 1$ may be *possible*.

3. Insert n Harmonical means between a and b.

(*a*) Given that $(x-y)pq+(z-x)pr+(y-z)qr = 0$, shew that p, q and r are the x^{th}, y^{th} and z^{th} terms respectively of an *H. P.*

4. Find for what value of r the number of combinations of n things taken r at a time is greatest.

5. There are 4 packs of cards, each pack containing 52 cards. Find the number of ways they may be distributed among 4 persons, so that each may have 52 cards, of which there are 4 kings assuming that the cards are all distinguishable from each other, and that the order among the persons remains unchanged.

6. Assuming the Binomial Theorem for an integer, prove it for a fractional quantity.

(*a*) When n is a positive integer, show that the sum of the squares of the co-efficients of the terms in the expansion of $(x+a)^n$ is equal to $\frac{\underline{|2n}}{(\underline{|n})^2}$.

1883.

1. Solve the following equation :—

$$\left.\begin{aligned} 4y^2+3xy-x^2 &= 0 \\ (y-a)(x-y) &= (x+2a)(x+y) \end{aligned}\right\}$$

2. Find for what relation between the co-efficients and what values of x the expression m^2x^2+bx+c is negative.

If x be real, show that the expression $\frac{m^2}{1+x}-\frac{n^2}{1-x}$ can have any real value. **(See page 169.)**

3. Find the sum of the cubes of the first n natural numbers.

Show that the sum of the cubes of any number of consecutive integers (*not* beginning with unity) is divisible by the sum of the integers.

4. Find the number of permutations of n things taken all together which are not all alike.

Show that 154 numbers less than 1000 and divisible by 5 can be formed with the ten digits, each digit not occurring more than once in each number. **(See page 308.)**

5. Find the greatest term in the expansion of $(x+a)^n$, n being a positive integer.

Show that the co-efficient of x^n in the expansion of $\frac{(1-2x)^n}{1-3x}$ is 1 and that of x^{n+r} is 3^r.

1884.

1. Solve $\left(x+m\right)^{\frac{2}{3}}+\left(x-m\right)^{\frac{2}{3}}=\left(n+\frac{1}{n}\right)\left(x^2-m^2\right)^{\frac{1}{3}}$. **(See page 141.)**

2. (*a*) If x is real, shew that ax^2+bx+c and a differ in sign only when the equation $ax^2+bx+c=0$ has real roots and x is taken to lie between them.

(*b*) If x is real, the value of $\frac{(x-1)(x+3)}{(x-2)(x+4)}$ does not lie between $\frac{4}{9}$ and 1. **(See page 151.)**

3. If a, b, c, d are proportionals, shew that

$\frac{ma+nb}{pa+qb}=\frac{mc+nd}{pc+qd}$; and that $\frac{a^2}{b}:\frac{c^2}{d}$ is inversely as $\frac{a}{b^2}:\frac{c}{d^2}$.

4. (*a*) Find the mth term of an *H. P.* whose first term is a, whose last term is c, and whose number of terms is n.

(*b*) Shew that the sum of n terms of a *G. P.* beginning with the pth term is r^{p-q} times the sum of an equal number of terms of the same series beginning with the q^{th} term. **(See page 242.]**

5. (*a*) Assuming the truth of the Binomial Theorem when the exponent is a positive integer, prove it for any positive exponent.

(*b*) Find the $(2n+1)^{th}$ term from the beginning in the expansion of $\left(x-\frac{1}{x}\right)^{3n}$.

1885.

1. (*a*) If $X = \sqrt[3]{r+\sqrt{r^2+q^3}} + \sqrt[3]{r-\sqrt{r^2+q^3}}$, find the value of $X^3+3qX-2r$. **(See page 81.)**

(*b*) Find the value of

$$\frac{}{(a-b)(a-c)(x-a)}+\frac{b}{(b-a)(b-c)(x-b)}+\frac{c}{(c-a)(c-b)(x-c)}.$$

2. Solve the equations :

(*a*) $3x^2+15x-2 = 2\sqrt{x^2+5x+1}$.

(*b*) $x^2-2xy = 7$, $2y^2-xy = -3$.

3. (*a*) If a is the first term of a Geometric series whose common ratio is r, find the sum of n terms.

(*b*) When are quantities said to be in Harmonic Progression ?

(*c*) If $a^p = b^x = c^r$ and a, b, c, are in Geometric Progression show that p. q, r are in Harmonic Progression.

4. (*a*) When is one quantity said to vary as another ?

(*b*) If $x \propto y$, when z is constant, and $x \propto z$ when y is constant, show that x will vary as yz when both y and z vary.

(*c*) *Apply this* to find how soon 20 men will earn Rs. 30, if 3 men earn Rs. 9 in 16 days.

5. (*a*) Find the number of combinations of n things taken r at a time, *without using* the corresponding formula for permutations.

(*b*) Find in how many ways groups of 8 persons, 4 ladies and 4 gentlemen, can be formed out of 8 ladies and 9 gentlemen, subject to the condition that two particular gentlemen are not *simultaneously* to be in the same group with a particular lady. **(See page 307.)**

6. (*a*) Write down the 10th term in the expansion of $(2x-y)^{11}$.

Show that the $(r+1)^{\text{th}}$ term in the expansion of $(1-2x)^{-\frac{1}{2}}$ is

$$\frac{\underline{|2r}}{2^r.(\underline{|r})^2}.x^r.$$

1886.

1. If $\sqrt{(x-\sqrt{a^2-b^2})^2+y^2}+\sqrt{(x+\sqrt{a^2-b^2})^2-y^2} = 2a$, shew that $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$. **(See page 83.)**

2. (*a*) If α, β are the values of x which make $ax^2+bx+c = 0$, prove that $ax^2+bx+c = a(x-\alpha)(x-\beta)$ for all values of x.

(*b*) Solve the equations :—

(i) $\sqrt[3]{2x^2(x+1)+10x+1}-(2x+1) = 0$;

(ii) $5xy-2x^2 = \quad 3y^2+xy = 18$.

3. (1) Shew that any magnitudes a, b, c are in *A.P.*, *G.P.*, or *H.P.* according as $\frac{a-b}{b-c} = \frac{a}{a}$, or $= \frac{a}{b}$, or $= \frac{a}{c}$.

(2) A person is employed to count Rs. 12,000. He counts at the rate of Rs. 150 per minute for an hour, at the end of which time he begins to count at the rate Rs. 2 less every minute than he did the previous minute. Find when he will finish his task, and explain the fact that two solutions occur.

4. (1) Define ratio. Shew that a ratio of greater inequality is diminished by adding the same quantity to both its terms.

(2) The value of a certain fraction is diminished in the ratio of 10 : 9 by adding 2 to its numerator and denominator, while its value is increased in the ratio of 2 : 3 by subtracting the same quantity from its numerator and denominator. Find the fraction.

5. (1) Find the number of permutations of n things taken all together which are not all different.

(2) How many different numbers each of six digits can be expressed by means of the digits of the number 121,202? **(See page 288.)**

6. (*a*) Find the numerically greatest term in the expansion of $(a+x)^n$, n being a positive integer.

(*b*) Obtain the value of $\sqrt[10]{1.004}$ correct to eight places of decimals by means of the Binomial Theorem.

1887.

1. Find when $x^n + y^n$ is divisible by $x+y$.

Shew that $b^3(a^3 + b^3 - c^3) + 3b^4(a-c)^2 + 3b^3(b^2 - ac)(a-c)$ is a perfect cube.

Resolve into simple factors the expression

$(y^3+1)(x^2+x+1)(x+1) - (x^3+1)(y^2+y+1)(y+1)$.

2. State the axioms on which operations with equations depend.

Solve the equations :—

$$(1)\quad \frac{1-\sqrt{x^2-1}}{1+\sqrt{x^2-1}} = \frac{x-\sqrt{x^2+8}}{x+\sqrt{x^2+8}};$$

$$(2)\quad \left.\begin{aligned} x^2+y^2+x+y &= 26 \\ 4(x+y) &= 3xy \end{aligned}\right\}.$$

(See page 201.)

3. If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f}$, prove that each of these ratios is equal to

$$\left(\frac{a^3 - 3ace + e^3}{b^3 - 3bdf + f^3}\right)^{\frac{1}{3}}.$$

If $\frac{a}{y+z} = \frac{b}{z+x} = \frac{c}{x+y}$, prove that $\frac{a(b-c)}{y^2-z^2} = \frac{b(c-a)}{z^2-x^2} = \frac{c(a-b)}{x^2-y^2}$.

(See page 99.)

4. Show how to insert a given number of Arithmetic means between two given quantities.

The p^{th} term of an *A. P.* is a and the q^{th} term is b. Show that the sum of the first $p+q$ terms is

$$\frac{p+q}{2}\left\{a+b+\frac{a-b}{p-q}\right\}.$$

5. Show how to find the number of combinations of n things taken r together.

In how many ways could a party of 2 ladies and 3 gentlemen be chosen from a company consisting of 5 ladies and 8 gentlemen?

6. Prove the Binomial Theorem for a positive integral exponent.

Find the co-efficient of x^r in the expansion of $\dfrac{(1+x)^2}{(1-2x)^3}$.

1888.

1. (*a*) Resolve into four factors the expression

$a^3(b+c)+b^3(c+a)+c^3(a+b)+2(b^2c^2+c^2a^2+a^2b^2)+4abc(a+b+c)$.

(*b*) If $a+b+c = 0$, prove that

$(bc+ca+ab)^3+(a^2-bc)(b^2-ca)(c^2-ab) = 0$

(*c*) Extract the square root of $x+i\sqrt{x^4+x^2+1}$, when $i = \sqrt{-1}$.

(See page 222).

2. Solve the equations :—

(1) $x^2+x+10\sqrt{x^4+3x}+16 = 2(20-x)$; **(See page 133.)**

(2) $$\left.\begin{aligned} 6x+5y &= \frac{6}{x}+\frac{5}{y}+29\frac{1}{3} \\ 3x+1y &= \frac{3}{x}+\frac{1}{y}+18\frac{2}{3} \end{aligned}\right\}.$$ **(See page 202.)**

3. (*a*) Give Euclid's definition of Proportion, and show that if four quantities a, b, c, d be in proportion according to this definition, then the quotients $\frac{a}{b}$, $\frac{c}{d}$ are equal.

(*b*) If $\sqrt{(x-x')^2+(y-y')^2} = \sqrt{x^2+y^2}-\sqrt{x'^2+y'^2}$ then x, y, x', y' are in proportion. **(See page 103.)**

4. (*a*) Find the sum of a series of terms in Arithmetical progression.

(*b*) The natural numbers are written as follows :—

1
2 3
4 5 6
7 8 9 10

Show that the sum of the numbers in the nth row is $\frac{1}{2}n(n^2+1)$.

5. What is the meaning of the expression $\underline{|n}$? For what reason is $\underline{|0}$ taken to be equal to 1 ?

Show that 2. 6. 10. 14. . . . . to n factors is equal to $(n+1)(n+2)(n+3)$ $(n+4)$ . . to n factors. **(See page 309.)**

6. (*a*) Find the number of permutations of n things taken all together, when the things are not all different.

(*b*) In how many ways can the letters of the word "*arrange*" be arranged ? How many arrangements can be made, (*a*) if the two *r*'s are not allowed to come together, (*b*) if neither the two *r*'s nor the two *a*'s are allowed to come together ? **(See page 285).**

7. (*a*) Assuming the truth of the Binomial Theorem for a positive integral index, prove it for a negative integral index.

(*b*) Prove that $\sqrt{2} = \frac{7}{5}\left(1+\frac{1}{10^2}+\frac{1.3}{1.2}\cdot\frac{1}{10^4}+\frac{1.3.5}{1.2.3}\cdot\frac{1}{10^6}+\&c. \right)$

(See page 345.)

1889.

1. Show how to determine the sign of the expression ax^2+bx+c for real values of x.

Prove that if x is real the expression $\frac{(x-a)(x-c)}{x-b}$ is capable of assuming all values, if a, b, c are in ascending order of magnitude. **(See page 168.)**

2. (*a*) Solve the equations :—

$$\left.\begin{aligned} y+\sqrt{x^2-1} &= 2 \\ \sqrt{x+1}+\sqrt{x-1} &= \frac{2}{\sqrt{y}}\,. \end{aligned}\right\}.$$

(*b*) The area of a rectangular field is $7\frac{1}{3}$ acres, and the sum of the lengths of adjacent sides exceeds the length of either diagonal by 110 yards. Find the lengths of the sides.

3. Assuming that the relation $a^m \times a^n = a^{m+n}$ is always true whatever m and n may be, show how to find the meaning of $a^{\frac{1}{n}}$, $a^{\frac{m}{n}}$, a^0, a^{-m} whatever m and n may be, and prove that $(a^m)^n = a^{mn}$ for all values of m and n.

Extract the square root of $\frac{(x+y)^2}{y} - x^{\frac{1}{2}}y^{-\frac{1}{2}}\left(x-\frac{1}{4}x^{\frac{1}{2}}y^{\frac{1}{2}}+y\right)$.

4. Show how to find the sum of the squares of the first n natural numbers. Prove that the sum of the squares of the first n odd numbers is $\frac{n(4n^2-1)}{3}$.

5. Show how to find the number of combinations of n different things taken r at a time, without assuming the formula for the number of permutations of n things taken r at a time. Find how many numbers greater than 1000 can be formed from the digits 112340, taken four at a time.

(See page 297.)

6. Show how to find the greatest term in the expansion of $(x+a)^n$, n being a positive integer. Expand $(a^2-2ax)^{\frac{3}{2}}$ in ascending powers of x as far as x^3, and write down the general term.

1890.

1. Show that $x^n - a^n$ is divisible by $x-a$ for all positive integral values of n, and hence, or otherwise, show that if any rational and integral expression which contains x vanish when a is put for x, the expression is divisible by $x-a$.

If $x-a$ is a factor of $a_1x^2+2b_1x+c_1$ and $x+a$ of $a_2x^2+2b_2x+c_2$, prove that $(c_2a_1-c_1a_2)^2+4(a_1b_2+a_2b_1)(b_1c_2+b_2c_1) = 0$.
(See page 170.)

2. Solve the equations :—

(1) $x(x-\sqrt{2}-\sqrt{3}-3)+\sqrt{2}+\sqrt{3}+2 = 0$;

(2) $x+\dfrac{3}{y} = 2,\ y+\dfrac{3}{x} = -2.$

3. If a, b, x be positive quantities and $a > b$, prove that $\dfrac{a+x}{b+x} < \dfrac{a}{b}$.

If a, b, c, d be in continued proportion, prove that

$$\sqrt{ab}-\sqrt{bc}+\sqrt{cd} = \sqrt{(a-b+c)(b-c+d)}.$$ **(See page 101.)**

4. Show how to insert n Geometric means between two quantities a and b.

α, β, γ are the Geometric means between ca, ab ; ab, bc ; bc, ca respectively. Prove that if a, b, c are in *A. P.*, α^2, β^2, γ^2 are also in *A. P.*, and $\beta+\gamma$, $\gamma+\alpha$, $\alpha+\beta$ are in *H. P.* **(See page 258.)**

5. Show how to find the number of permutations of n things taken r at a time. State how many combinations of r things each can be formed from n different things.

A cricket team consisting of eleven players is to be selected from two sets consisting of six and eight players respectively. In how many ways can the selection be made, on the supposition that the set of six shall contibute not fewer than four players ? **(See page 276).**

6. Assuming the truth of the Binomial Theorem for a positive integral index, prove the truth of the theorem when the index is any positive quantity.

Find the co-efficient of x^r in the expansion of $\left(\dfrac{1+x}{1-x}\right)^{\frac{1}{2}}$ in ascending powers of x.

1891.

1. Find the *G. C. M.* of $4x^6-209x^3+15$ and $15x^6-209x^3+4$.

2. Between two given numbers a and b insert :—(1) two arithmetic means ; (2) two harmonic means ; (3) two geometric means.

If x_1, x_2 be the arithmetic means, y_1, y_2 the harmonic means and z_1, z_2 the geometric means, shew that $x_1y_2 = x_2y_1 = z_1z_2$.

3. Solve the equations—

(1) $\sqrt{x^2+a}+\sqrt{x^2-a} = \sqrt{2a+b}+\sqrt{b}$.

(2) $\frac{x}{y+1}+\frac{y}{x+1} = \frac{5}{3}$; $x^2+y^2 = 2$.

4. Prove that the number of permutations of n things taken all together, of which p are alike and all the rest unlike is $|\underline{n} \div |\underline{p}$

In how many ways can the letters forming the word *plantain* be arranged so that the two a's do not come together ?

5. Write down the five terms in the expansion of $(a^2+x)^{\frac{1}{2}}$.

Prove that if M differ from N^2 by a small quantity the square root of M is approximately equal to $\frac{3}{2}N - \frac{(3N^2-M)^2}{8N^3}$. **(See page 352.)**

BOMBAY UNIVERSITY PREVIOUS EXAMINATION PAPERS.

1882.

1. Define an *expression*. When is an expression said to be *homogeneous* ?

Show that the value of the expression $y^2z - z^2y + z^2x - x^2z + x^2y - xy^2$ is not altered if any the same quantity be added to or subtracted from each of the quantities x, y and z.

2. Prove that the square root of a binomial, one of whose terms is a quadratic surd and the other rational, may sometimes be expressed by a binomial, one or each of whose terms is a quadratic surd. In what case is it useless to employ this method ?

Find the square root of $a+b+\sqrt{2ab+b^2}$.

3. Find the sum and product of the root of the quadratic—

$$px^2+qx+r = 0.$$

If α and β be the roots of this equation, shew that the roots of the equation $qrx^2+(pr+q^2)x+pq = 0$, are $\frac{1}{\alpha+\beta}$ and $\frac{1}{\alpha}+\frac{1}{\beta}$.

4. Prove that a ratio of greater inequality is diminished and of less inequality increased by adding any quantity to both its terms.

If four numbers be proportionals, shew that there is no number which being added to each will leave the resulting four numbers proportionals.

5. Find the sum of a *G. P.* to n terms and when possible to infinity.

If s_1, s_2, s_3 be the sums to n, $2n$, $3n$ terms respectively, prove that
$$s_1(s_3 - s_2) = (s_2 - s_1)^2.$$

6. If $(n)_r$ represent the number of combinations of n things taken r together, prove that independently of any formula that $r(n)_r = n(n-1)_{r-1}$.

Four persons are chosen by lot out of ten ; in how many ways can this be done and how often would any one person be chosen ? **(See page 276.)**

7. If m be any quantity whatever and $f(m)$ represent the series—
$$1 + mx + \frac{m(m-1)}{\underline{|2}}x^2 + \frac{m(m-1)(m-2)}{\underline{|2}}x^3 + \&c.,$$
prove that $f(m) \times f(n) = f(m+n)$. By what consideration does Fuler prove this relation and on what principle does he base his demonstration ? By the aid of this formula prove the Binomial Theorem for *positive fractional* exponent.

1883.

1. Prove that $(n+1)(n+2)(n+3) \ldots$ to n factors
$$= 2n \times 1.3.5 \ldots \text{to } n \text{ factors.}$$

2. Show that a factor may be found which will rationalise any binomial.

Reduce $\dfrac{\sqrt{1+x} - \sqrt{1-x}}{\sqrt{1+x} + \sqrt{1-x}}$ where $x = \dfrac{\sqrt{3}}{2}$. **(See page 85.)**

3. Distinguish between a quadratic *equation* and quadratic *expression* and shew that a quadratic has two and only two roots.

Solve the equation $\sqrt{x^2 - 3x + 8} = \frac{3}{4}(x^2 - 7x + 8)$.

4. Define *ratio, variation, proportional, commensurable.*

If A vary as B when C is constant, and A vary as C when B is constant, prove that A will vary as the product BC when both B and C are variable. Give fully a Geometrical illustration.

5. Sum the series :—

(1) $17\frac{1}{2}$, $14\frac{1}{6}$, $10\frac{5}{6}$ . . . to 24 terms.

(2) 1, 3, 6, 10, 15, . . . to n terms.

(3) $\frac{5}{7}$, $\frac{7}{21}$, $\frac{9}{63}$, $\frac{11}{189}$, . . . to infinity.

6. Investigate a formula for the number of permutations of n things taken all together which are not all different.

How many different numbers can be made out of all the figures of 111223 ?

7. Enunciate the *Binomial Theorem* and prove it for a positive integral exponent.

Write down the $(r+1)$th term in the expansion of $\sqrt{a^2 - x^2}$ and the co-efficient of x^{10} in $\dfrac{1+x}{(1-x)^3}$.

1884.

1. If $3s = a+b+c$, prove that $(s-a)^4+(s-b)^4+(s-c)^4$

$$= 2\{(s-a)^2(s-b)^2+(s-b)^2(s-c)^2+(s-c)^2(s-a)^2\}.$$

2. Shew that if two quadratic surds cannot be reduced to others which have the same irrational part, their product is irrational, and also that one quadratic surd cannot be made up of two others which have not the same irrational part.

3. If $a : b = c : d$, shew that

(α) $ma+nb : pa+qb = mc+nd : pc+qd.$

(β) $\frac{1}{ma}+\frac{1}{nb}+\frac{1}{pc}+\frac{1}{qd}=\frac{1}{bc}\left\{\frac{a}{q}+\frac{b}{q}+\frac{c}{n}+\frac{d}{m}\right\}$. **(See Page 102.)**

4. What do you understand by the *limit of an infinite series in Geometrical Progression*? Are Arithmetical series susceptible of limits? Find the sum of an infinite *G. P.*

Determine the value of $\sqrt{a\sqrt[3]{b\sqrt{a\sqrt[3]{b}}}}\ldots$ continued to infinity.

5. Insert a given number of Harmonical means between two given terms.

If a, b, c be in *G. P.*, and $a^x = b^y = c^z$, prove that x, y, z, are in *H. P.*

6. Find for what value of r the number of combinations of n things taken r at a time is greatest.

A Brahman, hospitably disposed, wishes to make up as many different parties as he can out of 40 friends, each party consisting of the same number; how many should he invite at a time?

7. Investigate the sum of the co-efficients of the terms in the expansion of $(1+x)^n$ and shew that the sum of the co-efficients of the odd terms is equal to the sum of the co-efficients of the even terms, n being a positive integer.

Prove that if n be a positive integer, $(1+x)^n(1+x^n) > 2^{n+1}.x^n$.

1885.

1. If $a+b+c+d = 2s$, prove that $4(ab+cd)^2-(a^2+b^2-c^2-d^2)^2$

$$= 16(s-a)(s-b)(s-c)(s-d).$$

2. Shew that in approximating to a cube root the number of figures may be nearly doubled by ordinary division.

If a be the greatest integer contained in $N^{\frac{1}{3}}$ and the difference be so small that its cube may be neglected, prove that a nearer approximate value of $N^{\frac{1}{3}}$ will be

$$\tfrac{1}{2}\left\{a+\left(\frac{4N-a^3}{3a}\right)^{\frac{1}{2}}\right\}.$$ **(See page 48.)**

3. If α, β be the roots of the quadratic equation $x^2-px+q = 0$, shew that $x^2-px+q = (x-\alpha)(x-\beta)$.

Shew also that $\frac{\alpha^2}{\beta^2}+\frac{\beta^2}{\alpha^2} = \frac{p^4}{q^2}-4\frac{p^2}{q}+2.$

If $a \propto b$ when c is invariable and $a \propto c$ when b is invariable, prove that $a \propto bc$ when both b and c are variable.

4. The value of a silver coin varies directly as the square of its diameter while its thickness remains the same, and directly as its thickness while its diameter remains the same. Two silver coins have their diameters in the ratio of $4 : 3$; find the ratio of their thickness if the value of the first be four times the value of the second. **(See page 115.)**

5. Investigate the sum of the squares of the first n natural numbers.

Sum the following series to n terms and write down the general term of each :—

$$\left.\begin{array}{ll}(a) & 2+7+14+23+\ldots\ldots \\ (b) & 2+5+10+17+\ldots\ldots\end{array}\right\}$$

6. Derive the number of permutations of n things r at a time from the number of their permutations $r-1$ at a time.

Find the number of different arrangements that can be made of bars of the seven prismatic colours so that the blue and the green shall never come together.

7. Investigate the greatest term in the expansion of $(x+a)^n$, where n is a positive integer.

Find the greatest term in $(1+x)^9$ when $x = \frac{1}{3}$.

1886.

1. If $x+y = p$ and $xy = q$, express x^2+y^2, x^3+y^3, x^4+y^4 in terms of p and q.

2. Simplify $\dfrac{a^2}{(a-b)(a-c)} - \dfrac{b^2}{(a-b)(b-c)} + \dfrac{c^2}{(a-c)(b-c)}$.

Show that $\left(\dfrac{1}{a-b}+\dfrac{1}{b-c}+\dfrac{1}{c-a}\right)^2 = \dfrac{1}{(a-b)^2}+\dfrac{1}{(b-c)^2}+\dfrac{1}{(c-a)^2}$.

3. Find x and y from the equations—

$$\left.\begin{array}{r}\sqrt{x}-\sqrt{y} = \sqrt{\frac{4}{3}} \\ \sqrt{xy} = 3\end{array}\right\}.$$

4. What is the condition that ax^2+bx+c shall be a perfect square with respect to x ?

For what value of n will the expression

$x^2-(n-1)x+n+\frac{1}{2}$ be a perfect square.

5. If a, b, c, d are proportionals—

$$a+b : a-b : : c+d : c-d \,;$$

$$\frac{a^2}{b^2}+\frac{c^2}{d^2} = 2.\frac{ac}{bd}\,;$$

$$(a+d)-(b+c) = \frac{(a-b)(a-c)}{a}.$$

6. Sum the series—

$$3-1+\frac{1}{3}-\frac{1}{9}+\&c., \text{ ad infinitum.}$$

$$3+6+11+20+\&c., \text{ to } n \text{ terms.}$$

If a, b, c be in Arithmetical Progression, b, c, d in Geometrical Progression and c, d, e in Harmonical Progression, prove that a, c, e are in Geometrical Progression. **(See page 258.)**

7. Show by mathematical induction that the sum of the squares of the first n natural numbers is $\frac{1}{6}n(n+1)(2n+1)$. **(See page 313.)**

Hence obtain an expression for the sum of the squares of the odd numbers in this series.

8. Prove that the number of combinations of n things taken r at a time is equal to the number of combinations taken $n-r$ at a time.

Find the number of different signals that can be made with six flags, two of which are white, two black, and two red ; six flags to be used in each signal.

9. Write down the co-efficient of x^r in the expansion of $(1+x)^n$ and $(1+x)^{-n}$ and the middle term of $\left(x+\frac{1}{x}\right)^{2n}$.

If P denote the sum of the odd terms and Q the sum of the even terms in the expansion of $(a+b)^n$, $P^2-Q^2=(a^2-b^2)^n$,

$$\text{and } 4PQ=(a+b)^{2n}-(a-b)^{2n}.$$

10. Find the 5th root of 35 correct to five places of decimals.

1887,

1. Find a value of x which will make the expression $x^5-8x^3+11x^2+7x-1789$ exactly divisible by x^2+7x-1.

2. When $n+1$ figures of a square root have been obtained by the ordinary method, shew that n more may be obtained by division only, supposing the whole number of figures in the square root to be $2n+1$.

By this method extract the square root of 5294745225.

3. Shew that a factor may be found which will rationalise any binomial.

Simplify $\dfrac{2\sqrt{8}\sqrt{3+\sqrt{5}}}{4+\sqrt{10}-\sqrt{2}}$. **(See page 84.)**

4. Shew that a quadratic equation cannto have more or less than two roots.

If α, β be the roots of $x^2+px+q=0$, form the equation whose roots are $\alpha^2+\alpha\beta$ and $\beta^2+\alpha\beta$. **(See page 159.)**

5. If $\dfrac{a}{b}=\dfrac{c}{d}=\dfrac{e}{f}$, prove that $\left(\dfrac{a+2c+3e}{b+2d+3f}\right)^2=\dfrac{ac+ce}{bd+df}$.

Solve the simultaneous equations—

$$\frac{1}{x}+\frac{1}{y}=\frac{12}{11};\quad \frac{x+y}{12}=\frac{7}{x+y-5}.$$

6. Investigate an expression for the n^{th} term of an Harmonical Progression of which the first and last terms and the number of terms are given.

If a, b, c are the p^{th}, q^{th}, r^{th} terms of an $A.\ P.$, shew that $(q-r)a+(r-p)b+(p-q)c=0$, and if a, b, c are the p^{th}, q^{th}, r^{th} terms of an $H.\ P.$, then $(q-r)bc+(r-p)ca+(p-q)ab=0$.

7. Find the sum of a Geometrical Progression when the number of terms is indefinitely great and the common ratio is a proper fraction.

Prove that any term is $>$, $=$, or $<$ the sum of all the succeeding terms according as the ratio is $<$, $=$, or $>$ $\frac{1}{2}$.

8. Investigate the number of permutations of n things taken r at a time.

Six examination papers are to be set in a certain order not to be divulged; it being discovered that this order has leaked out, in how many ways can the order be changed?

9. Write down the expansion of $(1+x)^n$, and find the sum of the co-efficients of all the terms.

If p_1, p_2, . . . p_n denote the co-efficients of x, x^2, . . . x^n in this expansion, shew that $p_1+2p_2+3p_3+\ldots+np_n=2^{n-1}.n$.

10. Obtain the general term in the expansion of $(1-x)^{-n}$.

Find the co-efficient of x^6 in the expansion of $\dfrac{(1+x)^2}{(1-x)^4}$.

1888.

1. Simplify :—$4\sqrt{147}-\dfrac{10}{\sqrt{3}}-3\sqrt{75}-2\sqrt{\tfrac{1}{3}}$,

and $\dfrac{2+\sqrt{3}}{\sqrt{2}+\sqrt{2+\sqrt{3}}}+\dfrac{2-\sqrt{3}}{\sqrt{2}-\sqrt{2-\sqrt{3}}}$. **(See page 80.)**

2. Prove that the product of any four consecutive even integers increased by 16 is a perfect square.

3. Shew how to obtain the sum and product of the two roots of a quadratic equation in the terms of the co-efficients and the last term.

If α and β are the roots of $ax^2+bx+c=0$, find the values of the sum and product of $a\alpha+b$ and $a\beta+b$.

4. Define *ratio* and *ratio of greater inequality*.

If $6x^2+6y^2=13xy$, what is the ratio of x to y?

If $a:b$ is a ratio of greater inequality, shew that $a:b$ is greater than $a^2+b^2:2ab$.

5. Define *direct*, *inverse* and *joint variation*; and give an illustration of each.

Given that $x+y$ varies as $z+\frac{1}{z}$, and $x-y$ varies as $z-\frac{1}{z}$, find the relation between x and z, if $z=2$, when $x=3$ and $y=1$. **(See page 116.)**

6. Find the nth term in an Arithmetical, a Geometrical, and an Harmonical Progression.

In a Geometrical progression, if the $(p+q)^{\text{th}}$ term is m, and the $(p-q)^{\text{th}}$ term is n, find the pth and qth terms.

7. Find the sum of an infinite Geometrical Progression, the common ratio being less than unity.

If S_1, S_2, S_3, . . . S_p are the sums of infinite Geometrical series, whose first terms are 1, 2, 3, . . . p, and whose common ratios are $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, . . $\frac{1}{p+1}$ respectively, prove that $S_1+S_2+S_3+\,..\,+S_p = \frac{1}{2}p(p+3)$.

8. Find the number of ways in which it is possible to make an arrangement of r things out of n, when in each permutation any of the things may be repeated once, twice, ... r times.

There are 3 candidates for a Professorship, and one is to be elected by the votes of 5 men; in how many ways can the votes be given?

(See page 289.)

9. Prove the Binomial Theorem, when the exponent is any negative quantity.

Find the $(r+1)^{\text{th}}$ term of $(1+x)^{-2}$ and $(1-nx)^{-\frac{1}{n}}$.

10. Find the greatest term in the expansion of $(1+x)^n$ when n is fractional and positive.

Find the greatest term in $\left(1+\frac{2}{3}\right)^{\frac{21}{2}}$.

1889.

1. If $\dfrac{1}{1+l+m}+\dfrac{1}{1+m+mn}+\dfrac{1}{1+n+nl}=1$,

prove that either $lmn=1$, or $(1+l)(1+m)(1+n)=-1$.

2. Prove that a factor can be found which will rationalise any binomial surd.

If $(x+\sqrt{x^2-bc})(y+\sqrt{y^2-ca})(z+\sqrt{z^2-ab})$

$$=(x-\sqrt{x^2-bc})(y-\sqrt{y^2-ca})(z-\sqrt{z^2-ab}),$$

shew that each of these expressions $=\pm abc$. **(See page 82.)**

3. When $n+2$ figures of a cube root have been obtained by the ordinary method, n more can be obtained by division only, supposing $2n+2$ to be the whole number.

If $x^6+3dx^5+ex^4+fx^3+gx^2+hx+k^3$ be a perfect cube find its cube root and determine the co-efficients e, f, g, h in terms of d and k.
(See page 47.)

4. If α, β are the roots of the equation $Ax^2+Bx+C=0$, then $\alpha+\beta = -\frac{B}{A}$ and $\alpha\beta = \frac{C}{A}$.

If α be a root of the equation $4x^2+2x-1=0$, prove that $4\alpha^3-3\alpha$ is the other root. **(See page 158.)**

5. If $a : b = c : d$, then $ma+nb : pa+qb = mc+nd : pc+qd$.

If $ab = cd = ef$, shew that

$$\frac{ac+ce+ea}{bdf(b+d+f)}+\frac{a^2+c^2+e^2}{d^2f^2+f^2b^2+b^2d^2}.$$ **(See page 100.)**

6. Find the sum of n terms of an Arithmetical Progression.

If s_1, s_2, s_3, denote the sum of n terms of three arithmetical series whose first terms are unity and their common differences in harmonic progression, prove that $n = \frac{2s_3s_1-s_1s_2-s_2s_3}{s_1-2s_2+s_3}$. **(See page 258.)**

7. Find the sum of n terms of the following series :—

$a, (a+b)r, (a+2b)r^2, (a+3b)r^3, \ldots\ldots\ldots\ldots$

If P, Q, R be the pth, qth, and rth terms of a Geometrical Progression, shew that $P^{q-r}.Q^{r-p}.R^{p-q} = 1$.

8. Find for what value of r the number of combinations of n things taken r at a time is greatest.

There are n points in a plane of which no three are in a straight line except m, which are all on a straight line. Find the number of triangles formed by joining the points. **(See page 277.)**

9. Write down the expansion of $(1-x)^{-n}$ and hence shew that

$$\sqrt{3}=1+\frac{1}{3}+\frac{1.3}{3.6}+\frac{1.3.5.}{3.6.9.}+\frac{1.3.5.7}{2.6.9.12}+\&c.$$ **(See page 350.)**

10. Find the sum of the co-efficients of the odd terms in the expansion of $(1+x)^n$ where n is a positive integer. **(See page 324.)**

If $c_0, c_1, c_2, c_3, \ldots\ldots c_n$ be the co-efficients in the expansion of $(1+x)^n$ where n is a positive integer, prove that

$$\frac{c_0}{1}+\frac{c_1}{2}+\frac{c_2}{3}+\frac{c_3}{4}+\ldots+\frac{c_n}{n+1}=\frac{2^{n+1}-1}{n+1}.$$

1890.

1. If $x = \frac{1}{2}(b+c-a)$, $y = \frac{1}{2}(c+a-b)$, $z = \frac{1}{2}(a+b-c)$, shew that $x^3+y^3+z^3-3xyz = \frac{1}{4}(a^3+b^3+c^3-3abc)$.

2. Find a factor which will rationalise a given binomial.

Bring $\frac{7}{2^{\frac{1}{2}}+2^{\frac{1}{4}}+1}$ to a form with a rational denominator. **(See page 70.)**

3. Find the sum and product of the roots of the quadratic $ax^2+bx+c=0$. When are the roots real and when imaginary ?

If x_1, x_2 be the roots of the equation $x^2+mx+m^2+n^2=0$, prove that $x_1^4+x_1^2x_2^2+x_2^4=n^2(2m^2+3n^2)$. **(See page 160).**

4. Define *proportion*, and if four quantities are in proportion, shew that the product of the means is equal to the product of the extremes.

If $(pa+qb+rc+sd)(pa-qb-rc+sd)$
$$=(pa-qb+rc-sd)(pa+qb-rc-sd),$$
shew that bc, ad, ps, qr are in proportion. **(See page 95.)**

5. Insert a given number of arithmetical means between two given quantities.

An Arithmetical Progression and an Harmonical Progression have the same first term, the same last term, and the same number of terms ; prove that the product of the rth term from the beginning in one series and the rth term from the end in the other is independent of r. **(See page 257.)**

6. Define *Geometrical Progression*, and obtain the relation which must hold among three quantities which are in Geometrical Progression.

From three numbers which are in *G. P.* three other numbers in *G. P.* are subtracted, and the remainders are found to be also in *G. P.* ; prove that the three series have the same common ratio.

7. Investigate the number of permutations of n things taken r at a time.

Find how many significant numbers can be formed by using the digits 0, 1, 2, 3, 4, but using each not more than once in any number.

8. Describe the method of proof called *Mathematical Induction*, and apply this method to prove that every even power of every odd number when divided by 8 leaves 1 for a remainder. **(See page 311.)**

9. The Binomial Theorem being proved for any positive integral value of the index, prove it for any negative integral value.

In the expansion of $\frac{3x-8}{4-4x+x^2}$ in ascending powers of x, prove that the co-efficient of x^4 is $-\frac{1}{4}$, and find the co-efficient of x^r. **(See page 347).**

10. Write down the general term in the expansion of $(1+x)^n$.

If a, b, c be three consecutive co-efficients in the expansion of a power of $1+x$, prove that the index of the power is $\frac{2ac+b(a+c)}{b^2-ac}$ and that the number of the term of which a is the co-efficient is $\frac{a(b+c)}{b^2-ac}$. **(See page 330.)**

1891.

1. Prove that if $x+y^{-1}=a$, $y+z^{-1}=b$, $z+x^{-1}=c$, then

$$\begin{aligned}(1-bc)x+(1-ab)x^{-1}+2b\\ &=(1-ca)y+(1-bc)y^{-1}+2c\\ &=(1-ab)z+(1-ca)z^{-1}+2a.\end{aligned}$$

2. Prove the rule for finding the L.C.M. of two Algebraical expressions.

If x^2+ax+b, $x^2+a'x+b'$ have an L. C. M. of the form x^3+px+q prove that $ab = a'b' = -aa'(a+a')$.

3. Show how to find the square root of a binomial of the form $a+\sqrt{b}$ where $\sqrt{b}$ is a surd.

Simplify $\dfrac{2\sqrt{5}-\sqrt{8}}{\sqrt{10}-2+\sqrt{(7-2\sqrt{10})}}$.

4. Find two independent relations between the roots and the co-efficients in a quadratic equation.

If y and z be the roots of the equation $A(x^2+m^2)+Axm+Cx^2m^2 = 0$, show that $A(y^2+z^2)+Ayz+Cy^2z^2 = 0$.

5. If $A \propto B$ when C is constant, and $A \propto C$ when B is constant, prove that $A \propto BC$ when both B and C vary.

If m sovereigns in a row stretch as far n pennies, and p sovereigns in a pile are as high as q pennies, compare the values of equal bulks of gold and copper, assuming that the area of a circle varies as the square of its radius.

6. Shew how to find the sum of n terms of an Arithmetical Progression.

If A_p, G_p be respectively the p^{th} terms of the series—

$$a,\ a+b,\ a+2b,\ \ldots\ldots$$
$$a,\ ar,\ ar^2,\ \ldots\ldots$$

find the sum of the first n terms of the series whose p^{th} term is $A_p\,G_p$.

7. When are three quantities said to be in *Harmonical Progression* ? Prove that the reciprocals of such quantities are in Arithmetical Progression.

If p, q, r be in Arithmetical Progression, prove that

$\dfrac{qr}{pq+pr}$, $\dfrac{rp}{qp+qr}$, $\dfrac{pq}{rp+rq}$ are in Harmonical Progression.

8. Explain the method of proof known as *Mathematical Induction.*

Shew by this method that the sum of n terms of the series,

$1+3+6+10+15+\ldots\ldots$ is $\frac{1}{6}n(n+1)(n+2)$. **(See page 310).**

9. Find an expression for the number of combinations of n dissimilar things taken r at a time.

There are m men and n monkeys, n being greater than m ; find the number of ways in which each man may become the owner of one monkey. If a man may have any number of monkeys, in how many ways may every monkey have a master ? **(See pages 267 and 290).**

10. Assuming the *Binomial Theorem* for positive exponents, shew it to be true for any exponent.

Shew that the co-efficient of x^r in the expansion of $\dfrac{(1+3x)^2}{(1+2x)^3}$ is $(-2)^{r-3}.(r-8)$.

1892.

1. If $a+b+c = 0$, prove that $a^5+b^5+c^5 = -5abc(ab+ca+ab)$.

2. In any equation which involves rational quantities and quadratic surds, the rational parts on each side are equal, and also the irrational parts.

Shew that $\sqrt[3]{(6\sqrt{3}+10)} - \sqrt[3]{(6\sqrt{3}-10)} = 2$.

3. Solve the equation $(x+2)(3x+1)(x-1)(3x+2) = 224$.

4. Shew how to resolve ax^2+bx+c into factors of the first degree in x. What will the factors be when $b^2 = 4ac$?

Find the values of a which make the expression $x^2-ax+1-2a^2$ always positive for real values of x.

5. If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f}$, prove that each of these ratios is equal to

$$\frac{pa+qc+re}{pb+qd+rf}.$$

If $\dfrac{x-\frac{yz}{x}}{1-yz} = \dfrac{y-\frac{zx}{y}}{1-zx}$, and x and y be unequal, then each of these ratios is equal to $x+y+z$ or $x^{-1}+y^{-1}+z^{-1}$.

6. Find the sum of the squares of the first n natural numbers.

Shew that $1+2^2+3+4^2+5+6^2+ \dots$ to n terms

$$= \tfrac{1}{12}(n+1)(2n^2+n+3) \text{ or } \tfrac{1}{12}n(2n^2+9n+4),$$ according as n is odd or even.

7. Shew how to insert a number of harmonic means between two given terms.

If x, y, z be in A. P., ax, by, cz in G. P., and a, b, c in H. P., then the harmonic mean of a and c is to the Geometric mean of a and c as the harmonic mean of x and z is to the Geometric mean of x and z.

8. Find the number of permutations of n things taken all together, of which p are of one sort, q of another, and the rest all different.

How many combinations and how many permutations can be made with the letters of the word *parabola* taken three at a time ?

9. Describe carefully the method of *Mathematical Induction*, and apply it to prove the Binomial Theorem for a positive integral exponent

Shew that $2^{4n} - 2^n(7n+1)$ is some multiple of the square of 14, where n is a positive integer greater than unity.

10. Find in its simplest form the general term in the expansion of $(1+x)^{-n}$.

If s = the sum of two quantities, p = their product, q = their quotient, shew that

$$p^2 = s^4\left(q^2-4q^3+\frac{4.5}{1.2}q^4-\frac{4.5.6}{1.2.3}q^5+\dots\dots\right)$$

1893.

1. If $a^2+a+1=0$, shew that $x^3-1=(x-1)(x-a)(x-a^2)$.

2. Prove that if two algebraical expressions have a common factor, the factor will divide the difference of the expressions.

The expressions x^2+6x+a and $x^2+12x+3a$ have a common factor; what numerical values can a have?

3. From an equation whose roots are α, β. How can you tell, without solving, whether the roots, supposed real, of a quadratic equation are positive or negative?

Prove that the positive root of $x^2-8x-8=0$ is greater than 8.

4. If a, b, x be positive quantities of which a is greater than b, prove that the ratio $a+x : b+x$ is less than the ratio $a : b$.

A is 24 years old, B is 15 years old; what is the least number of years after which the ratio of their ages will be less than $7 : 5$?

5. If A vary as B when C is constant and vary as C when B is constant, prove that A varies as BC when both B and C vary.

Supposing that the velocity of a steamer varies inversely as the area of its greatest section when the tonnage is constant, and inversely as the tonnage when the area is constant, and that a steamer whose section is 200 sq ft. and tonnage 1000, goes 15 miles per hour, find the velocity of a steamer whose section is 250 sq. ft. and tonnage 1200.

6. Find an expression for the sum of n terms of a geometrical progression and prove that in the continued product

$(1+x+x^2+\;.\;.\;.\;.\;+x^{2n})(1-x+x^2-\;.\;.\;.\;.\;+x^{2n})$ the co-efficients of odd powers of x are zero and of even powers unity.

7. Insert n harmonic means between a and b.

If $\dfrac{a-x}{px}=\dfrac{a-y}{qy}=\dfrac{a-z}{rz}$, and p, q, r be in A. P., shew that x, y, z are in H. P.

8. Explain the method of *Mathematical Induction.* By this method prove that either 2^n+1 or 2^n-1 is divisible by 3, where n is a positive integer.

9. Find the number of combinations of n things taken r at a time.

A gentleman invites a party of $m+n$ friends to dinner and places m at one round table and n at another; find the number of ways in which he can arrange them among themselves. **(See page 299.)**

10. Assuming the Binomial theorem to be true for a positive exponent, prove it for a negative exponent.

Prove that the co-efficient of x^n in the expansion of

$(1-9x+20x^2)^{-1}$ is $5^{n+1}-4^{n+1}$.

PUNJAB UNIVERSITY INTERMEDIATE EXAMINATION PAPERS.

1887.

1. Solve the equations :—

(i) $\left.\begin{aligned} 4x^2+7y^2 &= 148 \\ 12(x^2+y^2) &= 25xy \end{aligned}\right\}$; **(See page 179.)**

(ii) $x+y+3\sqrt{x+y} = x^2+y^2 = 10.$ **(See page 202.)**

2. The termini of a railway 126 miles long are at A and C, and the station B, at which a certain train stops 15 minutes, is 70 miles from A. The whole journey from A to C takes 15 minutes less than twice as long as the journey from A to B. Determine the average rate of the train, including all stoppages except that at B.

3. Prove that if the same quantity be added to the antecedent and consequent terms of a ratio of lesser inequality the ratio is increased.

A man engages a servant for 18 days in the month of January at Rs. 16 per mensem ; in paying him his wages, the man gives him $\frac{19}{32}\left(\text{instead of } \frac{18}{31}\right)$ of the month's pay. How much does he pay in excess of the stipulated amount ?

4. Prove the Binomial Theorem for positive integral indices. Write down the co-efficient of $\frac{1}{y^2}$ in the expansion of $\left(y+\frac{c^3}{y^2}\right)^{10}$.

1888.

1. Solve the equations :—

$$\text{(i)} \quad \frac{2x-b}{b} = \frac{2y+a}{b} = \frac{3x+y}{a+2b};$$

$$\text{(ii)} \quad \left(\frac{x-a}{x+a}\right)^2 - 5\left(\frac{x-a}{x+a}\right)+6 = 0.$$

2. Find the sum of n terms in a Geometrical Progression.

Find the sum of the following series :—

$$1-\frac{2}{3}+\frac{4}{9}-\&c. \ldots \text{ to 6 terms.}$$

$$4+\cdot8+\cdot16+\&c. \ldots \text{ to infinity.}$$

3. Enunciate and prove the exponential Theorem. Prove that the limit when n is infinite of $\left(1+\frac{x}{n}\right)^n$ is e^x.

1889.

1. Solve the equations :—

$$\text{(i)}\quad \left.\begin{aligned} \frac{2}{x}+\frac{7}{y} &= 28 \\ \frac{5}{x}-\frac{6}{y} &= 2 \end{aligned}\right\};$$

$$\text{(ii)}\quad \left.\begin{aligned} bx+ay &= a^2+b^2 \\ \frac{x^2}{a^2}+\frac{y^2}{b^2} &= \frac{b^2}{a^2}+\frac{a^2}{b^2} \end{aligned}\right\}.$$

2. What must be added to $a^2x^2+abxy+c^2y^2$ in order to make it a perfect square ?

3. Find the sum of any number of terms of an arithmetical Progression. How many terms of the series $5+7+9+$&c. must be taken in order that the sum may be 480 ?

4. Prove the Binomial Theorem for positive integral indices. Show that the co-efficient of x^n in the expansion of $(1+x)^{2n}$ is double the co-efficient of x_n in the expansion of $(1+x)^{2n-1}$.

5. What is logarithm and a mantissa ? How is the characteristic of a logarithm determined by inspection ?

1890.

1. Find the sum of the roots of the equation $ax^2+2bx+c=0$.

If a and β be the roots of the equation $ax^2+2bx+c=0$, form the equation whose roots are $\frac{a+\beta}{a\beta}$ and $\frac{1}{a\beta}$.

2. Solve the equations :—

(i) $\sqrt{x+2}+\sqrt{x-3} = 5$;

(ii) $\frac{3+2x}{1+2x}-\frac{5+2x}{7+2x} = 1-\frac{4x^2-2}{7+16x+4x^2}$.

3. Find the sum of n terms of a Geometrical series, and show that, when the number of terms is even, the product of two terms equidistant from the beginning and end is equal to the product of the two middle terms.

Sum the series :— $2+\sqrt{2}+1+$&c. to n terms.

4. Find the number of permutations of n things taken r at a time.

A company of 80 men are to be selected from a regiment of 900, find the number of ways in which it can be done so that the same ten men may be always included.

Prove that

$a^x = 1+(\log a)x+\frac{(\log a)^2}{\underline{|2}}x^2+$&c. In what base is log a taken ?

1891.

1. Trace the changes in the value of ax^2+bx+c as x varies from a very large positive to a very large negative quantity.

Construct a quadratic equation with rational co-efficients of whose roots one will be $p+\sqrt{q}$.

2. Solve the equations :—

(i) $\dfrac{x-7}{x-9}-\dfrac{x-9}{x-11}=\dfrac{x-13}{x-15}-\dfrac{x-15}{x-17}$;

(ii) $3\sqrt{x}-1=\dfrac{5}{3\sqrt{x}+7}+6$;

(iii) $\dfrac{x-a}{x-b}+\dfrac{x-b}{x-a}=\dfrac{a}{b}+\dfrac{b}{a}$.

3. Show how to insert two harmonical means between a and b.

Sum the series :—

(i) $\dfrac{1}{\sqrt{3}}, 1, \dfrac{3}{\sqrt{3}}$, &c., to 18 terms ;

(ii) $(a+x)^2$, a^2+x^2, $(a-x)$, &c., to n terms.

4. Prove the Binomial Theorem for any exponent.

Expand $(4+3x)^{\frac{5}{2}}$ to four terms, and write down the general term.

1892.

1. Prove that the algebraical expression $x^2+2bx+c$ is greater than, equal to, or less than $(x+b)^2$ according as the roots of the equation $x^2+2bx+c=0$ are imaginary, equal or real.

Form the equation whose roots are $\dfrac{1}{3+\sqrt{5}}, \dfrac{1}{3-\sqrt{5}}$.

2. Solve the equations :—

(i) $2\sqrt{4x+5}-\sqrt{8x-4}=\sqrt{2x+11}$;

(ii) $\dfrac{a}{x+a}+\dfrac{b}{x+b}=\dfrac{a-c}{x+a-c}+\dfrac{b+c}{x+b+c}$.

3. Show how to insert n Geometrical means between a and b.

If a, b, c be in Geometric Progression, and x, y be the Arithmetic means between a, b and b, c respectively, prove that

$$\frac{2}{b}=\frac{1}{x}+\frac{1}{y}\text{, and } 2=\frac{a}{x}+\frac{c}{y}.$$

4. Find the number of permutations of n things taken all together, which are not all different.

A person has 15 acquaintances of whom 5 are relatives. In how many ways may he invite 13 guests from among them so that 3 of these may be relatives ?

5. Write down the two middle terms of $(a+x)^{2n+1}$.

'6. Prove that

$$\log_e \frac{n+1}{n} = 2\left\{\frac{1}{2n+1} + \frac{1}{3(2n+1)^3} + \frac{1}{5(2n+1)^5} + \&c.\right\}.$$

1893.

1. (*a*) Prove that a quadratic equation cannot have more than two *roots.*

(*b*) What do you understand by the *roots* being *real*, *imaginary* or *impossible*, and state the conditions of each.

2. Solve the following equations :—

(*a*) $x^2 + \sqrt{x^2 - 5} = 11$;

(*b*) $\dfrac{2x(a-x)}{3a-2x} = \dfrac{a}{4}$;

(*c*) $x^3 - 3x = 2$.

3. When are quantities said to be—

(*a*) in Arithmetical Progression ;

(*b*) in Geometrical Progression ;

(*c*) in Harmonical Progression ?

4. If there are 6 terms, prove algebraically—

(*a*) That in the case of Arithmetical Progression the sum of 1st and last is equal to the sum of 3rd and 4th.

(*b*) That in the case of Geometrical Progression the product of 1st and last is equal to the product of 3rd and 4th.

5. At an Election where every voter may vote for any number of candidates not greater than the number to be voted, there are 6 candidates and 3 members to be chosen ; in how many ways may a man vote ?

6. Find the relation between the logarithms of the same number to different bases.

ALLAHABAD UNIVERSITY INTERMEDIATE EXAMINATION PAPERS.

1889.

1. Solve the equations :—

(*a*) $x(x+1) + 3\sqrt{2x^2+6x+5} = 2(12-x)+1$;

(*b*) $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 3x,\quad x^{\frac{1}{2}} + y^{\frac{1}{3}} = x$;

(*c*) $x(y+z) = 100,\quad y(z+x) = 144,\quad z(x+y) = 154.$

2. Transform the equation $x^4 - 4x^3 + 13x^2 - 18x - 52 = 0$ into one whose roots shall be the same except that each is increased by unity, and by this means solve the original equation.

3. The Geometrical mean between a and b is to their Arithmetical mean as m is to n; show that

$$a : b :: n+\sqrt{n^2-m^2} : n-\sqrt{n^2-m^2}.$$

4. Out of 7 white and 8 black sailors, 5 are to be selected for a boat's crew, which must always consist of three white and 2 black men; in how many ways may the crew be formed?

5. Expand by the Binomial Theorem to 5 places—

$$\frac{1}{(1+\sqrt[3]{x})^6}.$$

6. State and prove the Exponential Theorem.

7. Find the square root of $12-6\sqrt{3}$.

8. Water is admitted into a cistern by three cocks, two of which are exactly equal. When they are all open, five-twelfths of the cistern is filled in four hours; and if one of the equal cocks is stopped, seven-ninths of the cistern is filled in ten hours and forty minutes. In how many hours would each cock fill the cistern?

1890.

1. Show how to determine whether the roots of the equation $ax^2+bx+c=0$ are real, equal, or impossible.

If α, β are the roots of the equation $x^2+px+q=0$, then $\alpha+\frac{1}{\beta}$ and $\beta+\frac{1}{\alpha}$ are the roots of the equation $qx^2+p(1+q)x+(1+q)^2=0$.

2. Solve the equations :—

$$\text{(i)}\quad \frac{x+2}{x-2}+\frac{2x-3}{2(x-1)}=\frac{23}{6};$$

$$\text{(ii)}\quad \left.\begin{aligned} xy+\frac{x}{y} &= 10 \\ xy^2-x &= 6y \end{aligned}\right\}.$$

3. If $a : b :: c : d$, prove that

$$a^2+c^2 : b^2+d^2 :: ac : bd.$$

4. Shew how to find a Harmonic mean between a and b. If $2(y-a)$ is a Harmonic mean between $y-x$ and $y-z$, then $x-a$, $y-a$, $z-a$, form a Geometrical Progression. **(See page 258.)**

5. Find the total number of permutations of n things taken all together when there are p of one kind, q of another kind, and the rest are unlike.

6. Show that when a Binomial is expanded, terms equally distant from the beginning and the end have the same co-efficients.

Write down the two middle terms of the expansion of $(a-b)^9$.

1891.

1. Prove that $x^4+ax^3+bx^2+cx+d$ will be a perfect square for all values of x, if $(a^2-4b)^2=64d$, and $c^2=a^2d$.

2. Solve the equations :—

(i) $3(1+x+x^2) = \sqrt{21(1+x^2+x^4)}$;

(ii) $\left.\begin{aligned} xy+x+y &= 27 \\ \frac{1}{x}+\frac{1}{y} &= \frac{1}{3} \end{aligned}\right\}$

3. A man travels 84 miles, and finds that he could have made the journey in 5 hours less if he had travelled 5 miles an hour faster ; at what rate did he travel ?

4. Find the sum of a Geometrical Progression in terms of the first term, the last term, and the common ratio.

The sum of 3 quantities in Geometrical Progression is $24\frac{4}{5}$, and their product is 64 ; find them. **(See page 250.)**

5. How many different words can be made out of the letters which form the word *Allahabad* ? In how many will the vowels occupy the even places ? **(See page 287.)**

6. In the expansion of $(1+x)^n$ where n is a positive integer, both the sum of the co-efficients of the odd terms and the sum of the co-efficients of the even terms are equal to 2^{n-1}.

Find the greatest term in the expansion of $(1+5x)^9$, when $x = \frac{1}{4}$.

1892.

1. If α, β be the roots of $ax^2+bx+c = 0$, form the quadratic whose roots are $\frac{1}{\alpha+\beta}$ and $\frac{1}{\alpha}+\frac{1}{\beta}$.

Prove that $(y-1)(y-3)(y-4)(y-6)+10$ is positive for all real values of y.

2. Solve, giving all the roots :—

(i) $\frac{a-x}{\sqrt{a}+\sqrt{a-x}}+\frac{a+x}{\sqrt{a}+\sqrt{a+x}} = \sqrt{a}$;

(ii) $\left.\begin{aligned} (1+x)(1+y) &= 10 \\ x^2y+xy^2 &= 18 \end{aligned}\right\}$;

(iii) $xyz = \frac{x+y}{3} = \frac{y+z}{4} = \frac{z+x}{5}$.

3. Insert four Harmonical means between 1 and 30.

In a Harmonical progression if the p^{th} term $= qr$, the q^{th} term $= pr$, prove that the r^{th} term $= pq$.

4. Write down the n^{th} terms and sum the following series :—

(i) $1+\frac{3}{2}+\frac{5}{4}+\frac{7}{8}+$&c. to n terms ;

(ii) $1^3+2^3+3^3+4^3+$&c. to n terms ; **(See page 231.)**

(iii) $2.1^2+3.2^2+4.3^2+5.4^2+$&c. to n terms.

5. Find the number of permutations of n letters all together, when p of them are of one sort, q of another sort, and the rest all different.

6. If the r^{th} term in the expansion of $(x+1)^{20}$ has its co-efficient equa to that of the $(r+4)^{th}$ term, find r.

7. State (without proof) the Exponential Theorem and assuming its truth, prove that

$$\log(1+x) = x+\frac{x^2}{2}+\frac{x^3}{3}+\&c. \text{ to infinity.}$$

1893.

1. Eliminate a, b, c from the following equations :—

$$bz+cy = a,\ cx+az = b,\ ay+bx = c.$$

2. Solve the following equations :—

(i) $$\frac{ax+b}{cx+b}+\frac{bx+a}{cx+a} = \frac{(a+b)(x+2)}{cx+a+b};$$

(ii) $$\frac{x}{b+y} = \frac{y}{a+x},\ ax+by = (x+y)^2.$$

3. Find the sum of any number of terms of a series in Arithmetical progression of which two particular terms are known.

4. Investigate the order of magnitude of the Arithmetical, Geometrical and Harmonical means between two given numbers.

5. Prove that the number of combinations of n things taken r together is equal to the number taken $n-r$ together.

Prove that if the number of combinations of n things taken r together be equal to the number taken r together, either $r = s$, or $r+s = n$.

6. Take any number, the one next to it, and a third equal to the product of the first two. Add together the squares of the three numbers and prove that the result will always be a perfect square, whatever the number you choose to start with.

7. Prove the Exponential Theorem.

1894.

1. (*a*) Write down the roots of the equation $ax^2+bx+c = 0$ and show when the roots are (1) real and unequal, (2) real and equal, (3) imaginary, and (4) rational and unequal.

(*b*) Solve the equations :—

(i) $\sqrt{(x^2-8x+15)}+\sqrt{(x^2+2x-15)} = \sqrt{(4x^2-18x+18)}$;

(ii) $(x^2+y^2)(x+y) = 272, \quad (x^2-y^2)(x-y) = 32.$

2. (*a*) Deduce the formula for the sum of n terms of an Arithmetical progression of which the first term is a and common difference d.

(*b*) On the ground are placed a basket and 12 stones in a straight line. The first stone is one yard from the basket, the second stone is

3 yards from the first, the third stone 5 yards from the second, and so on, the distances between the stones increasing in Arithmetical progression. How many yards will a man run, who starting from the basket, picks up the stones one by one, returning each time he picks up a stone to deposit it in the basket?

3. (*a*) Find for what value of r the number of combinations of n things taken r at a time is the greatest.

(*b*) A police post, consisting of 5 mounted men and 9 foot policemen, has to furnish a daily guard consisting of 2 from each class. How many days will elapse before the same guard recurs after all possible selections have been made?

4. (*a*) In the expansion of $(1+x)^n$ prove that the sum of the co-efficients of the odd terms = the sum of the co-efficients of the even terms $=2^{n-1}$.

(*b*) Find the co-efficient of x^6 in the expansion of $(1+2x)^{\frac{5}{2}}$.

(*c*) Having given

$$\log 2 = \cdot 30103,$$
$$\log 3 = \cdot 47712,$$
$$\text{and } \log 7 = \cdot 84510,$$

solve the equations $2^x 7^y = 80{,}000$; $3^y = 500$; the values of x and y to be given correct to 5 decimal places.

ANSWERS TO UNIVERSITY PAPERS.

Cal. 1888.

1. (i) 31 or 17 ; (ii) $\left.\begin{array}{l} x = 14 \\ y = -29 \end{array}\right\}$, $\left.\begin{array}{l} x = 2 \\ y = -1 \end{array}\right\}$;

(iii) $x = 2$ or $-3 \pm \sqrt{-15}$.

Cal. 1890.

1. (i) -24, 37 ; (ii) $\left.\begin{array}{l} x = 3 \\ y = 2 \end{array}\right\}$, $\left.\begin{array}{l} x = -3 \\ y = -2 \end{array}\right\}$,

$$\left.\begin{array}{l} x = \sqrt{\dfrac{195}{7}} \\ y = 5\sqrt{\dfrac{15}{7}} \end{array}\right\}, \quad \left.\begin{array}{l} x = -\sqrt{\dfrac{195}{7}} \\ y = -5\sqrt{\dfrac{15}{7}} \end{array}\right\};$$

(iii) $(C_1A_2 - C_2A_1)^2 = (A_1B_2 - A_2B_1)(B_1C_2 - B_2C_1)$.

4. $\sqrt[3]{\dfrac{3}{2}}$.

Cal. 1891.

1. (α) $6\frac{1}{2}$ or $-1\frac{1}{3}$; (β) $\left.\begin{array}{l} x = \dfrac{2ab}{\pm\sqrt{a^2+4b^2}+a} \\ y = \dfrac{2ab}{\pm\sqrt{a^2+4b^2}-a} \end{array}\right\}$.

Cal. 1892.

1. (i) $\frac{12}{5}$ or $-\frac{17}{7}$;

(ii) $\left.\begin{array}{l} x = \pm 1 \\ y = \pm 3 \end{array}\right\}$, $x = \pm\dfrac{13}{\sqrt{21}}$, $y = \pm\dfrac{2}{\sqrt{21}}$.

4. 10, 8, $6\frac{2}{5}$, &c. **5.** $\dfrac{\underline{|2n}}{(\underline{|n})^2}x^n$.

Cal. 1893.

1. $\dfrac{2b}{a^2}(3c-4b)$; $-\dfrac{2b}{a^6}\{4(4b^2-ac)^2 - ac(4b^2-2ac) - 4a^2b^2\}$.

3. $\dfrac{\underline{|2n}}{(\underline{|n})^2}x^n$. **5.** 1·648.

Cal. 1894.

1. (i) $5\frac{1}{2}$ or $-3\frac{2}{5}$; (ii) $\frac{25}{147}$ or 27. **3.** 1728 ; 25.

Cal. 1895.

2. (i) $\dfrac{-(a+b+c)\pm\sqrt{a^2+b^2+c^2-2ab-2ac-2bc}}{2}$ or 0 ;

(ii) $\left.\begin{array}{l} x = 0 \\ y = 0 \end{array}\right\}, \quad \left.\begin{array}{l} x = 2 \\ y = 3 \end{array}\right\}.$

Cal. 1896.

1. (i) $-2\pm2\sqrt{2}, 1, -5$; (ii) $3, -1, 1\pm\sqrt{30}$.

2. $x^2+x-30=0$. **4.** (ii) $2(1+6x+x^2)$.

Cal. 1897.

2. (i) -5 or 3 ; (ii) $\left.\begin{array}{l} x = 2 \\ y = 1 \end{array}\right\}, \quad \left.\begin{array}{l} x = \frac{1}{2} \\ y = \frac{1}{8} \end{array}\right\}.$

4. $a+(n+1)b$. **6.** $\dfrac{6\sqrt{2}\{1.3.5.\ \dots \text{ to } (r-2) \text{ terms}\}}{4^r \underline{|r}}$.

Cal. 1898.

1. (i) $\dfrac{a+b+2(\sqrt{ab}+2)}{a+b-2(\sqrt{ab}+2)}$ or 1 ; (ii) $x = \pm4, y = \pm9$.

3. $\dfrac{(b-2a)\pm\sqrt{(b-2a)^2+8sb}}{2b}$. **4.** (i) $\frac{n}{6}(2n^2+3n+7)$. **7.** 1·04139.

Cal. 1899.

1. (i) $2a$ or b ; (ii) $\left.\begin{array}{l} x = \pm1 \text{ or } \pm3 \\ y = \pm\frac{1}{3} \text{ or } \pm1 \end{array}\right\}.$ **2.** $x^2-x+2=0$.

3. (2) $-\cdot75$; $-\cdot5$; $-\cdot1\dot{6}$. **4.** 36.

5. (i) $\dfrac{(n+1)(2n+1)(3n+1)\ \dots\ \{(r-1)n+1\}}{n^r \underline{|r}}x^r$; (ii) 1·006578.

Cal. 1900.

1. $x^2+2(2q-p^2)x+p^2(4q-p^2)=0$.

2. (i) ±5 ; (ii) $\left.\begin{array}{l} x = \pm3 \text{ or } \pm2 \\ y = \pm2 \text{ or } \pm1 \end{array}\right\}.$

Cal. 1901.

1. $\pm(a+\sqrt{-1})$; -1 or $\dfrac{1\pm\sqrt{-3}}{2}$.

2. (1) $-\frac{a}{3}, a, \frac{a}{2}$; (2) $\left.\begin{array}{l} x = 9 \text{ or } 1 \\ y = 1 \text{ or } 9 \end{array}\right\}.$ **5.** $1-4x+13x^2$.

Cal. 1902.

1. $a^2c^2x^2+(2ac-b^2)(a^2+c^2)x+(2ac-b^2)^2=0$.

2. (1) $3a$ or $-4a$; (2) $\left.\begin{array}{l} x=0,\ 3,\text{ or } 9 \\ y=0,\ 9,\text{ or } 3 \end{array}\right\}$.

Cal. 1903.

1. (1) $b=0$; (2) $c=a$; (a) $bx^2-2ax+a=0$.

2. (1) 1 ; (2) $\left.\begin{array}{l} x=3 \\ y=1 \end{array}\right\}$, $\left.\begin{array}{l} x=1 \\ y=3 \end{array}\right\}$, $\left.\begin{array}{l} x=-3 \\ y=-1 \end{array}\right\}$, $\left.\begin{array}{l} x=-1 \\ y=-3 \end{array}\right\}$.

3. $\dfrac{n(n+1)(2n+7)}{6}$.

Cal. 1904.

1. $b=0$; (a) the expression under the radical sign $=-(ab'-a'b)^2$, which shews that x is real only when $ab'-a'b=0$.

2. (1) $\dfrac{-1\pm\sqrt{-3}}{2}$, $\dfrac{a\pm\sqrt{a^2-4}}{2}$;

   (2) $\left.\begin{array}{l} x=0 \\ y=0 \end{array}\right\}$, $\left.\begin{array}{l} x=\pm\dfrac{1}{a}\sqrt{a^2+1} \\ y=\pm\sqrt{a^2+1} \end{array}\right\}$.

4. $\dfrac{\underline{|m+n+p+q}}{\underline{|m}\,\underline{|n}\,\underline{|p}\,\underline{|q}}$. 5. 2 or 7.

Cal. 1905.

1. $2(a-x)(x+\sqrt{x^2+b^2})=a^2+b^2-(a-x)^2-(x^2+b^2)$ $+2(a-x)\sqrt{x^2+b^2}=a^2+b^2-\{(a-x)-\sqrt{x^2+b^2}\}^2$, which shews &c. *Otherwise*, putting m for $2(a-x)(x+\sqrt{x^2+b^2})$, we get $4x^2(m-b^2)$ $-4ax(m-2b^2)+m^2-4a^2b^2=0$; whence, the expression under the radical sign in the value of $x=m^2(a^2+b^2-m^2)$; hence, &c.

2. (1) a, $-9a$, $(-4\pm\sqrt{-15})a$;

   (2) $\left.\begin{array}{l} x=1 \\ y=\frac{1}{2} \end{array}\right\}$, $\left.\begin{array}{l} x=-1 \\ y=-2 \end{array}\right\}$.

4. $\dfrac{\underline{|2n}}{n^2}$ or $\dfrac{\underline{|2n}}{4n^2}$, according as every two arrangements in which all the persons sitting at a round table have the same neighbours, be counted as two or one.

Cal. 1906.

2. (1) (i) a, $\dfrac{1}{a}$; (iii) 1, -3, $-\dfrac{7}{5}$;

   (ii) $\left.\begin{array}{l} x=3 \\ y=6 \end{array}\right\}$, $\left.\begin{array}{l} x=6 \\ y=3 \end{array}\right\}$. (2) -60.

Cal. 1907.

2. (1) $0, \dfrac{2ab}{b-a}$; (2) $x = \pm\dfrac{a^2}{\sqrt{a^2+b^2+c^2}}$, &c.

3. 6. **5.** $1+\dfrac{1}{4}x+\dfrac{1.3}{4.8}x^2+\dfrac{1.3.5}{4.8.12}x^3+\ldots\ldots\ldots$

$+\dfrac{1.3.5.7\ldots.(2r-1)}{4.8.12.16\ldots.4r}x^r+$&c.

Cal. 1908.

2. $ax^2+(2a+b)x+(a+b+c) = 0.$

3. (1) $1, \dfrac{-1\pm\sqrt{-3}}{2}$; (2) $1, -2, \dfrac{-1\pm\sqrt{-19}}{2}$;

(3) $\left.\begin{array}{l} x = 4 \\ y = 2 \end{array}\right\}$, $\left.\begin{array}{l} x = 2 \\ y = 4 \end{array}\right\}$. **4.** $p+q-m.$

6. $1-\dfrac{1}{3}x-\dfrac{1.2}{3^2\underline{|2}}x^2-\dfrac{1.2.5}{3^3\underline{|3}}x^3-\ldots\ldots-\dfrac{1.2.5.8.11\ldots\ldots.(3r-4)}{3^r\,\underline{|r}}x^r-$&c.;

$\cdot 999.$

Cal. 1909.

2. (i) $\left.\begin{array}{l} x = a \\ y = b \end{array}\right\}$; (ii) $\left.\begin{array}{l} x = 1 \\ y = 2 \end{array}\right\}$, $\left.\begin{array}{l} x = -\frac{1}{3} \\ y = -\frac{2}{3} \end{array}\right\}$.

3. 120. **4.** (i) $\dfrac{3.7.11.15.19.23.27}{4.8.12.16.20.24.28}\cdot 5^7$;

(ii) $-\dfrac{3}{4}-\dfrac{5}{4.3^8}$. **5.** (i) -238; (ii) 1.

Mad. 1881.

1. $-\frac{1}{2}, -3, 1.$ **6.** $2^n.$

Mad. 1882.

1. $0, 1, \left(\dfrac{p^2+3q^2}{p^2-q^2}\right)^{12}$. **2.** $a^2+b^2+c^2-2ab-2bc-2ca$ not less than zero.

5. $\dfrac{\underline{|192}\,.\,\underline{|16}}{(\underline{|48}\,.\,\underline{|4})^4}$.

Mad. 1883.

1. $\left.\begin{array}{l} x = 0, -\dfrac{32a}{17}, -a \\ y = 0, -\dfrac{8a}{17}, a. \end{array}\right\}$.

Mad. 1884.

4. $\frac{ac(n-1)}{a(m-1)-c(m-2)}$. 5. (*b*) $\frac{\lfloor\underline{3n}}{\lfloor\underline{2n}\,\lfloor\underline{n}} \cdot \frac{1}{x^n}$.

Mad. 1885.

1. (*b*) $\frac{x(a-b)(b-c)(c-a)-3abc}{(a-b)(b-c)(c-a)(x-a)(x-b)(x-c)}$.

2. (*a*) $0, -5, \frac{-15 \pm \sqrt{265}}{6}$; (*b*) $\left.\begin{array}{l} x = 7, -7 \\ y = 3, -3 \end{array}\right\}$.

4. 8 days. 6. $-220\, x^2 y^2$.

Mad. 1886.

2. (*b*) (i) $0, \frac{1}{3}, -2$; (ii) $\left.\begin{array}{l} x = \pm 3 \\ y = \pm 2 \end{array}\right\}$, $x = \pm\frac{2\sqrt{6}}{\sqrt{11}}$, $y = \pm\frac{3\sqrt{6}}{\sqrt{11}}$.

4. (2) $\frac{1}{3}$. 6. 1·00039857.

Mad. 1887.

1. $2(1+x)(1+y)(1-xy)(x-y)$. 2. (1) ± 2 or $\pm\sqrt{2}$.

5. 560. 6. $2^{r-3}\left\{ 9r^2 + 15r + 8 \right\}$.

Mad. 1888.

1. (*a*) $(a+b)(b+c)(c+a)(a+b+c)$.

Mad. 1889.

2. (*a*) $\left.\begin{array}{l} x = \pm 1 \\ y = 2 \end{array}\right\}$, $\left.\begin{array}{l} x = \pm \frac{5}{3} \\ y = \frac{2}{3} \end{array}\right\}$; (*b*) $x = 220$, $y = 165$.

3. $xy^{-\frac{1}{2}} - \frac{1}{2}x^{\frac{1}{4}}y^{\frac{1}{4}} + y^{\frac{1}{2}}$.

6. $\frac{3}{\lfloor\underline{r}}\left\{1.\,3.\,5\ldots\ (2r-5)\right\}a^{\frac{4}{3}-r}x^r$.

Mad. 1890.

2. (i) $1, \sqrt{2}+\sqrt{3}+2$; (ii) $\left.\begin{array}{l} x = 3 \\ y = -3 \end{array}\right\}$, $\left.\begin{array}{l} x = -1 \\ y = 1 \end{array}\right\}$.

6. $\frac{1.\,3.\,5\ldots \text{ to } \frac{r}{2} \text{ terms}}{2^{\frac{r}{2}}\,\lfloor\underline{\tfrac{r}{2}}}$, when r is even;

$\frac{1.\,3.\,5\ldots \text{ to } \frac{r-1}{2} \text{ terms}}{2^{\frac{r-1}{2}}\,\lfloor\underline{\tfrac{r-1}{2}}}$, when r is odd.

Mad. 1891.

1. x^2-4x+1.   **3.** (1) $\pm\sqrt{a+b}$;

(2) $\left.\begin{aligned} x &= -1 \\ y &= -1 \end{aligned}\right\}$, $\left.\begin{aligned} x &= \tfrac{7}{5} \\ y &= -\tfrac{1}{5} \end{aligned}\right\}$, $\left.\begin{aligned} x &= -\tfrac{1}{5} \\ y &= \tfrac{7}{5} \end{aligned}\right\}$.

4. 7560.

Bom. 1882.

2. $\sqrt{\frac{2a+b}{2}}+\sqrt{\frac{b}{2}}$.

Bom. 1883.

3. $7, \frac{8}{3}, \frac{13\pm\sqrt{73}}{6}$.   **5.** (i) 710; (iii) $1\frac{2}{7}$.   **6.** 60.

7. $-\frac{1.3.5.\ldots.(2r-3)}{2^r a^{2r-1} \lfloor r}x^{2r}$ 21.

Bom. 1884.

4. $a^2b^{\frac{1}{3}}$.   **6.** 20.

Bom. 1885.

6. 3600.   **7.** 3rd term.

Bom. 1886.

1. p^2-2q; p^3-3pq; $(p^2-2q)^2-2q^2$.

2. 1.   **3.** $x=6, y=\frac{3}{2}$.

4. $b^2-4ac=0$; 0 or 6.   **6.** (i) $4\frac{1}{2}$.   **8.** 90.

9. $\frac{\lfloor n}{\lfloor r\,\lfloor n-r}$; $(-1)^r\frac{n(n+1)(n+2)\ldots(n+r-1)}{\lfloor r}$; $\frac{\lfloor 2n}{\lfloor n\,\lfloor n}$

10. 2·02345.

Bom. 1887.

1. $x=1$.   **2.** 72765.

5. $\left.\begin{aligned} x &= 11 \\ y &= 1 \end{aligned}\right\}$, $\left.\begin{aligned} x &= 1 \\ y &= 11 \end{aligned}\right\}$,

$\left.\begin{aligned} x &= -\tfrac{7}{2}\pm 2\sqrt{\tfrac{11}{3}} \\ y &= -\tfrac{7}{2}\mp 2\sqrt{\tfrac{11}{3}} \end{aligned}\right\}$.

8. 719.   **9.** 2^n.   **10.** 231.

Bom. 1888.

1. (i) $9\sqrt{3}$. **3.** b ; ac. **4.** $\frac{2}{3}$ or $\frac{3}{2}$.

6. $a\left(\frac{m}{n}\right)^{\frac{p-1}{2q}}$; $a\left(\frac{m}{n}\right)^{\frac{q-1}{2q}}$ **9.** $\left(-1\right)^r.\left(r+1\right)x^r$;

$\frac{\qquad\qquad}{\lfloor r}x$. **10.** 3rd ter

Bom. 1890.

7. 96.

Bom. 1891.

3. $2(\sqrt{2}-1)$. **5.** $240pm^2 : qn^2$.

Bom. 1892.

3. $2,\ -3,\ \frac{-3\pm\sqrt{-143}}{6}$.

4. $\left(x+\frac{b}{2a}\right)^2$; a must lie between $\frac{2}{3}$ and $-\frac{2}{3}$. **8.** 26 ; 136.

10. $(-1)^r\frac{n(n+1)(n+2)\ldots(n+r-1)}{\lfloor r}x^r$.

Bom. 1893.

2. 0 or 9. **4.** 8. **5.** 10 miles per hour.

Pun. 1887.

2. $61\frac{17}{19}$ miles per hour. **3.** $\frac{13}{62}$ Rs. **4.** $210c^{12}$.

Pun. 1888.

1. (i) $x=\frac{3b^2+ab}{4a}$, $y=\frac{3b^2-ab-2a^2}{2a}$;

(ii) $-3a,\ -2a$. **2.** $\frac{133}{243}$; 5.

Pun. 1889.

1. (i) $x=\frac{47}{182}$, $y=\frac{47}{136}$; (ii) $x=\frac{a^2}{b}$; $y=\frac{b^2}{a}$;

or, $x=b$, $y=a$. **2.** $-abxy\pm2acxy$. **3.** 24.

Pun. 1890.

1. $c^2x^2+(2b-a)cx-2ab = 0.$

2. (i) 7; (ii) $\frac{7}{8}$. **3.** $2^2+2^{\frac{3}{2}}-2^{\frac{4-n}{2}}-2^{\frac{3-n}{2}}$.

4. $\dfrac{\underline{|890}}{\underline{|70}\,\underline{|820}}$.

Pun. 1891.

1. $x^2-2px+p^2-q^2 = 0.$

2. (i) 13; (ii) 6; (iii) $a+b$ or 0.

3. (i) $9841(\sqrt{3}+1)$; (ii) $n(a^2+x^2+3ax-nax)$.

4. $(-1)^{r-1}\dfrac{5.3.1.3.5\ldots(2r-7)}{\underline{|r}}2^{5-3r}.(3x)^r$.

Pun. 1892.

1. $4x^2-6x+1 = 0.$ **2.** (i) $2\frac{1}{7}$ or $1\frac{9}{14}$; (ii) 0 or $-\frac{a+b}{2}$.

4. 10. **5.** $\dfrac{\underline{|2n+1}}{\underline{|n}\,\underline{|n+1}}a^{n+1}x^n$; $\dfrac{\underline{|2n+1}}{\underline{|n}\,\underline{|n+1}}a^nx^{n+1}$.

Pun. 1893.

2. (*a*) $5\frac{8}{11}$; (*b*) $\frac{3a}{4}$ or $\frac{a}{2}$; (*c*) -1 or 2. **5.** 41.

All. 1889.

1. (*a*) $\dfrac{-3\pm\sqrt{241}}{2}$, -5, 2;

(*b*) $\left.\begin{matrix} x = 0 \\ y = 0 \end{matrix}\right\}$; $\left.\begin{matrix} x = 4 \\ y = 8 \end{matrix}\right\}$; (*c*) $x = \pm 5$, $y = \pm 9$, $z = \pm 11$.

2. $x^4-8x^3+31x^2-60x-16 = 0$; $1\pm\sqrt{3}$ or $1\pm 2\sqrt{-2}$. **4.** ·980.

5. $1-6x^{\frac{1}{3}}+21x^{\frac{2}{3}}+56x+126x^{\frac{4}{3}}$.

7. $\pm\sqrt{3}(\sqrt{3}-1)$. **8.** 32 and 24.

All. 1890.

2. (i) $\dfrac{27\pm\sqrt{29}}{11}$; (ii) $\left.\begin{matrix} x = 0 \\ y = 0 \end{matrix}\right\}$, $\left.\begin{matrix} x = 4 \\ y = 2 \end{matrix}\right\}$, $\left.\begin{matrix} x = -4 \\ y = -2 \end{matrix}\right\}$.

6. $-126a^4b^5$.

All. 1891.

2. (i) $\frac{-1\pm\sqrt{-3}}{2}$, $\frac{7\pm3\sqrt{-7}}{14}$;

(ii) $\left.\begin{array}{l}x=6\\y=3\end{array}\right\}$, $\left.\begin{array}{l}x=3\\y=6\end{array}\right\}$. **3.** 7 miles per hour

6. $384\frac{267}{512}$.

All. 1892.

1. $bcx^2+(b^2+ac)x+ab=0$.

2. (ii) $\left.\begin{array}{l}x=3+\sqrt{6}\\y=3-\sqrt{6}\end{array}\right\}$, $\left.\begin{array}{l}x=3-\sqrt{6}\\y=3+\sqrt{6}\end{array}\right\}$;

(iii) $\left.\begin{array}{l}x=0\\y=0\\z=0\end{array}\right\}$ $\left.\begin{array}{l}x=2\sqrt{6}\\y=\sqrt{6}\\z=3\sqrt{6}\end{array}\right\}$ $\left.\begin{array}{l}x=-2\sqrt{6}\\y=-\sqrt{6}\\z=-3\sqrt{6}\end{array}\right\}$.

3. $1\frac{29}{121}$, $1\frac{29}{46}$, $2\frac{8}{21}$, $4\frac{7}{17}$.

4. (i) $\frac{3{,}2^n-2n-3}{2^{n-1}}$; (iii) $\frac{n(n+1)(n+2)(3n+1)}{12}$. **6.** 8.

All. 1893.

1. $x^2+y^2+z^2+2xyz=1$.

2. (i) 0, $\frac{c(a^2+b^2)-(a^3+b^3)}{c(a^2+b^2)-c^2(a+b)}$;

(ii) $\left.\begin{array}{l}x=0\\y=0\end{array}\right\}$, $x=\frac{b\sqrt{a}}{\sqrt{a}+\sqrt{b}}$, $y=\frac{a\sqrt{b}}{\sqrt{a}+\sqrt{b}}$.

All. 1894.

1. (i) 0, 3 or $5\frac{2}{3}$; (ii) $x=5$, $y=3$. **2.** 1300 yds.

3. 360. **4.** (*b*) $-\frac{5}{16}$; (*c*) $x=\cdot40706$, $y=5\cdot65679$.

APPENDIX.

CHAPTER I.

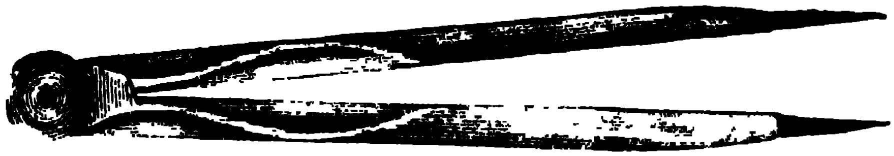
GRAPHS.

1. Instruments required. The student should first of all provide himself with the following instruments and acquire skill in manipulating them with accuracy and neatness.

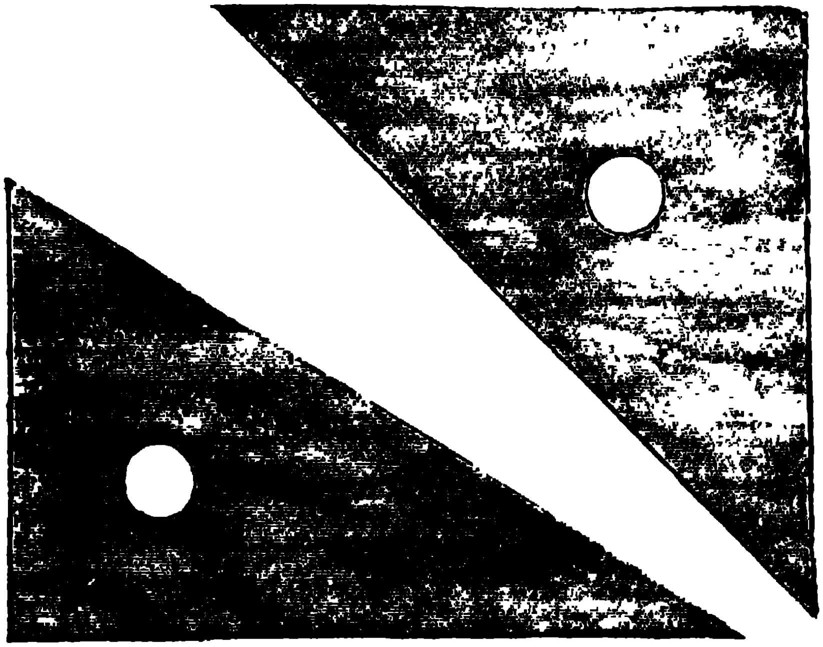
(1) A Hard Pencil.

Note. It must be well sharpened so that the lines drawn may be very fine.

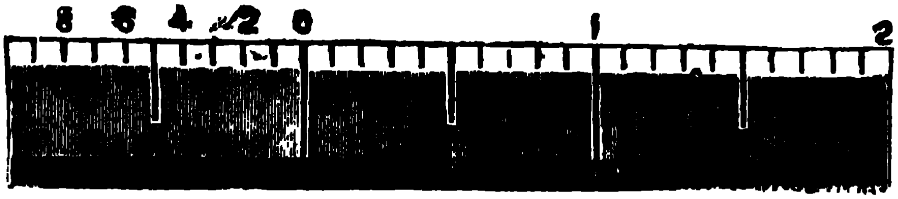
(2) A Pair of Compasses (also called **Dividers**).



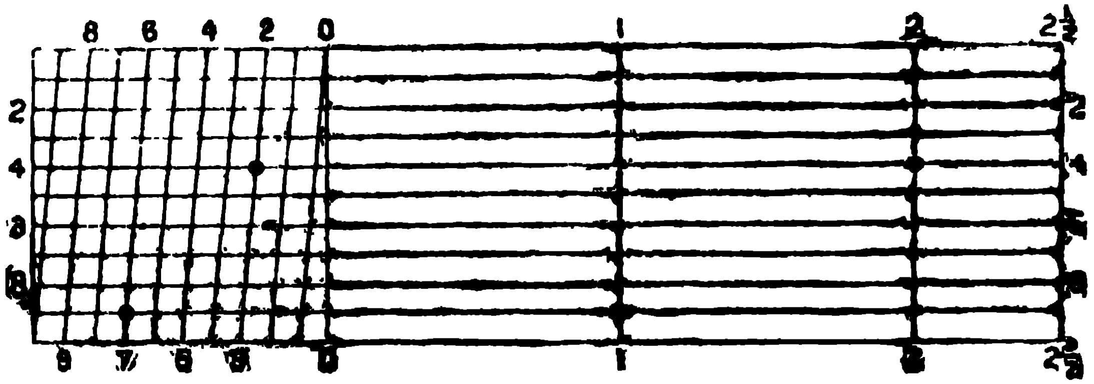
(3) Two Set-squares.



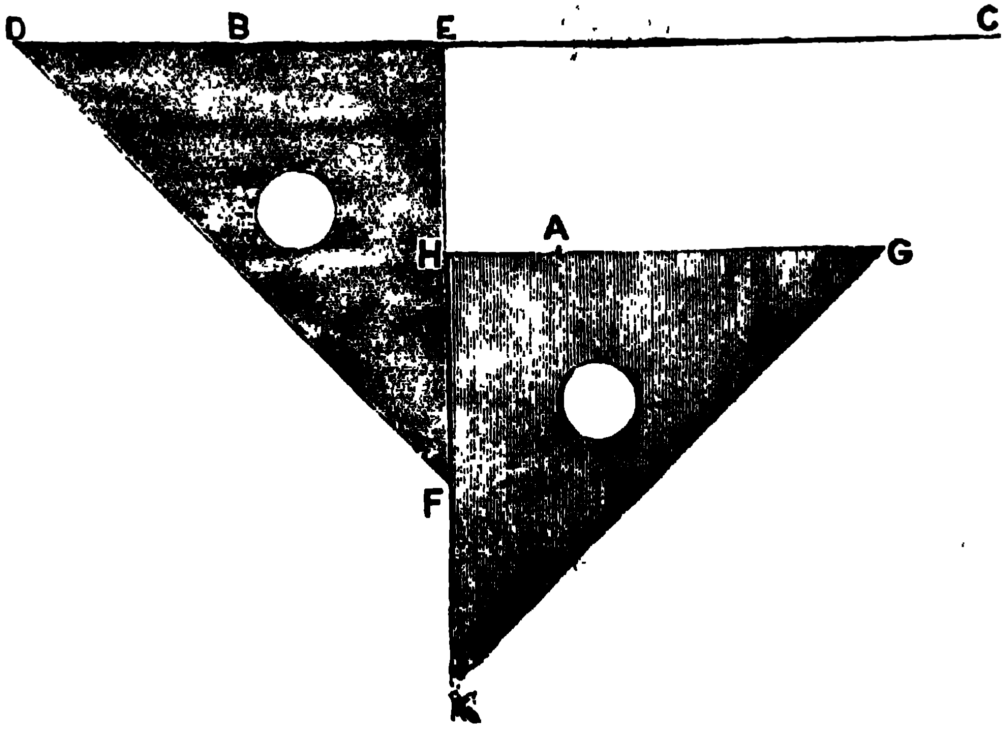
(4) **A graduated Flat Ruler** (of moderate length) **shewing tenths of an inch.**



(5) **A Diagonal Scale, giving hundredths of an inch.**

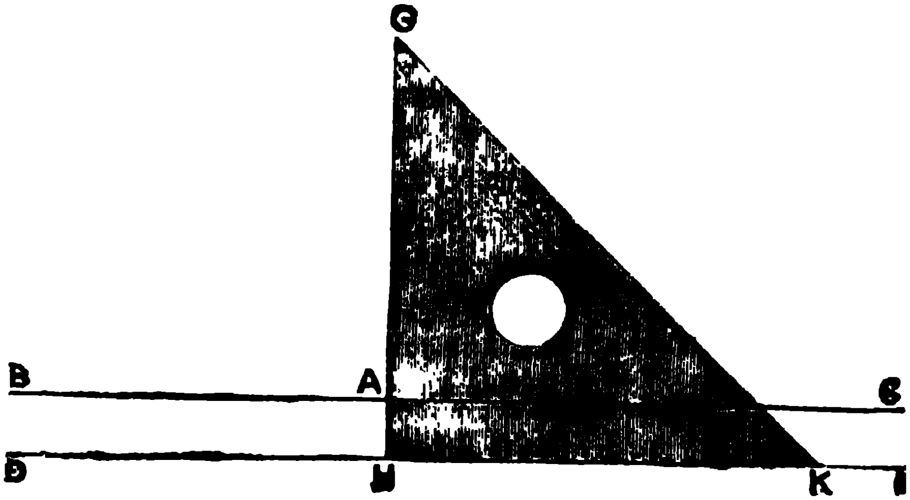


Example 1. Through the point A draw a straight line parallel to BC.



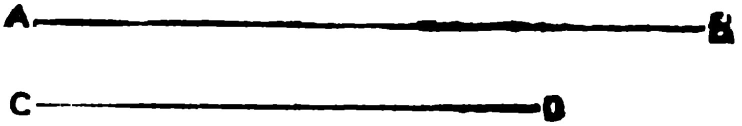
Place the Set-square DEF in such a way that the edge DE may fall along BC. Then, slip the other Set-square GHK into the position shewn in the diagram, so that HG may pass by A. Now trace a line along HG, which will evidently be parallel to BC.

Example 2. Through the point A in the straight line BC draw a straight line perpendicular to BC.



First trace a line DE parallel to BC. Then place the Set-square GHK in such a way that HK may fall along DE and GH may pass by A. Now trace a line along HG, which will evidently be perpendicular to BC.

Example 3. Find the lengths of the straight lines AB and CD :—

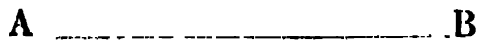


(1) By means of the Pair of Compasses and the Diagonal Scale we find that the length of AB is equal to the distance between the two points marked on the line 4—4 in the diagram. Hence the required length = 2·24 inches.

(2) The length of CD is found to be equal to the distance between the two points marked on the line 9—9 in the diagram. Hence the required length = 1·69 inches.

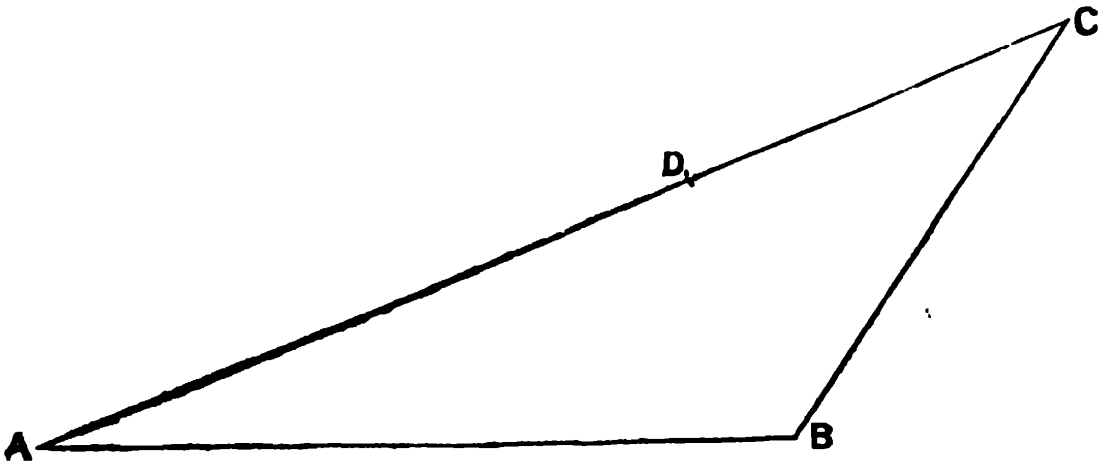
Exercise (I).

1. Produce the straight line AB to double its length :—



2. On a given straight line AB a point D is taken supposing it to be the middle point. By means of a Pair of Compasses however it is found that AD is a trifle shorter than BD. How is the mistake to be corrected ?

3. ABC is a triangle and D a point on AC, as in the following diagram. Through D draw, towards AB, a straight line parallel to CB.



4. In the same diagram, through D draw, away from AB, a straight line parallel to BC.

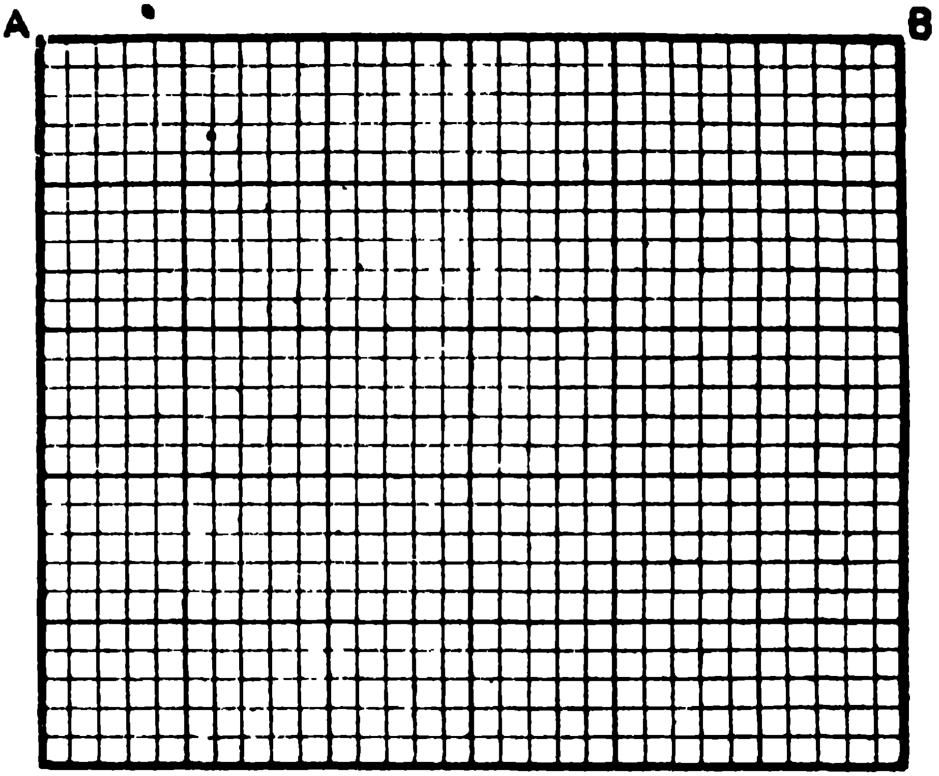
5. In the diagram of example 3, through B draw a straight line parallel to AC.

6. From the vertices of a given triangle draw perpendiculars to its opposite sides.

7. In example 3, measure the lengths of the sides of the triangle, and also measure the lengths of AD and DC.

2. Squared paper. A specimen of a sheet of squared paper is given on the next page.

We have two sets of parallel straight lines on the paper. One set being parallel to the length, and the other, parallel to the breadth, of the paper, it is clear that every line of the first set is perpendicular to every line of the second. The distance between every two consecutive parallels is one-tenth of an inch, whilst every two consecutive *thick* parallels are half an inch apart. The whole paper is thus divided into a large number of small squares which are equal to one another, each side of each square being one-tenth of an inch in length. The paper is also divided into a number of thick-bordered squares, each side of each such square being half an inch in length. It is clear also that twentyfive of the small squares are contained in each of the thick-bordered squares.

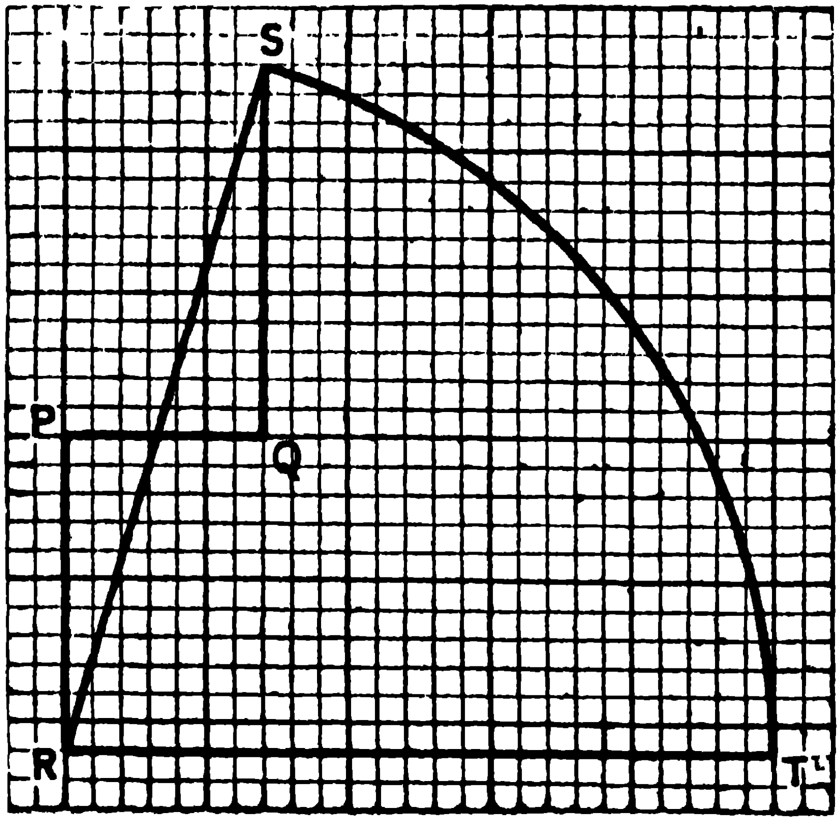


Note 1. Lines parallel to AB may be regarded as *east-and-west* lines, and those parallel to AD, as *north-and-south* lines. They may also be considered as *horizontal* and *vertical* lines respectively.

Note 2. For the sake of convenience the length of a side of a smal square may be denoted by the symbol *a*.

Note 3. The paper may also be ruled so that the length of a side of a small square is only one-tenth of a centimetre (*i.e.* a millimetre), instead of one-tenth of an inch. In that case the distance between every two consecutive *thick* parallels is evidently half a centimetre or 5 millimetres. (One centimetre is approximately equal to ·39 of an inch).

Example 1. P, Q, R, S are four stations such that Q is 7 miles east of P, R is 11 miles south of P and S is 13 miles north of Q. Find the distance between R and S.



Taking the length of a side of a small square (*i.e.*, a) to represent one mile, we have P, Q, R, S as in the above figure, where PQ = $7a$, PR = $11a$ and QS = $13a$.

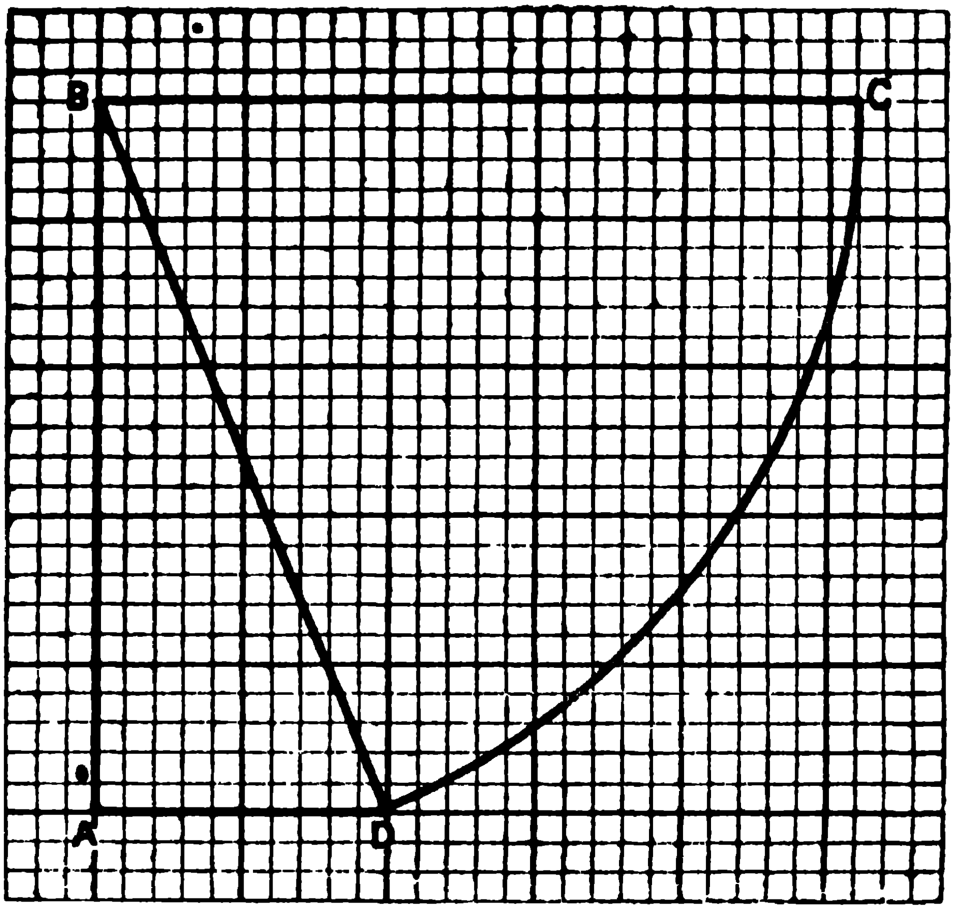
With R as centre and RS as radius describe an arc of a circle cutting the east-and-west line through R at T.

Now as RT = $25a$, we have RS also = $25a$. Hence the required distance = 25 miles.

Example 2. An upright post is 8 feet high. A string of length $8\frac{2}{3}$ feet has one end attached to the top of the post and is held tight with the other end in contact with the ground. How far is this end from the foot of the post?

Let $3a$ (*i.e.* 3 times the length of a side of a small square) represent one foot. Then 8 feet will be represented by $24a$ and $8\frac{2}{3}$ feet by $26a$.

Let AB represent the post, so that AB = $24a$. Take a point C on the horizontal line through B such that BC = $26a$.



With B as centre and BC as radius describe an arc of a circle cutting the horizontal line through A at D. Join BD; then BD represents the string.

Now, AD is equal to $10a$, which is $9a+a$. Hence the required distance = $\underline{3\frac{1}{3}\text{ feet}}$.

Exercise (2).

1. A is $5\frac{1}{3}$ units of length east of O, and P is 4 units of length north of A. How far is P from O?

2. B is 3 feet west of O, and Q is $7\frac{1}{5}$ feet south of B. How far is Q from O?

3. C is 2 yards north of O, and R is $6\frac{6}{7}$ yards west of C. How far is R from O?

4. D is 2·1 inches south of O, and S is 2·8 inches east of D. How far is S from O?

5. A is 2·7 feet east of O. P is north of A and 4·5 feet from O. How far is P from A?

6. Q is 2·4 feet south of B. O is east of B and 2·5 feet from Q. How far is B from O?

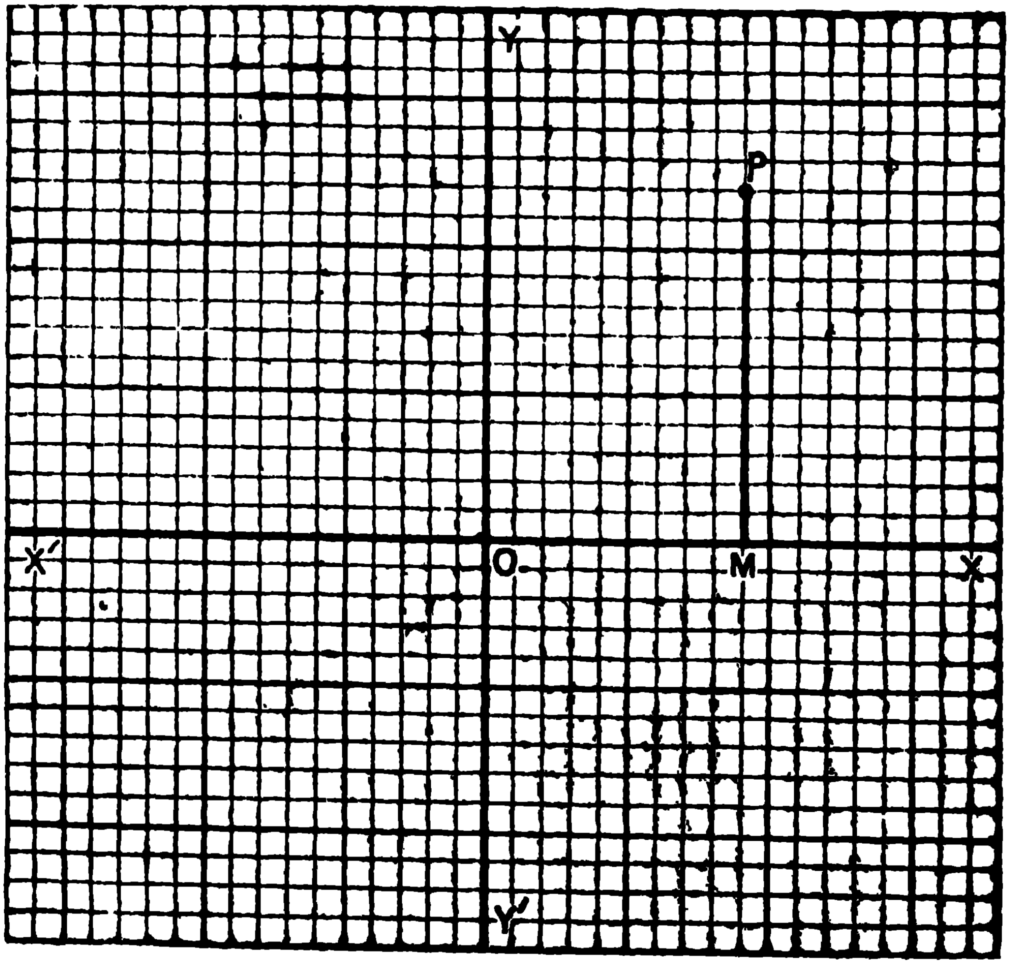
7. B is $4\frac{4}{5}$ yards east of A. C is $\frac{3}{5}$ yard north of A, and D is 2 yards north of B. How far is D from C?

8. B is 25 feet north of A. P is 40 feet west of A, and Q is 20 feet east of B. How far is Q from P?

9. Two vertical posts, 14 feet and $3\frac{4}{5}$ feet high, are $13\frac{3}{5}$ feet apart. Find the distance between the tops of the posts.

10. A ladder 30 feet long has its foot at a distance of 10 feet from a vertical wall. How far up the wall does it reach? (The Diagonal Scale may be used if necessary).

3. If in a plane, a point and two straight lines passing through it at right angles to each other be given, the position of any point in the plane can be easily defined.



In the plane of the paper as shewn in the last diagram, let XOX′ and YOY′ be the two given straight lines at right angles to each other. If P be any point in the plane, how to know its position ?

We may regard XOX′ as the *east-and-west* line, and YOY′ as the *north-and-south* line. Draw PM parallel to YOY′ meeting XOX′ at M. Evidently then M is due east of O, and P, due north of M. Hence if OM and MP be known, we know the position of P at once.

Taking the length of a side of a small square as the unit of length, we have OM = 9 units of length and MP = 12 units of length. Hence the position of P may be briefly defined as follows :—

9 units east, 12 units north.

Note 1. If Q be a point whose position is defined to be **5 units east, 8 units north,** to find Q all that we have to do is to take a point 5 units due east of O and thence proceed 8 units northwards.

Note 2. If R be a point whose position is defined to be **7 units west, 4 units south,** to find R all that we have to do is to take a point 7 units due west of O and thence proceed 4 units southwards.

Exercise (3).

[SQUARED PAPER IS TO BE USED IN EVERY CASE.]

1. Find the points whose positions are defined as follows :—

(1) 5 units east, 7 units north.
(2) 8 units west, 5 units north.
(3) 10 units west, 12 units south.
(4) 15 units east, 6 units south.
(5) 8 units west, 13 units north.
(6) 14 units east, 15 units south.

2. It is clear that "6 units west" is the same as "– 6 units east ", and "8 units south" is the same as "– 8 units north". Hence find the points whose positions are defined as follows :—

(1) 7 units east, – 8 units north.
(2) – 10 units east, 6 units north.
(3) – 9 units east, – 13 units north.

3. In defining the position of a point the words "east" and "north" may be omitted if it is accepted as a rule that the distance measured towards the east should invariably be mentioned first. On this convention, find the points whose positions are defined as follows :—

(1) 8 units, 9 units. (2) 6 units, −11 units.
(2) −12 units, 15 units. (4) −10 units, −14 units.

4. We may define the position of a point still more briefly if the word "units" be omitted. Find, then, the points whose positions are defined as follows :—

(1) 6, 4. (2) 13, 8. (3) −7, 6.
(4) 8, −6. (5) −10, −13. (9) −9, −15.

4. Definitions. The student is referred to the diagram of the last article. The given lines XOX′ and YOY′ with reference to which the positions of all points in the plane are defined, are called the **axes of co-ordinates**; and the point O, where these lines intersect, is called the **origin.**

The straight line XOX′ is called the **axis of x** and the straight line YOY′, the **axis of y.**

The lengths OM and MP which define the position of the point P are called its **co-ordinates**, OM being called the **abscissa** (or **x co-ordinate**) and MP, the **ordinate** (or **y co-ordinate**).

"The point (x, y)" or simply "(x, y)" means "the point whose abscissa $= x$ units of length, and ordinate $= y$ units of length."

Note 1. When we speak of the "x and y" of a point, we mean its "abscissa and ordinate".

Note 2. The abscissa is positive or negative according as M is on the right or on the left of O. The ordinate is positive or negative according as P is above or below XOX′.

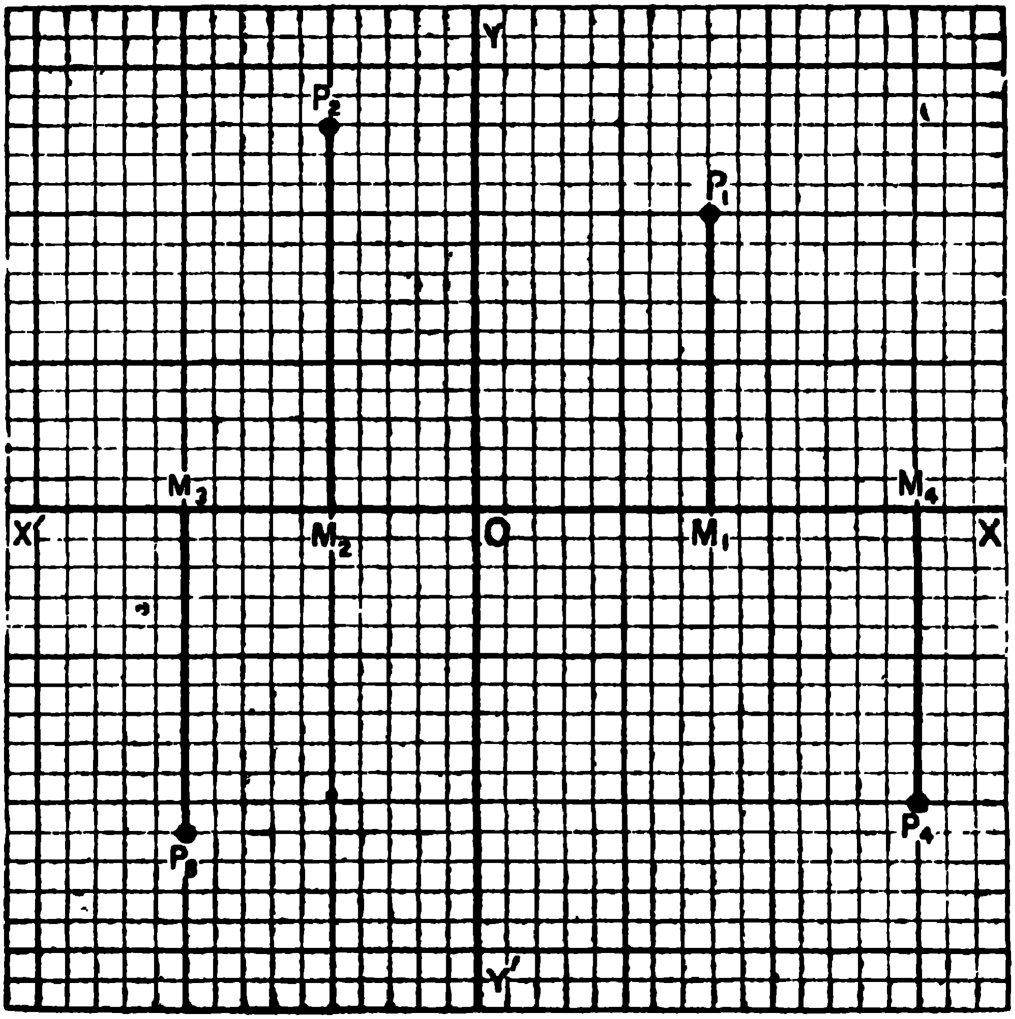
Note 3. "To **plot** a point" is to find the position of a point when its co-ordinates are given.

Example 1. In the diagram given on the next page, write down the co-ordinates of the points P_1, P_2, P_3, P_4.

The figure explains itself. Take the length of a side of a small square as the unit of length.

(1) $OM_1 = 8$ units and M_1 is on the *right* of O; $M_1P_1 = 10$ units and P_1 is *above* the line XOX′. Hence the co-ordinates of P_1 are 8 and 10.

(2) $OM_2 = -5$ units and M_2 is on the *left* of O; $M_2P_2 = 13$ units and P_2 is *above* the line XOX′. Hence the co-ordinates of P_2 are -5 and 13.

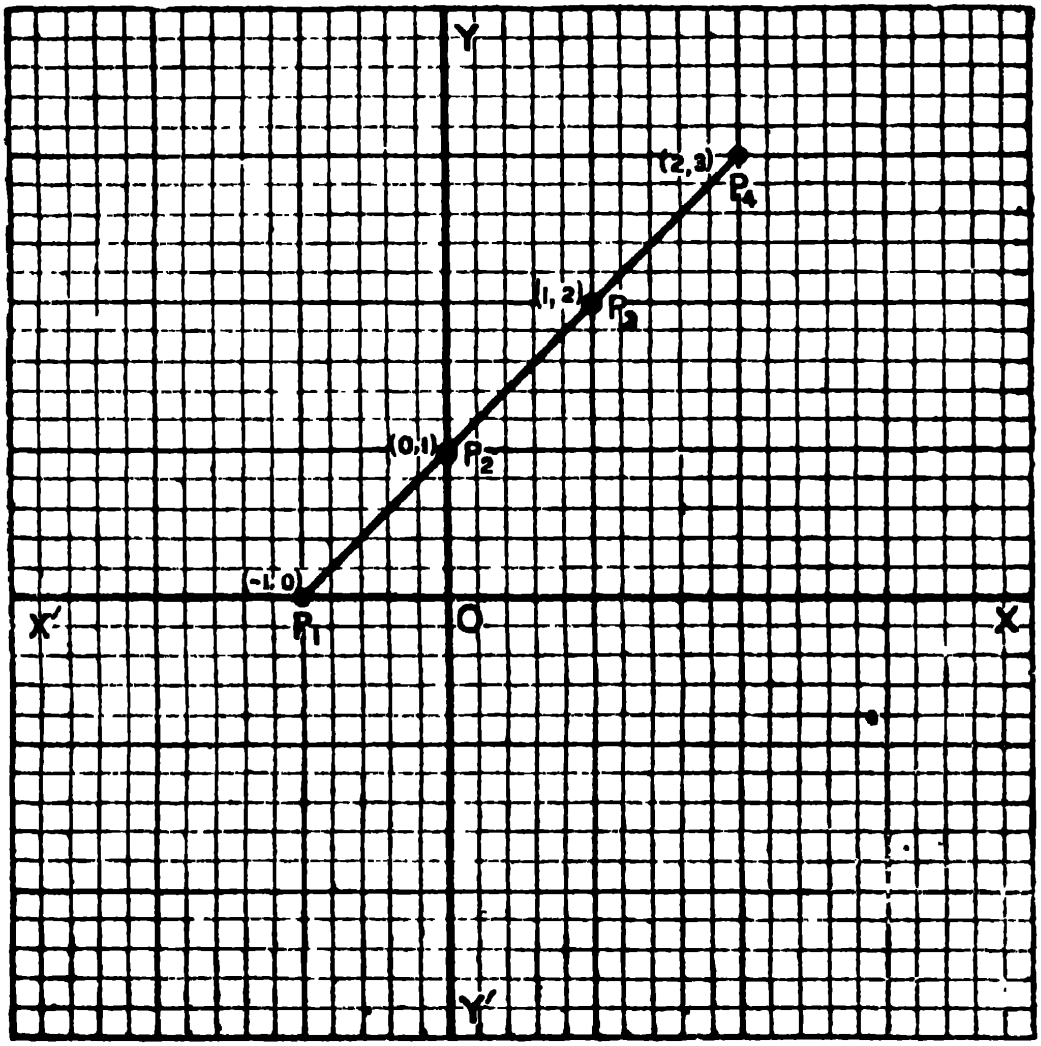


(3) $OM_3 = 10$ units and M_3 is on the *left* of O; $M_3P_3 = 11$ units and P_3 is *below* the line XOX′. Hence the co-ordinates of P_3 are -10 and -11.

(4) $OM_4 = 15$ units and M_4 is on the *right* of O; $M_4P_4 = 10$ units and P_4 is *below* the line XOX′. Hence the co-ordinates of P_4 are 15 and -10.

Example 2. Plot the points $(-1, 0)$, $(0, 1)$, $(1, 2)$ and $(2, 3)$; and shew that they all lie in a straight line.

Let 5 times the side of a small square represent the unit of length, and let P_1, P_2, P_3, P_4, respectively denote the four given points. Then the positions of the points will be as shown in the figure.

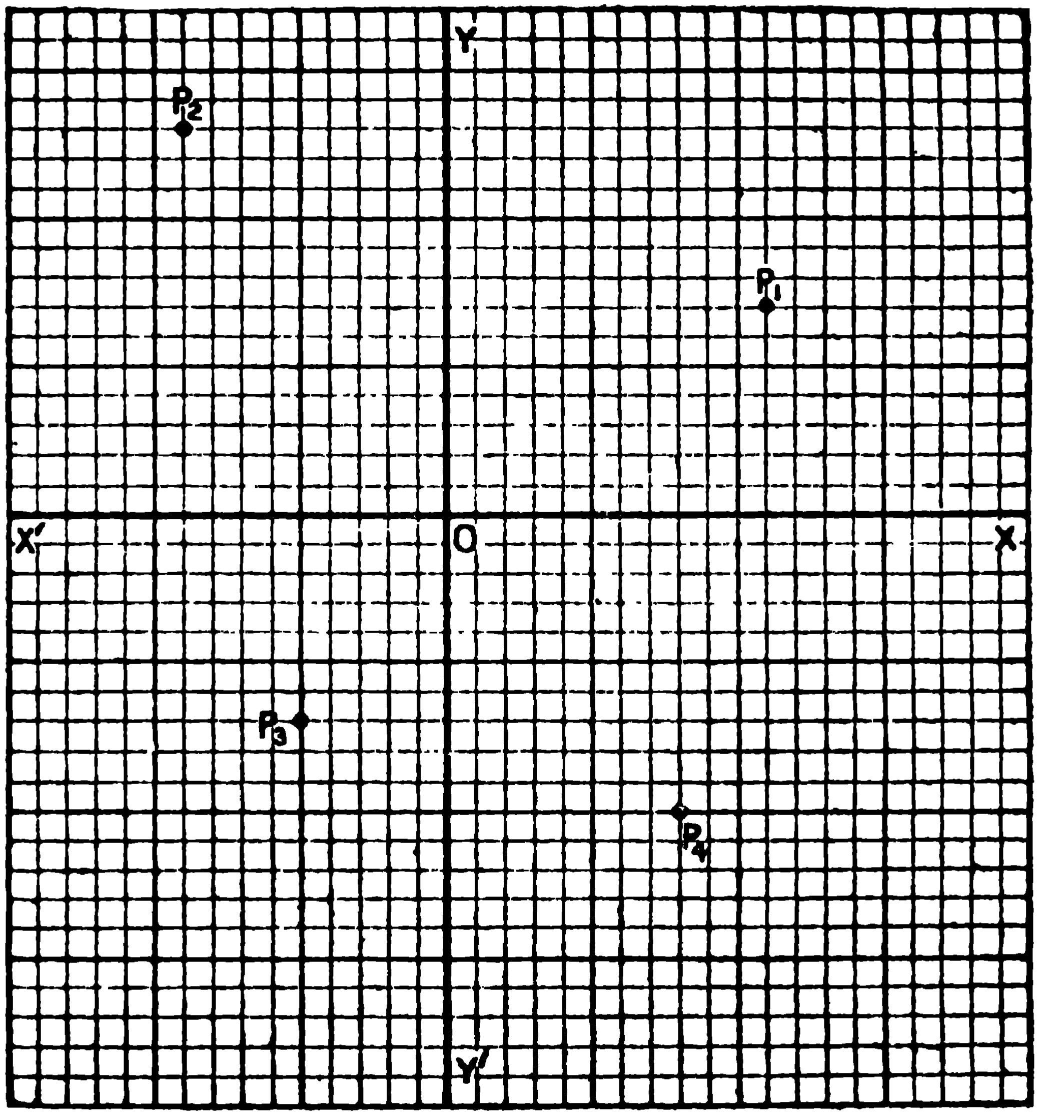


Now we find that a Flat Ruler may be so placed that its edge will pass through all the four points. Hence they all lie in the same straight line.

Exercise (4).

1. In the diagram given on the next page, what are the co-ordinates of the points P_1, P_2, P_3, P_4, (i) when the unit of length is represented by a side of a small square, (ii) when the unit of length is represented by 5 times the side of a small square?

2. What will be the co-ordinates of the same points if the unit of length be represented by three times the side of a small square ?



3. Plot the points (– 4, – 4), (7, 7), (13, 13) and ,satisfy yourself that they lie in a straight line passing through the origin.

4. Plot the points (– 8, 4) and (10, – 5), and satisfy yourself that the straight line joining them passes through the origin.

5. Plot the points (8, 5) and (– 4, – 11), and find the distance between them.

6. Plot the points (- 7, 9) and (– 12, 21), and find the distance between them.

7. Plot the points $(-11, 13)$ and $(3, -35)$, and find the distance between them.

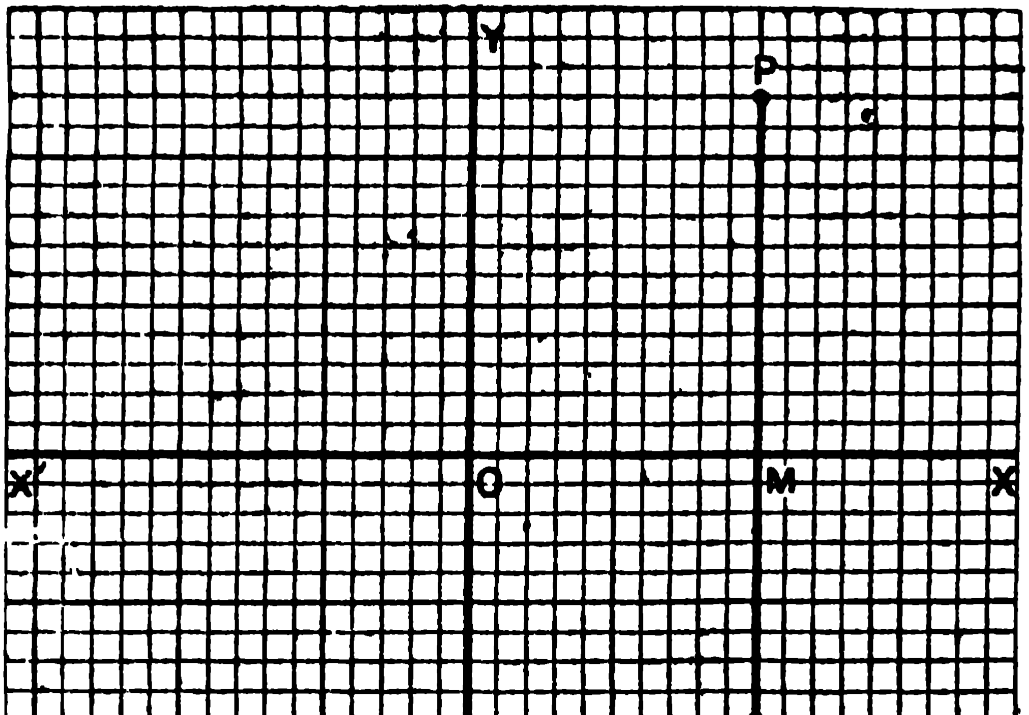
8. Join the points $(0, 0)$ and $(5, 5)$, and produce the straight line both ways. Find the ordinate of the point on this straight line whose abscissa is 11, and the abscissa of the point whose ordinate is -13.

9. Join the points $(0, 7)$ and $(12, 0)$, and produce the straight line both ways. Find the ordinate of the point on the straight line whose abscissa is -18, and the abscissa of the point whose ordinate is -14.

10. Join the points $(-4, 0)$ and $(0, -8)$, and produce the straight line both ways. Find the ordinate of the point on the straight line whose abscissa is -10, and the abscissa of the point whose ordinate is -24.

5. Graphs of Simple Equations. The following examples will make the subject clear.

Example 1. If a point moves in such a manner that its abscissa is always equal to 5 units of length, find the path along which the point will move.



Let twice the side of a small square represent the unit of length. (The figure is on page 14).

On OX take the point M such that OM = 5 units of length; through M draw the straight line PMP′ parallel to YOY′.

Now, if any point be taken on the straight line PMP′ its x will evidently be equal to 5 units of length; but this will *not* be so if the point be taken on either side of the line PMP′.

Hence the moving point will always be on the line PMP′.

We see therefore that if a point moves in such a manner that its x is always equal to 5 units of length, the path along which the point will move is the straight line PMP′. This fact is briefly expressed by saying that **the straight line PMP′ is the Graph of the equation $x = 5$.**

Note 1. From the above it is clear that the graph of the equation $y = 5$ is a straight line parallel to XOX′.

Note 2. Generally speaking, the graph of the equation $x = a$ is a straight line parallel to the axis of y and passing through a point on the axis of x which is at a distance of a units of length from the origin, and the graph of the equation $y = b$ is a straight line parallel to the axis of x and passing through a point on the axis of y which is at a distance of b units of length from the origin.

Note 3. Evidently therefore the graph of the equation $x = 0$ is the axis of y itself, and the graph of the equation $y = 0$ is the axis of x itself.

Example 2. If a point moves in such a manner that its x and y are always connected by the relation $y = 3x$, find the path along which the point will move.

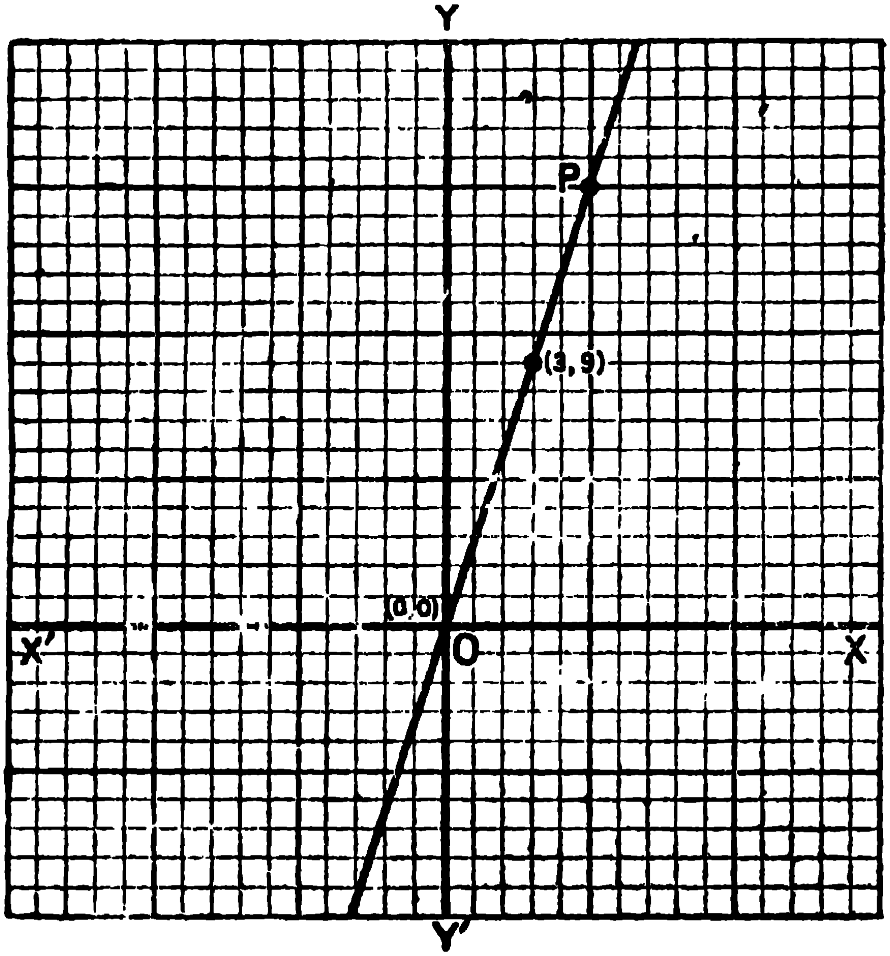
$$\text{Since } y = 3x, \text{ when } \left.\begin{aligned} x &= 0 \\ \text{we have } y &= 0 \end{aligned}\right\}, \quad \text{and when } \left.\begin{aligned} x &= 3 \\ \text{we have } y &= 9 \end{aligned}\right\}$$

Evidently therefore (0, 0) and (3, 9) are two positions of the moving point.

Take the length of a side of a small square as the unit of length. (The figure is on the next page).

Join the points (0, 0) and (3, 9), and produce the straight line both ways. Then this straight line will be the required path.

Take any point P on this straight line. The co-ordinates of P are found to be 5 and 15, which evidently satisfy the given relation. Similarly the co-ordinates of any other point on this straight line may be shewn to satisfy the given relation. But the co-ordinates of a point which is outside the line OP will *not* satisfy the given relation, as can be easily verified.



Hence the moving point will always be on the line OP and never stray out of it.

Thus it is found that if a point moves in such a way that its x and y are invariably connected by the relation $y = 3x$, the path along which the point will move is the straight line OP. In other words **the line OP is the Graph of the equation $y = 3x$**

Note Generally speaking, the graph of the equation $y = mx$, where m is any given number, is a straight line passing through the origin.

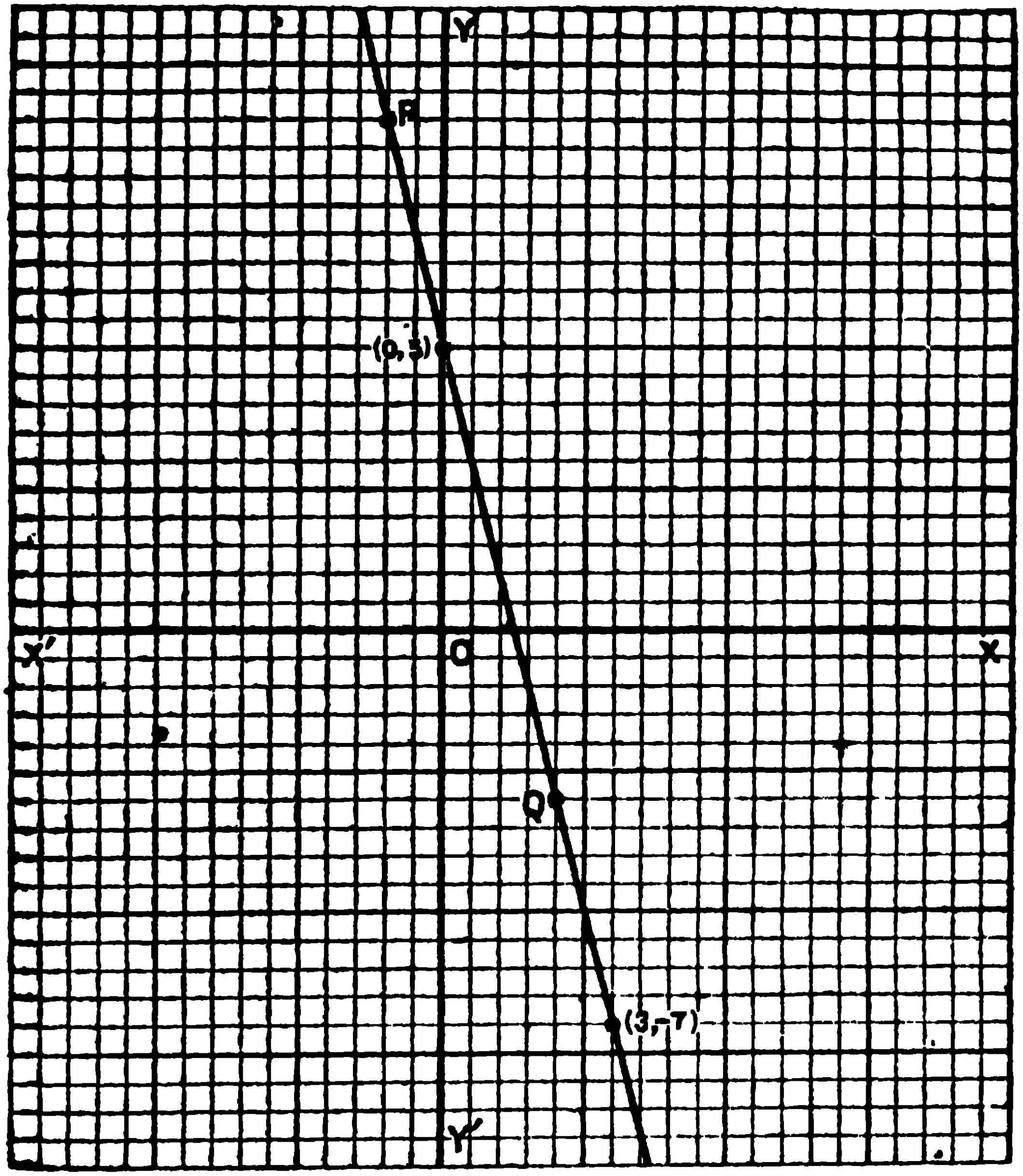
Example 3 If a point moves in such a way that its x and y are invariably connected by the relation $y = -4x+5$, find the path along which the point will move.

From the given relation,

$$\left.\begin{array}{l}\text{when } x = 0\\ \text{we have } y = 5\end{array}\right\}, \qquad \left.\begin{array}{l}\text{and when } x = 3\\ \text{we have } y = -7\end{array}\right\}.$$

Evidently therefore (0, 5) and (3, -7) are two positions of the moving point.

Let twice the side of a small square represent the unit of length. Join the points (0, 5) and (3, -7), and produce the straight line both ways. Then this straight line will be the required path.



Take a point P on this straight line. The co-ordinates of P, which are found to be -1 and 9, satisfy the given relation. Take another point Q on the straight line; its co-ordinates also, which are found to be 2 and -3, satisfy the given relation. Similarly the co-ordinates of any other point on this straight line may be shewn to satisfy the given relation. But if a point be taken outside the line PQ, its co-ordinates will *not* satisfy the given relation, as can be easily seen. Hence the moving point will always be on the line PQ and never stray out of it.

Thus it is found that if a point moves in such a manner that its co-ordinates always satisfy the equation $y = -4x+5$, the path along which the point will move is the line PQ. In other words, **the line PQ is the Graph of the equation $y = -4x+5$.**

Note 1. Generally speaking, the graph of the equation $y = mx+c$ where m and c are any given numbers, is a straight line passing through the point (o, c).

Note 2. As every equation of the first degree in x and y can be reduced to the form $y = mx+c$, it is clear that **graphs of all simple equations are straight lines.**

Note 3. The graph of the *equation* $y = mx+c$ is also said to be the graph of the *expression* $mx+c$.

Note 4. The graph of any given equation may be defined to be **the path described by a point which moves in such a manner that in every position of the point its co-ordinates satisfy the given equation.**

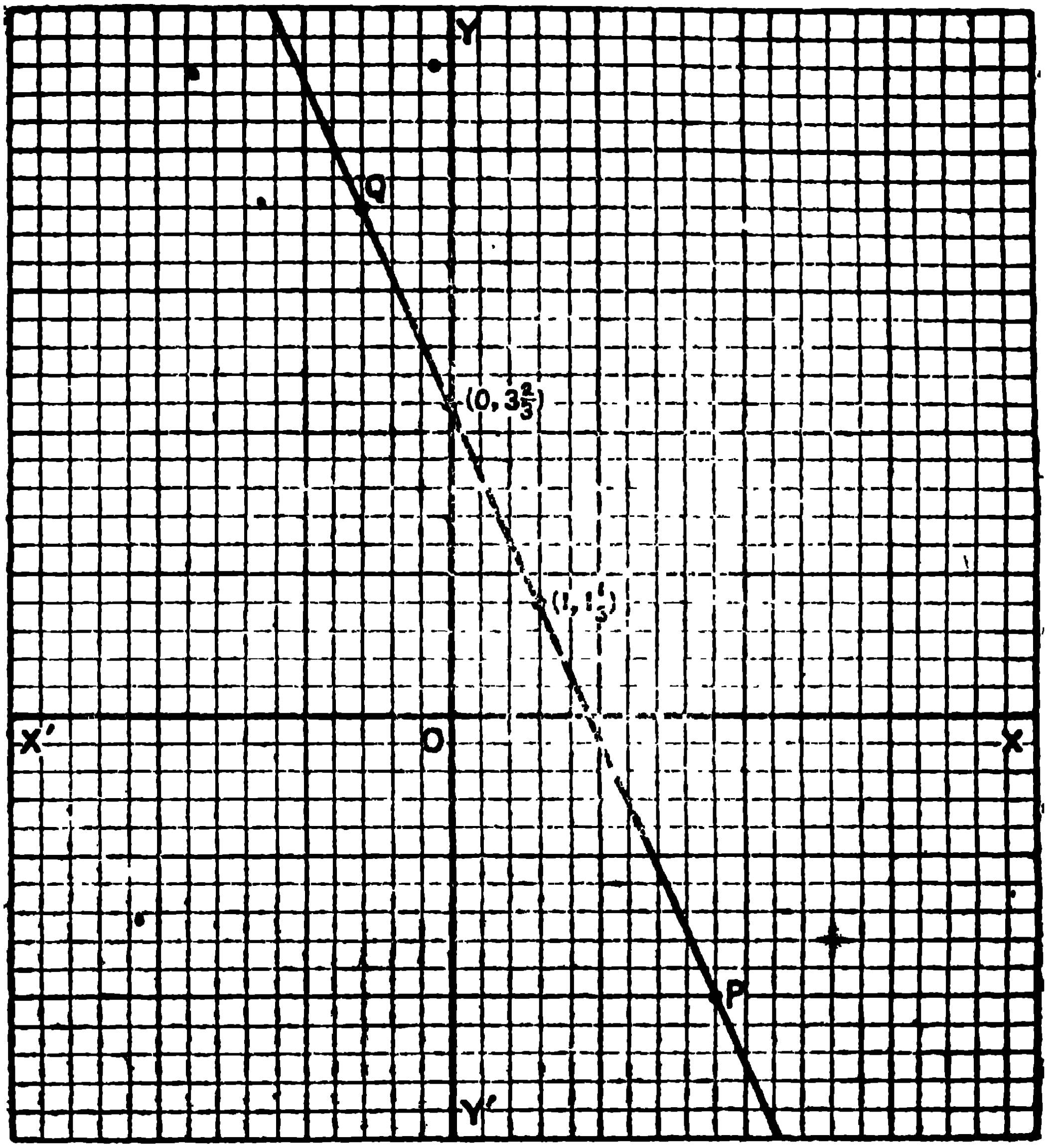
Example 4. Draw the graph of the equation $7x+3y=11$.

When $\left.\begin{array}{l} x = 0 \\ y = 3\frac{2}{3} \end{array}\right\}$, when $\left.\begin{array}{l} x = 1 \\ y = 1\frac{1}{3} \end{array}\right\}$.

Evidently therefore $(0, 3\frac{2}{3})$ and $(1, 1\frac{1}{3})$ are two points on the graph.

Let 3 times the side of a small square represent the unit of length. Join the points $(0, 3\frac{2}{3})$ and $(1, 1\frac{1}{3})$, and produce the straight line both ways. Then this straight line will be the required graph. (See diagram on page 529).

Take any point P on the line; its co-ordinates, which are found to be 3 and $-3\frac{1}{3}$, satisfy the given relation. Take any other point Q on the line; its co-ordinates also, which are found to be -1 and 6, satisfy the given relation. Similarly it may be shown that the co-ordinates of any point that may be taken on the line PQ will satisfy the given relation, but the co-ordinates of any point which is outside PQ will *not*. Hence the line PQ is the required graph.



Note 1. The graph of the *equation* $7x + 3y = 11$ is also said to be the graph of the *expressio* $\dfrac{11 - 7x}{3}$.

Note 2. The straight line PQ being the graph of the equation $7x + 3y = 11$, this equation is said to be **the equation of the straight line PQ**.

Note 3. Hence the equation of a given straight line means **the equation which is satisfied by the co-ordinates of every point on that straight line.**

Example 5. Find the equation of the straight line which passes through the points (1, 1) and $(3, -\frac{1}{2})$.

Let $y = mx + c$ be the required equation.

This equation being satisfied by $(1, 1)$ and also by $(3, -\frac{1}{2})$, we must have

$$\left.\begin{aligned} 1 &= m+c \\ \text{and } -\tfrac{1}{2} &= 3m+c \end{aligned}\right\} . \quad \begin{aligned} &\text{Hence, } 2m = -\tfrac{3}{2}, \text{ and } \therefore\ m = -\tfrac{3}{4}\,; \\ &\text{whence } c = 1+\tfrac{3}{4} = \tfrac{7}{4}. \end{aligned}$$

Thus the required equation is $y = -\frac{3}{4}x + \frac{7}{4}$,

$$\text{or} \qquad 3x+4y = 7.$$

Exercise (5).

1. Draw the graphs of the following equations :—

(1) $x = 8$. (2) $x = 13$. (3) $x+11 = 0$.

(4) $y = -7$. (5) $y-9 = 0$. (6) $y+10 = 0$.

2. Draw the graphs of the following equations :—

(1) $y = x$. (2) $y = -x$. (3) $y = 2x$.

(4) $y+2x = 0$. (5) $y = -3x$. (6) $3y = 5x$.

(7) $7y+8x = 0$. (8) $6y+13x = 0$.

3. Draw the graphs of the following equations :—

(1) $y = 3x+4$. (2) $y = 7x-8$. (3) $y = -5x+9$.

(4) $y = -8x-11$. (5) $3y = 7x+4$. (6) $-6y = 7x-10$.

4. Draw the graphs of the following equations :—

(1) $2x+7y = 10$. (2) $4x-5y-7 = 0$.

(3) $5x+6y+8 = 0$. (4) $-3x+7y+8 = 0$.

(5) $10y-9x = 13$. (6) $8x-11y+13 = 0$.

5. Draw the graphs of the following equations :—

(1) $\frac{x}{3}+\frac{y}{4} = 1$. (2) $\frac{x}{7}+\frac{y}{-9} = 1$. (3) $\frac{x}{-8}+\frac{y}{13} = 1$.

(4) $y = \frac{5-7x}{6}$. (5) $y = \frac{9x-13}{4}$. (6) $\frac{3x}{4}-\frac{4y}{3} = 1$.

6. Draw the graphs of the following equations :—

(1) $x-3$. (2) $3x+4$. (3) $-7x+8$. (4) $\frac{7-4x}{3}$.

(5) $\frac{5x-9}{4}$. (6) $\frac{8x+11}{5}$.

7. Find the equation of the straight line which passes through each of the following pairs of points :—

(1) (0, 0), (5, 6). (2) (0, 5), (7, 0).
(3) (6, −8), (−7, 5). (4) (−4, 8), (−9, −13).
(5) (−11, 0), (7, −10).

6. Graphical Solution of Equations.

Example. Solve graphically—

$$\left.\begin{aligned} 2x - 7y + 12 &= 0 \\ 3x + 2y &= 32 \end{aligned}\right\}.$$

Let us draw the graphs of the two equations.

We find that

$\left.\begin{aligned} x &= -6 \\ y &= 0 \end{aligned}\right\}, \left.\begin{aligned} x &= 1 \\ y &= 2 \end{aligned}\right\}$ are points on the graph of the 1st. equation ;

whilst

$\left.\begin{aligned} x &= 0 \\ y &= 16 \end{aligned}\right\}, \left.\begin{aligned} x &= 6 \\ y &= 7 \end{aligned}\right\}$ are points on the graph of the 2nd. equation.

Hence, taking the length of a side of a small square as the unit of length, the two graphs are as shewn on the next page.

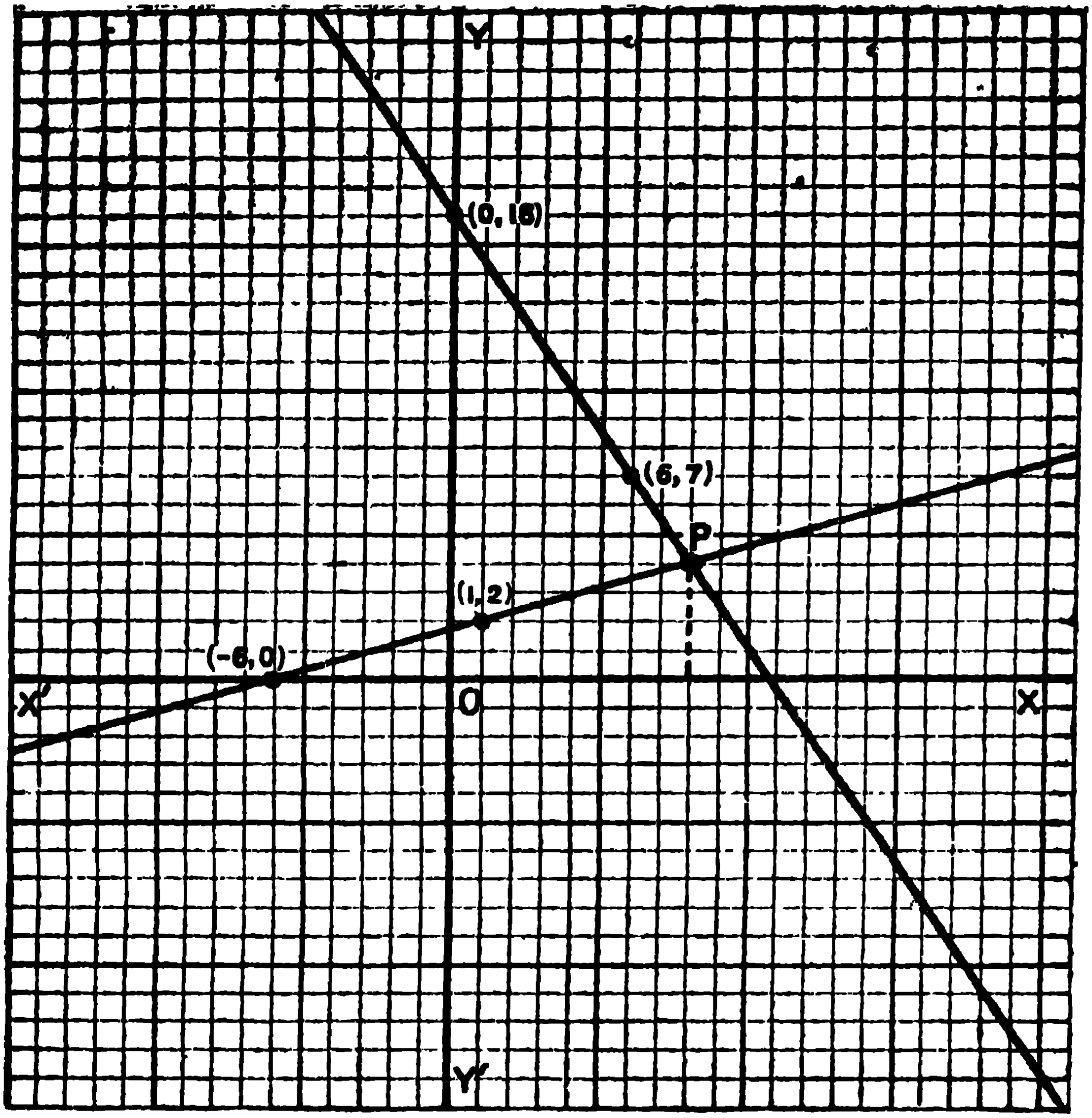
Let P be the point where the two graphs intersect. P being common to the graphs, its co-ordinates will satisfy both the given equations.

Now the co-ordinates of P are found to be 8 and 4.

Hence, $\left.\begin{aligned} x &= 8 \\ y &= 4 \end{aligned}\right\}$ is the required solution.

Note 1. By actual verification we find that both the equations are satisfied when $x = 8$ and $y = 4$.

Note 2. If it is required to "solve graphically the equation $\frac{x-3}{5} = \frac{3x-22}{2}$", all that we have to do is to draw the graphs of the expressions $\frac{x-3}{5}$ and $\frac{3x-22}{2}$ and take the abscissa of the point common to the two graphs.



Exercise (6).

Solve the following equations graphically :—

1. $x + y = 9,\ 3x - 2y = 7.$ **2.** $4x + 3y = 13,\ 3x + 2y = 11.$

3. $\frac{x}{4} + \frac{y}{3} = 4,\ 4x - 5y = 2.$

4. $y - x = 2,\ 3x - 2y = 5.$

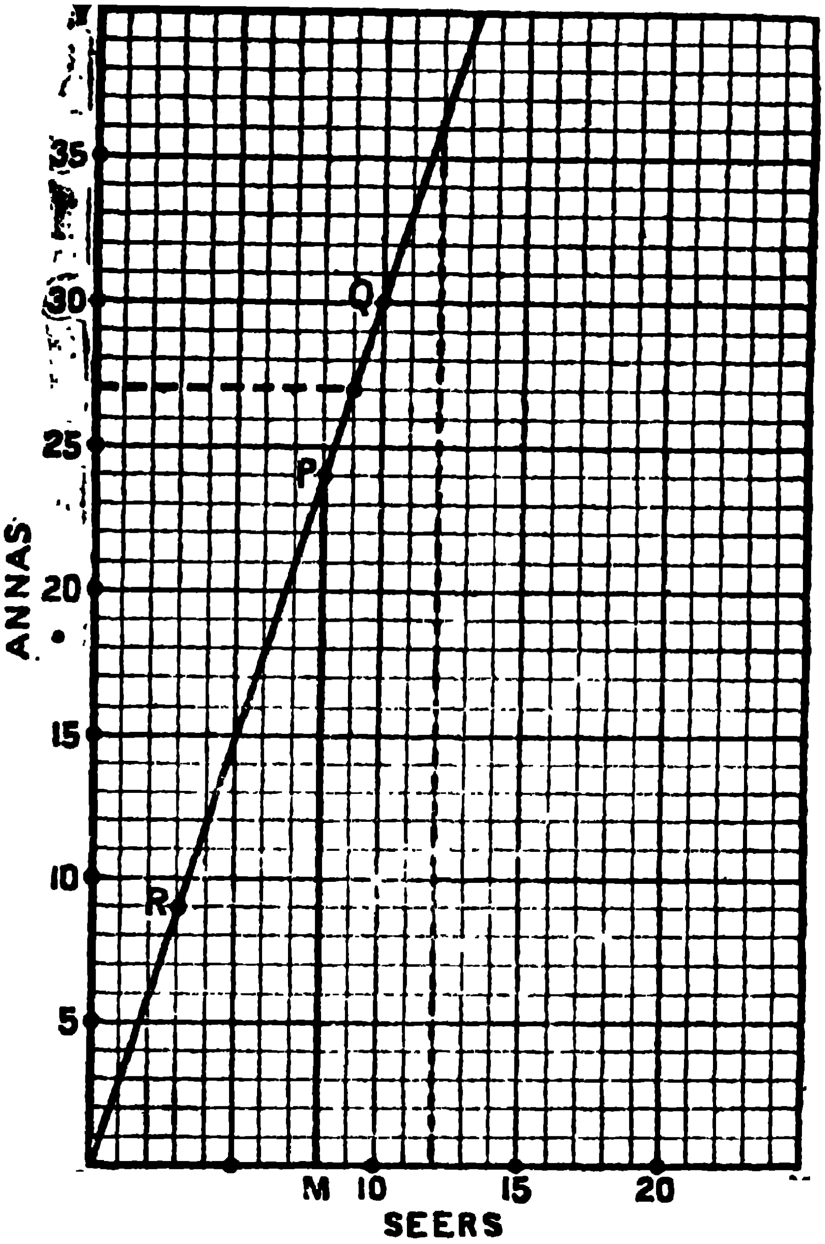
5. $5x - 3y = 11,\ 2y - 3x + 4 = 0.$

6. $x - 2 \qquad -5x + 4$ **7.** $\frac{2x + 7}{3} = \frac{3x - 7}{2}.$

8. $\frac{4x - 3}{5} = \frac{6x}{7} - 1.$

7. Graphical Problems.

Example 1. Given that the price of a seer of rice is three annas, shew that a straight line can be drawn such that if any point be taken on it, the abscissa of the point will represent the quantity of rice of which the price is represented by the ordinate.



In the above figure let the length of a side of a small square measured along OX represent one seer, and let an equal length measured along OY represent one anna. Then the meaning of the figures along OX and OY is clear.

Since the price of a seer is 3 annas, the price of 8 seers must be 24 annas. Clearly therefore P is a point such that its abscissa OM represents a quantity of rice of which the price is represented by the ordinate PM.

Join OP and produce it. Then this is the straight line every point on which will satisfy a condition similar to that satisfied by P.

Q is the point (10, 30); consequently its abscissa represents a quantity of rice of which the price is represented by its ordinate. R is the point (3, 9); its abscissa therefore represents a quantity of rice of which the price is represented by its ordinate. Similarly this is true of every point on the line OP.

Hence OP is the required straight line.

Note 1. The line *OP* is called the graph of the price of rice, or more simply the **price-graph** of rice.

Note 2. The graph enables us to determine readily the price of any given number of seers of rice. For instance, if the abscissa be taken to be 12, the ordinate is immediately found to be 36; thus we know that the price of 12 seers of rice is 36 annas. Similarly for any other abscissa the corresponding ordinate can be immediately found.

Note 3. The graph also enables us to determine quickly the number of seers of rice that can be had for any given price. For instance, if the ordinate is taken to be 27, the corresponding abscissa is immediately found to be 9, which shows that we can have 9 seers of rice for 27 annas.

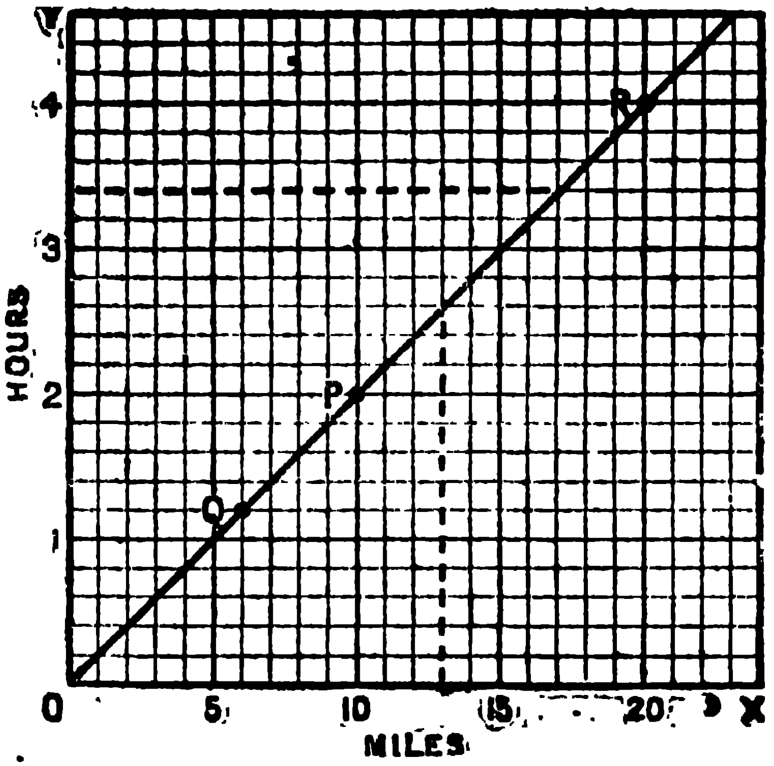
Example 2. A person, named B, starting from a given place, travels at the rate of 5 miles an hour. Shew that a straight line can be drawn such that if any point be taken on it, the abscissa of the point will represent the number of miles that B travels in the time represented by the ordinate.

In the figure on the next page let the length of a side of a small square measured along OX represent one mile, and let an equal length measured along OY represent 12 minutes. Then the meaning of the figures along OX and OY is clear.

Since B travels 5 miles in one hour, he travels 10 miles in 2 hours. Clearly therefore P is a point such that its abscissa represents the number of miles that the person travels in the time represented by its ordinate.

Join OP and produce it. Then this is the straight line every point on which will satisfy a condition similar to that satisfied by P.

Let Q be any point on the line. Its abscissa represents 6 miles and ordinate represents 1 hour 12 minutes; but we *know* that the person travels 6 miles in 1 hour 12 minutes. Hence Q satisfies the condition above mentioned.



Let R be some other point on the line. Its abscissa represents 20 miles and ordinate represents 4 hours; but we *know* that the person travels 20 miles in 4 hours. Hence R also satisfies the proposed condition.

Similarly for any other point on the line. Hence OP is the required straight line.

Note 1. The line *OP* is called **the graph of B's motion,** or the **motion-graph of B.**

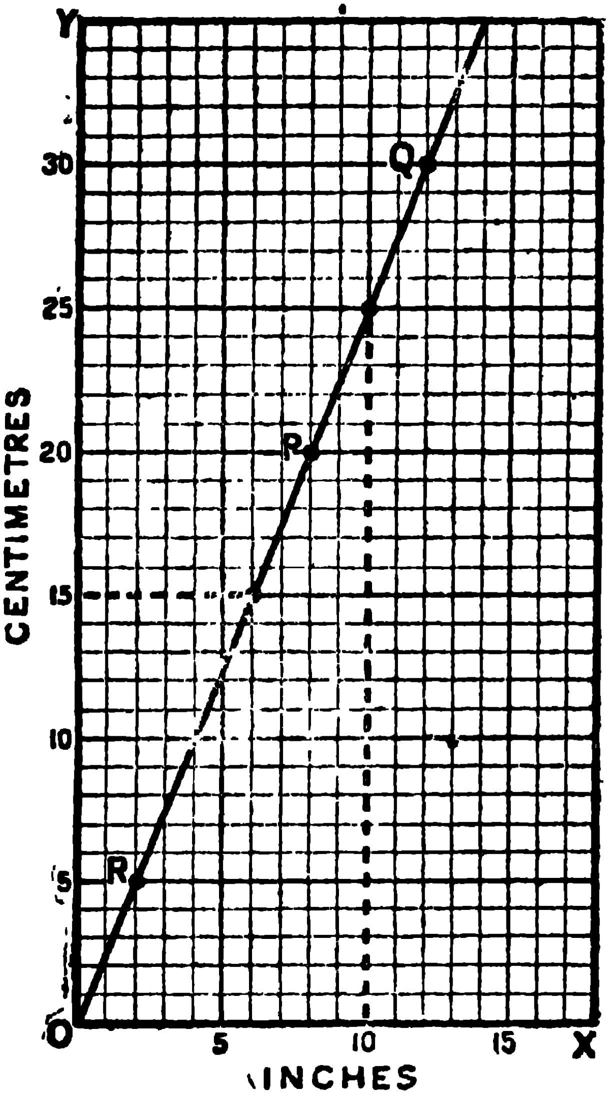
Note 2. The graph enables us to determine readily the time in which *B* travels any given number of miles. For instance, if the abscissa be taken which represents 13 miles, the corresponding ordinate is immediately found to be that which represents 2 hours 36 minutes; thus it is known that the time taken by the person to travel 13 miles is 2 hours 36 minutes.

Note 3. The graph also enables us to determine readily the number of miles that the person travels in any given time. For instance, if the ordinate be taken which represents 3 hours 24 minutes, the corresponding abscissa is immediately found to be that which represents 17 miles; thus it is known that in 3 hours and 24 minutes the person travels 17 miles.

Example 3. If one inch be equal in length to 2·5 centimetres, shew that a straight line can be drawn such that the abscissa of any point on the line will represent the number of inches that are equivalent to the number of centimetres represented by the ordinate.

In the figure on the next page let the length of a side of a small square measured along OX represent one inch, and let an equal length measured along OY represent one centimetre. Then the meaning of the figures along OX and OY is clear.

Since 1 inch = 2·5 centimetres, we have 8 inches = 20 centimetres. Clearly therefore P is a point such that its abscissa represents the number of inches that are equivalent to the number of centimetres represented by its ordinate.



Join OP and produce it. Then this is the straight line every point on which will satisfy a condition similar to that satisfied by P.

Let Q be any point on the line. Its abscissa represents 12 inches, whilst its ordinate represents 30 centimetres ; but we *know* that these two are equivalent. Hence Q satisfies the condition above mentioned.

Let R be some other point on the line. Its abscissa represents 2 inches, whilst its ordinate represents 5 centimetres ; but we *know* that these two are equivalent. Hence R also satisfies the proposed condition.

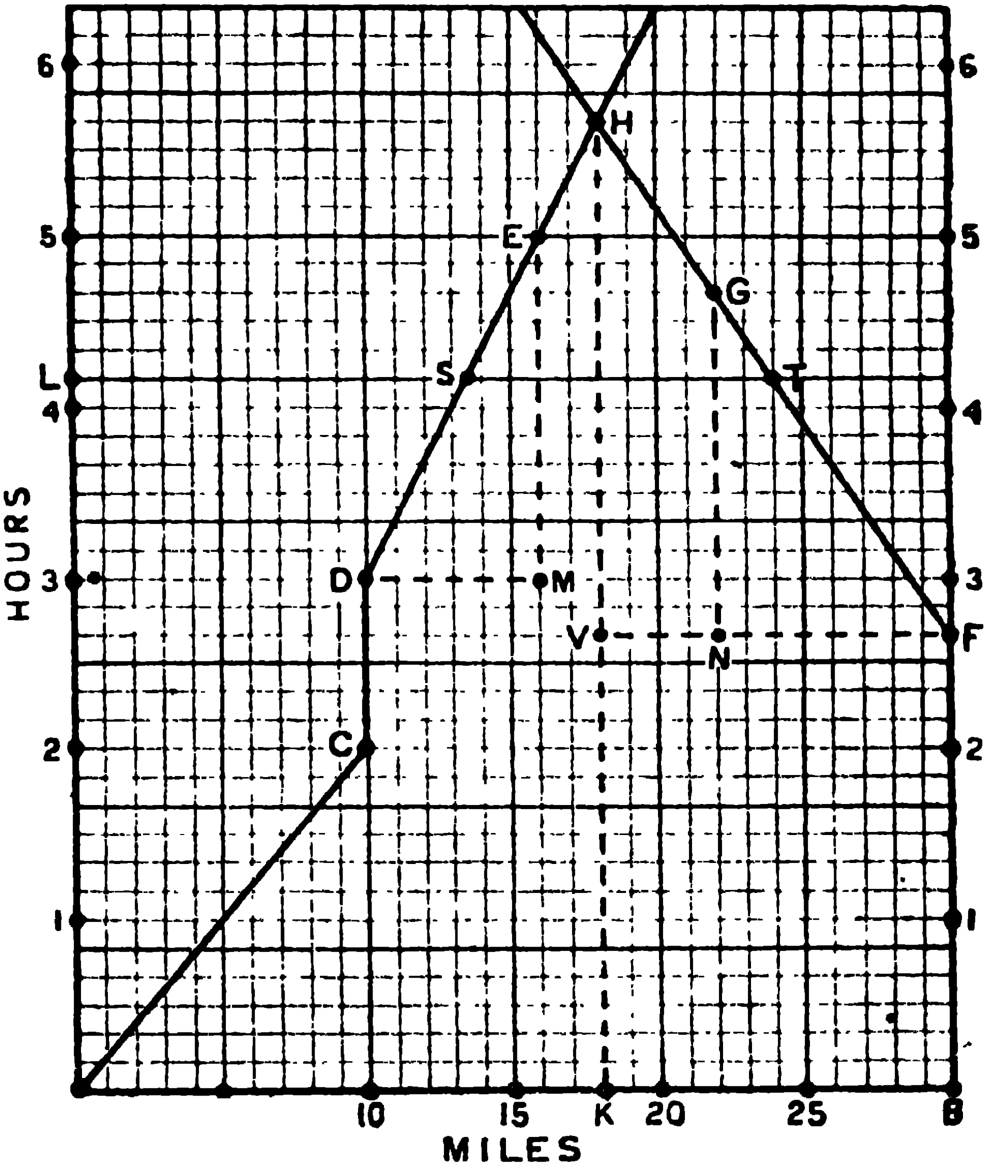
Similarly for any other point on the line. Hence OP is the required straight line.

Note 1. The line *OP* is called the graph for converting inches into centimetres and *vice versa*, or more briefly the **conversion-graph for inches and centimetres.**

Note 2. The graph enables us to determine readily the number of centimetres that are equivalent to any given number of inches. For instance, if the abscissa be taken which represents 10 inches, the corresponding ordinate is immediately found to be that which represents 25 centimetres ; thus it is known that 10 inches are equivalent to 25 centimetres.

Note 3. The graph also enables us to determine readily the number of inches that are equivalent to any given number of centimetres. For instance, if the ordinate be taken which represents 15 centimetres, the corresponding abscissa is immediately found to be that which represents 6 inches ; thus it is known that 15 centimetres are equivalent to 6 inches.

Example 4. A and B are two stations 30 miles apart. P starts from A and travels towards B at the rate of 5 miles an hour; at the end of 2 hours he takes rest for one hour, and then resumes his journey at the rate of 3 miles an hour. Q leaves B 2 hours 40 minutes after P leaves A, and travels towards A, without stoppage, at the rate of 4 miles an hour. When and where will the two travellers meet?



Let the length of a side of a small square measured horizontally represent one mile, and let an equal length measured vertically represent 10 minutes. Then the meaning of the figures along the lines in the above diagram is clear.

(i) P starts from A, and travelling at the rate of 5 miles an hour, completes 10 miles in 2 hours. Hence if the point C be taken such that its co-ordinates respectively represent 10 miles and 2 hours, AC is the graph of P's motion for the first two hours.

The graph for the 3rd hour must be such that the abscissa of any point on it may represent 10 miles, because P is supposed to be at rest throughout this hour. Hence CD drawn vertically to represent one hour, as in the diagram, will be the graph of P's rest.

After the 3rd hour P travels at the rate of 3 miles an hour. Hence if DM be taken to represent 6 miles and ME to represent 2 hours, the straight line DE is the graph of P's motion after the 3rd hour.

Thus the broken line ACDE is the complete graph of P's motion.

(ii) Q starts from B 2 hours 40 minutes after P leaves A. Hence if BF be measured vertically to represent 2 hours 40 minutes, BF may be regarded as the graph of Q's rest at B.

When Q leaves B he moves towards A at the rate of 4 miles an hour. Hence if FN be taken to represent 8 miles and NG to represent 2 hours, the straight line FG will be the graph of Q's motion.

(iii) Let the two graphs intersect at H, and draw HK perpendicular to AB. Produce FN to meet HK at V.

Now it is clear that at the end of time HK, P will have gone a distance AK towards B, and Q will have gone a distance BK (*i.e.* FV) towards A. Hence they will meet at this instant. Thus the required time of meeting = that represented by HK = 5 hours 40 minutes after the commencement of P's motion.

Also, the distance of the place of meeting from A = that represented by AK = 18 miles.

Note 1. As HV represents 3 hours, it is clear that P and Q meet at the end of 3 hours after Q starts from B.

Note 2. The horizontal line through L meets the graphs at the points S and T. As AL represents 4 hours 10 minutes and ST represents $10\frac{1}{2}$ miles, it is clear that at the end of 4 hours 10 minutes from the commencement of P's motion, P and Q are at a distance of $10\frac{1}{2}$ miles from each other.

Exercise (7).

1. If milk sells for 4 annas per seer, construct the price-graph of milk, giving the price of any quantity of milk up to 5 seers. From the graph read off the price of 3 seers and 5 chattacks of milk, and also the quantity of milk that can be had for 10 annas and 9 pies.

2. If *Fazli* mangoes be worth one rupee two annas a dozen, construct a price-graph for mangoes, giving the price of any number up to 30. Read off from the graph the price of 17 mangoes and also the number of mangoes that can be had for 1 Re. 12 as. 6 p.

3. If a man walks at the rate of 4 miles an hour, construct a graph of his motion. Read off from the graph the time in which he travels 13 miles and also the number of miles he travels in $4\frac{3}{4}$ hours.

4. If one cubit be equal to 1·5 feet, construct a conversion-graph for cubits and feet. Read off from the graph the number of feet that are equivalent to $5\frac{3}{4}$ cubits and also the number of cubits that are equivalent to $6\frac{3}{4}$ feet.

5. A starts from a place and walks in a given direction at the rate of 3 miles an hour; B starts from the same place one hour later and moves in the same direction at the rate of 5 miles an hour. Draw the motion-graphs of A and B, and find when and where B overtakes A.

6. A and B are two stations 20 miles apart. P starts from A and travels towards B at the rate of 3 miles an hour; whilst Q starting from B travels towards A at the rate of 2 miles an hour. Construct the motion-graphs of P and Q, and find when and where they meet.

7. Fifty articles of the same kind cost 3 Rs. 2 as. Construct a graph from which you can read off the cost of any number of articles up to 50. Hence find the cost of 19 articles, and the number of articles that you would get for 2 Rs. 7 as.

8. Given that 1 kilogramme = 2·2 lbs., construct a graph which will enable you to read off the number of kilogrammes that are equivalent to any given number of lbs. up to 15 lbs. Read off the number of kilogrammes in 11 lbs.

9. A man travels for 3 hours at the rate of 2 miles an hour, at the end of which he takes rest for an hour and a half, and then starts to walk at the rate of two and a half miles an hour. Construct the graph of his motion.

10. A man starts from a place B to walk towards C at the rate of 4 miles an hour. After 3 hours he changes his mind and walks back towards B at the rate of 3 miles an hour. At the end of 2 hours again he suddenly changes his mind and begins to run towards C at the rate of 7 miles an hour. Draw a graph of his motion.

11. A, B and C are three stations in order on the same road, the distance between A and B being 6 miles. Q starts from B at noon to walk towards C at the rate of 3 miles an hour, and at 1-30 P. M. P starts from A to run towards B at the rate of $6\frac{1}{2}$ miles an hour. Draw graphs of their motion, and find when and where P will overtake Q.

CHAPTER II.

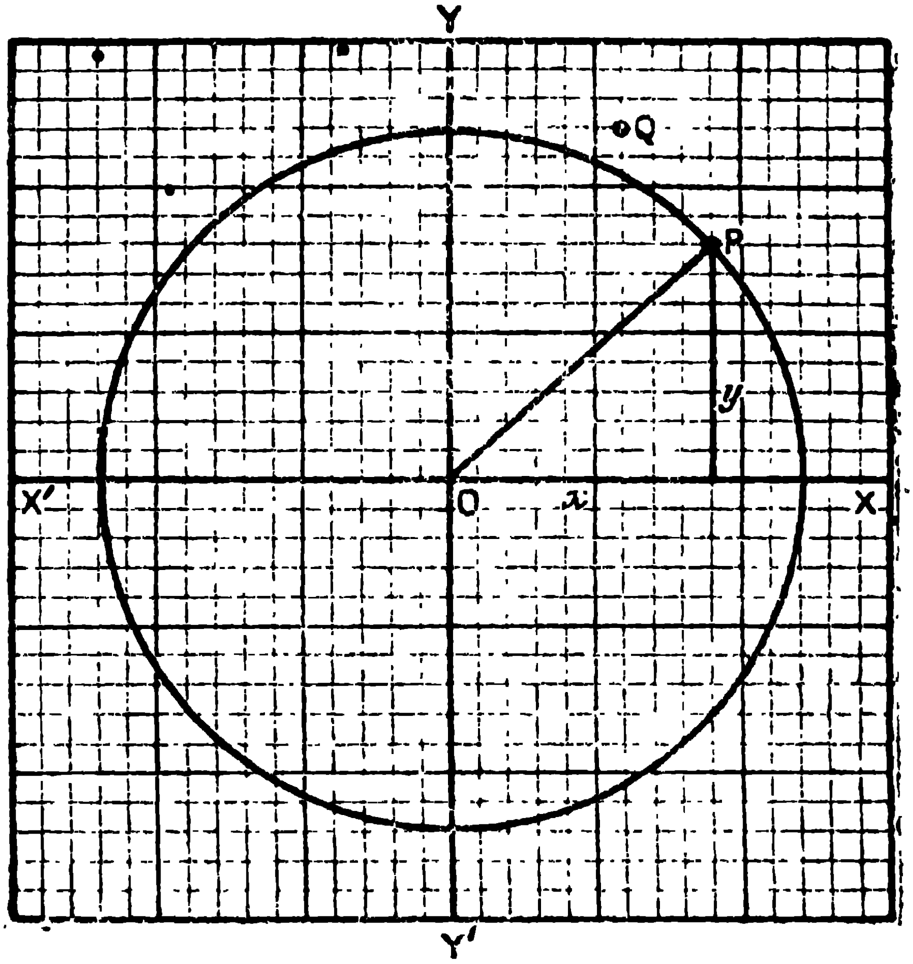
GRAPHS (*Continued*).

1. Draw the graph of the equation $x^2 + y^2 = 36$.

Let twice the length of a side of a small square represent the unit of length.

With centre O and a radius equal to 6 units of length describe a circle, as in the diagram on the next page. Then this circle will be the required graph.

Take *any* point P on the circle, and let its co-ordinates, be denoted by x and y; evidently then $x^2 + y^2 = OP^2 = 36$. But if a point, such as Q, be taken anywhere *not on the circle*, it is easy to see that its co-ordinates will *not* satisfy the given equation.



Thus it is shewn that the co-ordinates of every point on the circle, and of no other point, satisfy the given equation. Hence the circle drawn is the required graph.

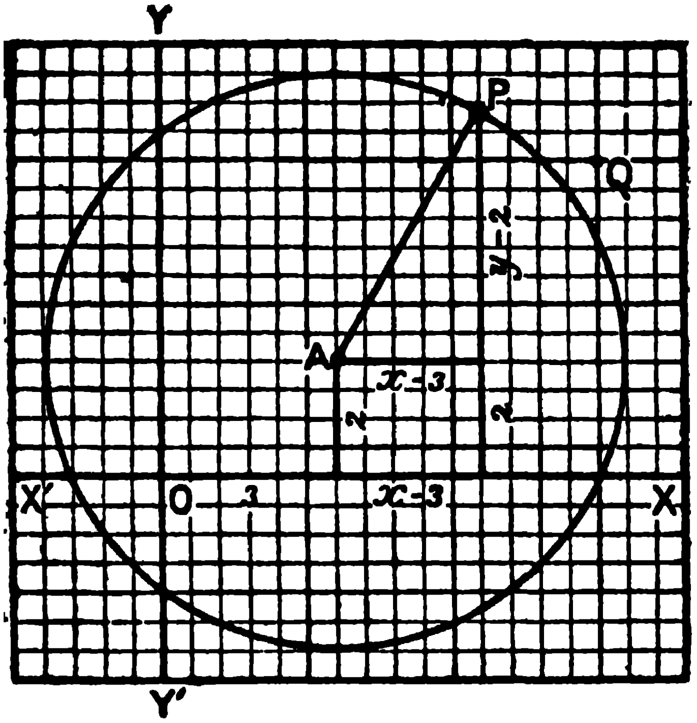
2. Draw the graph of the equation $(x-3)^2+(y-2)^2=25$.

Let twice the length of a side of a small square represent the unit of length.

Let *A* be the point (3, 2). With centre *A* and a radius equal to 5 units of length describe a circle as in the diagram on the next page. Then this circle will be the required graph.

Take *any* point *P* on the circle, and let its co-ordinates be denoted by x and y. Now from the diagram it is clear that *AP* is the hypotenuse of a right-angled triangle of which the sides are $(x-3)$ and $(y-2)$ units of length respectively.

Hence $(x-3)^2+(y-2)^2 = AP^2 = 25$, which shews that the co-ordinates of P satisfy the given equation. But if a point such as Q, be taken any where *not on the circle*, it is easy to see that its co-ordinates will *not* satisfy the given equation.



Thus it is clear that the co-ordinates of every point on the circle and of no other point, satisfy the given equation. Hence the circle described is the required graphs.

Note 1. It may be similarly shewn that the graph of the equation $(x+2)^2+(y+5)^2 = 49$ is a circle of which the centre is the point $(-2, -5)$ and the radius is equal to 7 units of length.

Note 2. The equation $x^2+y^2-8x+10y+25 = 0$ can be easily reduced to the form $(x-4)^2+(y+5)^2 = 16$. Hence its graph is a circle of which the centre is the point $(4, -5)$ and the radius is equal to 4 units of length.

3. Draw the graph of the equation $y^2 = 4x^2$.

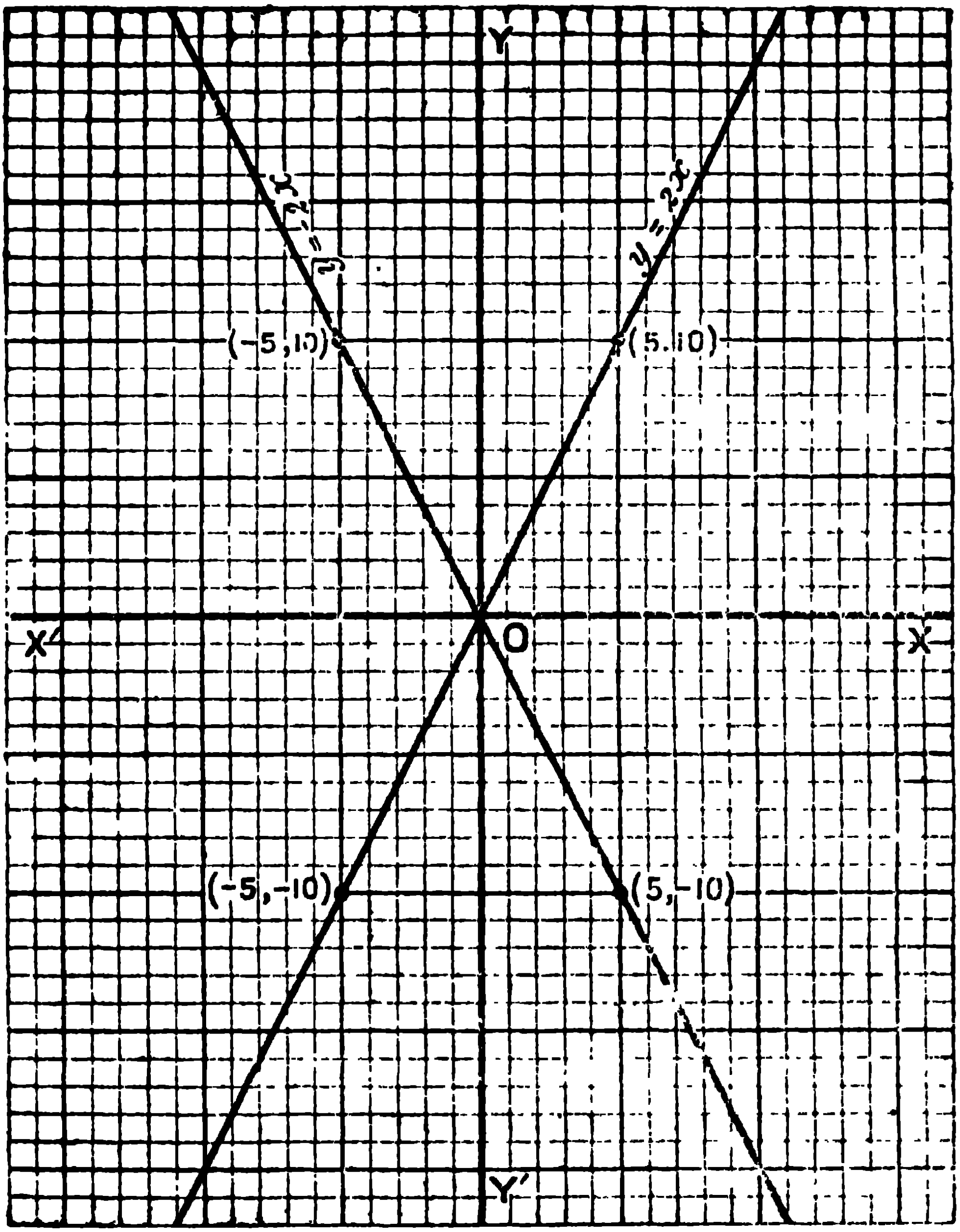
From the given equation we have

$$\left.\begin{aligned} y^2-4x^2 &= 0 \\ \text{or,}\quad (y+2x)(y-2x) &= 0 \end{aligned}\right\}.$$

Hence, it is clear that the given equation is satisfied by (1) all those points which satisfy the equation $y+2x = 0$, and also (2) by all those points which satisfy the equation $y-2x = 0$.

Hence, the required graph consists of *two straight lines*, one being the graph of the equation $y+2x = 0$, and the other being the graph of the equation $y-2x = 0$.

Hence the required graph is as shewn below :—



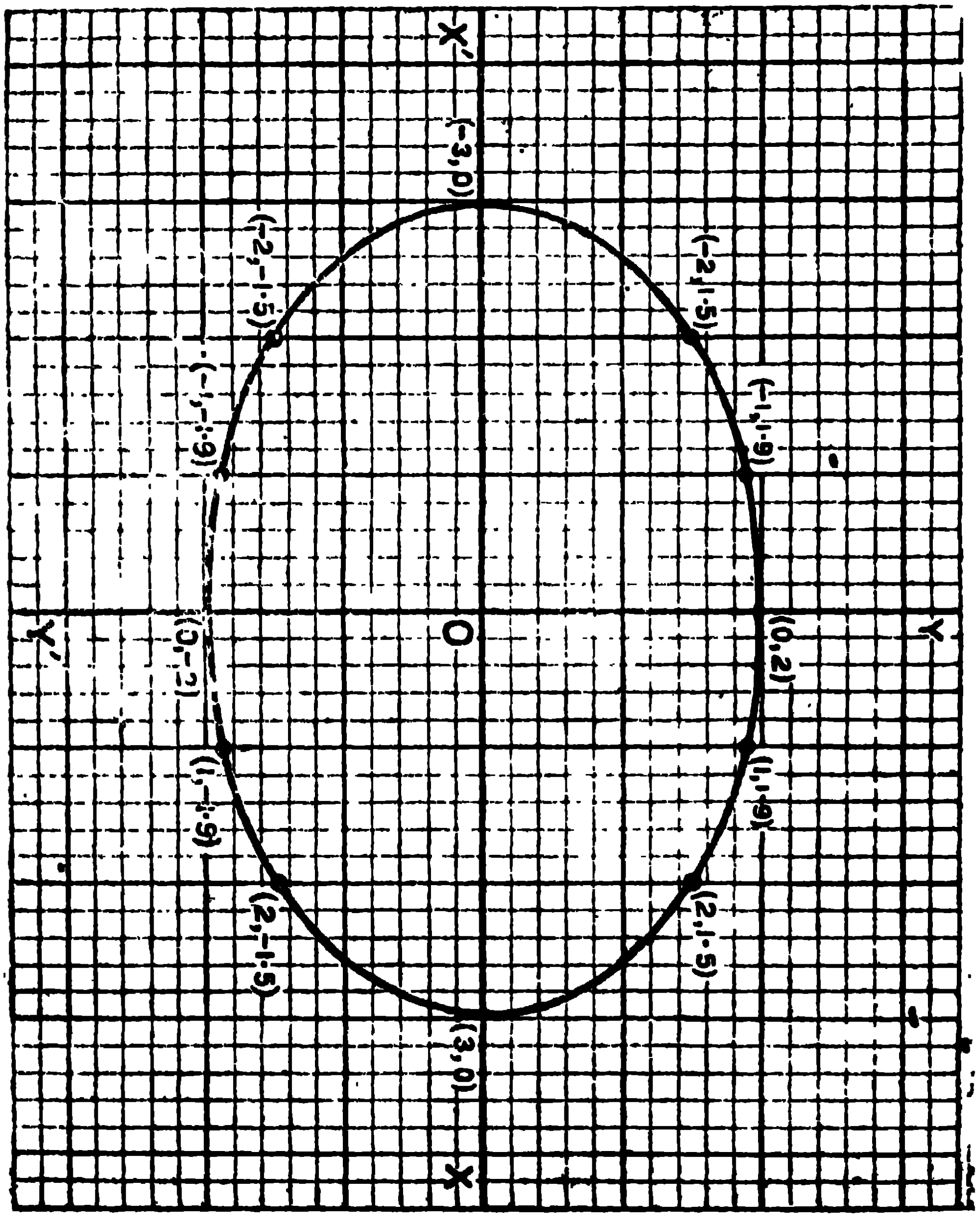
4. Draw the graph of the equation $4x^2+9y^2 = 36$.

(1) When $x = 0$, we have $y^2 = 4$, and therefore $y = \pm 2$. Hence the points (0, 2) and (0, − 2) are on the required graph.

(2) When $y = 0$, we have $x^2 = 9$, and therefore $x = \pm 3$. Hence the points (3, 0) and (− 3, 0) are on the required graph.

(3) When $x = \pm 1$, we have $9y^2 = 32$, and therefore $y = \pm \frac{4}{3}\sqrt{2} = \pm \frac{4 \times 1\cdot414...}{3} = \pm \frac{5\cdot656...}{3} = \pm 1\cdot885... = \pm 1\cdot9$ approximately. Hence the four points (1, 1·9), (1, −1·9), (−1, 1·9) and (−1, −1·9) are on the required graph.

(4) When $x = \pm 2$, we have $9y^2 = 20$, and therefore $y = \pm \frac{2}{3}\sqrt{5} = \pm \frac{2 \times 2\cdot236...}{3} = \pm \frac{4\cdot472...}{3} = \pm 1\cdot490... = \pm 1\cdot5$ nearly. Hence the four points (2, 1·5), (2, −1·5), (−2, 1·5) and (−2, −1·5) are on the required graph.



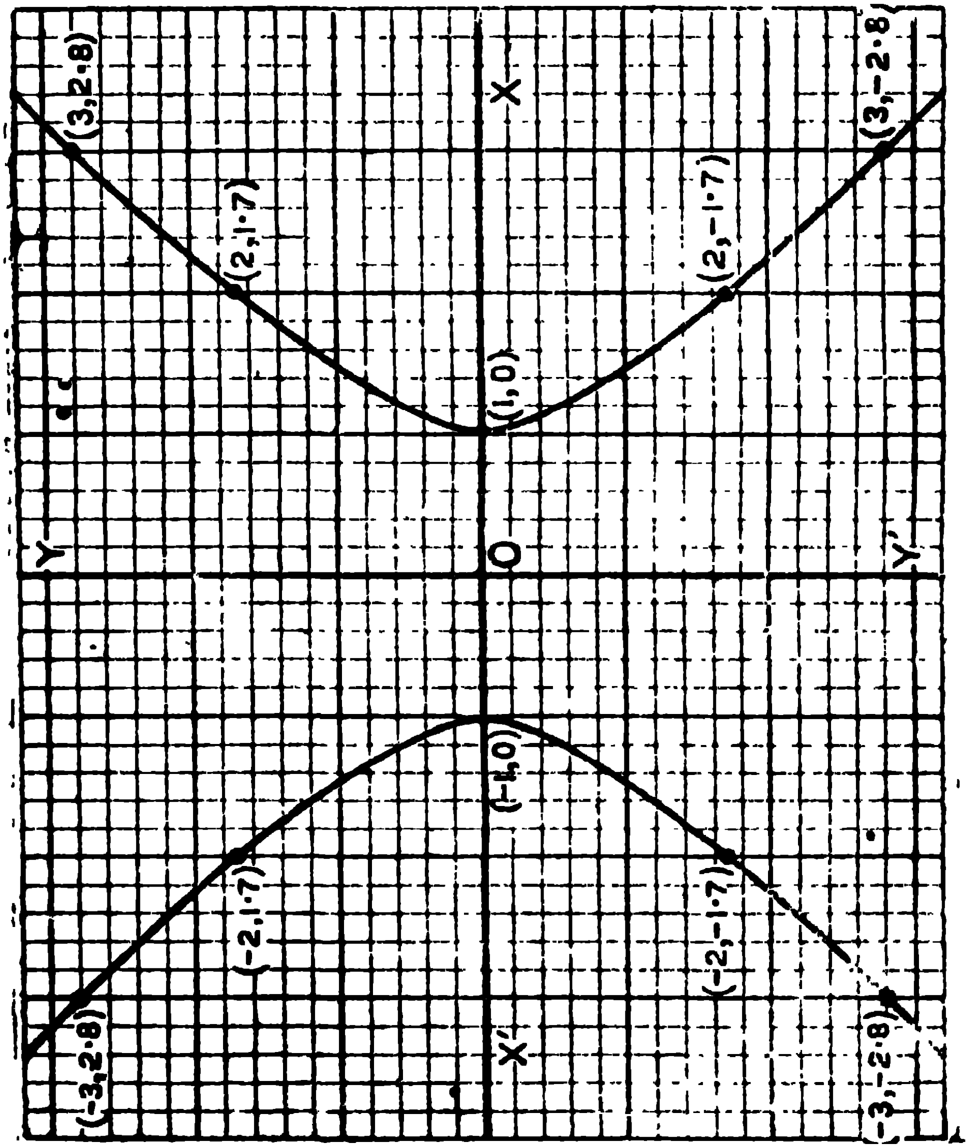
Let us now plot the twelve points as found above (taking 5 times the side of a small square as the unit of length) and draw a free-hand curve through them, as in the diagram on the last page.

The curve so drawn is the required graph.

Note 1. Evidently the curve is *symmetrical* about the axis of x, *i.e.* every chord at right angles to the axis of x is bisected by it. Similarly, the curve is also symmetrical about the axis of y.

Note 2. The curve lies entirely within the space enclosed by the four straight lines $x=3$, $x=-3$, $y=2$, $y=-2$. A curve of this class is called an **Ellipse.**

5. Draw the graph of the equation $x^2 - y^2 = 1$.



(1) When $x = 0$, we have $y^2 = -1$, and therefore y is *imaginary*. This shews that the graph does not cut the axis of y.

(2) When $y = 0$, we have $x^2 = 1$, and therefore $x = \pm 1$. Hence the points (1, 0) and (– 1, 0) are on the required graph.

(3) When $x = \pm 2$, we have $y^2 = 3$, and therefore $y = \pm\sqrt{3} = \pm 1{\cdot}732\ldots = \pm 1{\cdot}7$ approximately. Hence the four points (2, 1·7), (2, – 1·7), (– 2, 1·7) and (– 2, – 1·7) are on the required graph.

(4) When $x = \pm 3$, we have $y^2 = 8$, and therefore $y = \pm 2\sqrt{2} = \pm 2 \times 1{\cdot}414\ldots = \pm 2{\cdot}828\ldots = \pm 2{\cdot}8$ approximately. Hence the four points (3, 2·8), (3, – 2·8), (– 3, 2·8) and (– 3, – 2·8) are on the required graph.

Let us now plot the ten points as found above (taking 5 times the side of a small square as the unit of length) and draw a free-hand curve through them, as in the diagram on the last page.

The curve so drawn is the required graph.

Note 1. The curve so drawn is evidently symmetrical about the axis of x and also about the axis of y.

Note 2. The curve consists of two branches, one lying entirely on the right of the line $x = 1$ and the other lying entirely on the left of the line $x = -1$. A curve of this class is called a **Hyperbola.**

Note 3. From articles 1, 3, 4, and 5 it may be easily seen that if the equation $ax^2 + by^2 = c$ be taken, (1) the graph is *two straight lines passing through the origin* **when c is zero, and a and b are of different signs**; (2) the graph is a *circle* **when $\frac{a}{c}$ and $\frac{b}{c}$ are positive and equal**; (3) the graph is an *Ellipse* **when $\frac{a}{c}$ and $\frac{b}{c}$ are positive and unequal**; and (4) the graph is a *Hyperbola* **when $\frac{a}{c}$ and $\frac{b}{c}$ are of different signs (their absolute values being either equal or unequal).**

6. Draw the graph of the equation $y = x^2$, taking the unit for measuring y equal to half that for measuring x.

Evidently the following points are on the required graph :—

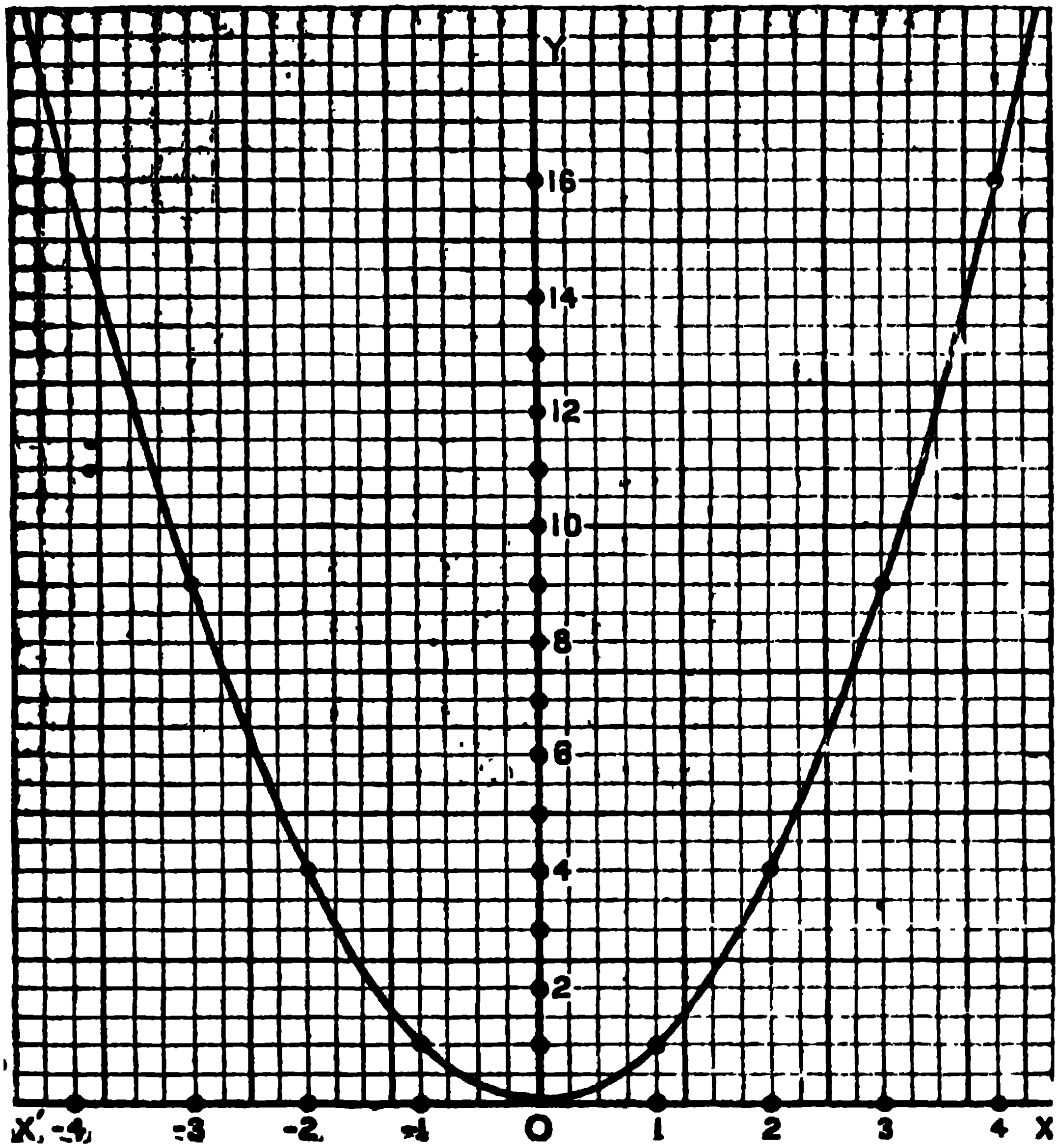
$$\left.\begin{array}{l} x = 0 \\ y = 0 \end{array}\right\}, \qquad \left.\begin{array}{l} x = 1 \\ y = 1 \end{array}\right\}, \qquad \left.\begin{array}{l} x = -1 \\ y = 1 \end{array}\right\},$$

$$\left.\begin{array}{l} x = 2 \\ y = 4 \end{array}\right\}, \quad \left.\begin{array}{l} x = -2 \\ y = 4 \end{array}\right\}, \quad \left.\begin{array}{l} x = 3 \\ y = 9 \end{array}\right\},$$

$$\left.\begin{array}{l} x = -3 \\ y = 9 \end{array}\right\}, \quad \left.\begin{array}{l} x = 4 \\ y = 16 \end{array}\right\}, \quad \left.\begin{array}{l} x = -4 \\ y = 16 \end{array}\right\}.$$

Let four times the side of a small square (*i.e.* ·4 of an inch) be the unit for measuring x and twice the side of a small square (*i.e.* ·2 of an inch), the unit for measuring y.

Let us now plot the points found above and draw a curve through them free-hand, as in the following diagram :—



The curve so drawn is the required graph.

Note 1. If the unit for measuring y were the same as that for measuring x (*i.e.* ·4 of an inch), the curve drawn would be the graph of the equation $y = \frac{1}{2}x^2$, or that of $2y = x^2$.

Note 2. Every chord drawn perpendicular to OY is bisected by it, as can be easily verified. Hence the curve is symmetrical about the axis of y. This is also evident from the fact that if the paper be folded about OY the left-hand portion of the curve entirely coincides with the right-hand portion.

Note 3. The curve lies entirely above the axis of x, and extends upwards to infinity. It is easy to see that the graph of the equation $y = -x^2$ would be a curve lying entirely *below* the axis of x and extending downwards to infinity.

Note 4. The abscissa of any point on the curve is evidently the square root of the ordinate. Hence when the graph of the equation $y = x^2$ is drawn, by measuring the abscissa of any point on the graph we can determine the square root of the number which represents the ordinate.

Note 5. A curve of this class is called a **Parabola**.

7. Draw the graph of the expression: $3 - 4x - 2x^2$.

The required graph is the same as that of the equation $y = 3 - 4x - 2x^2$.

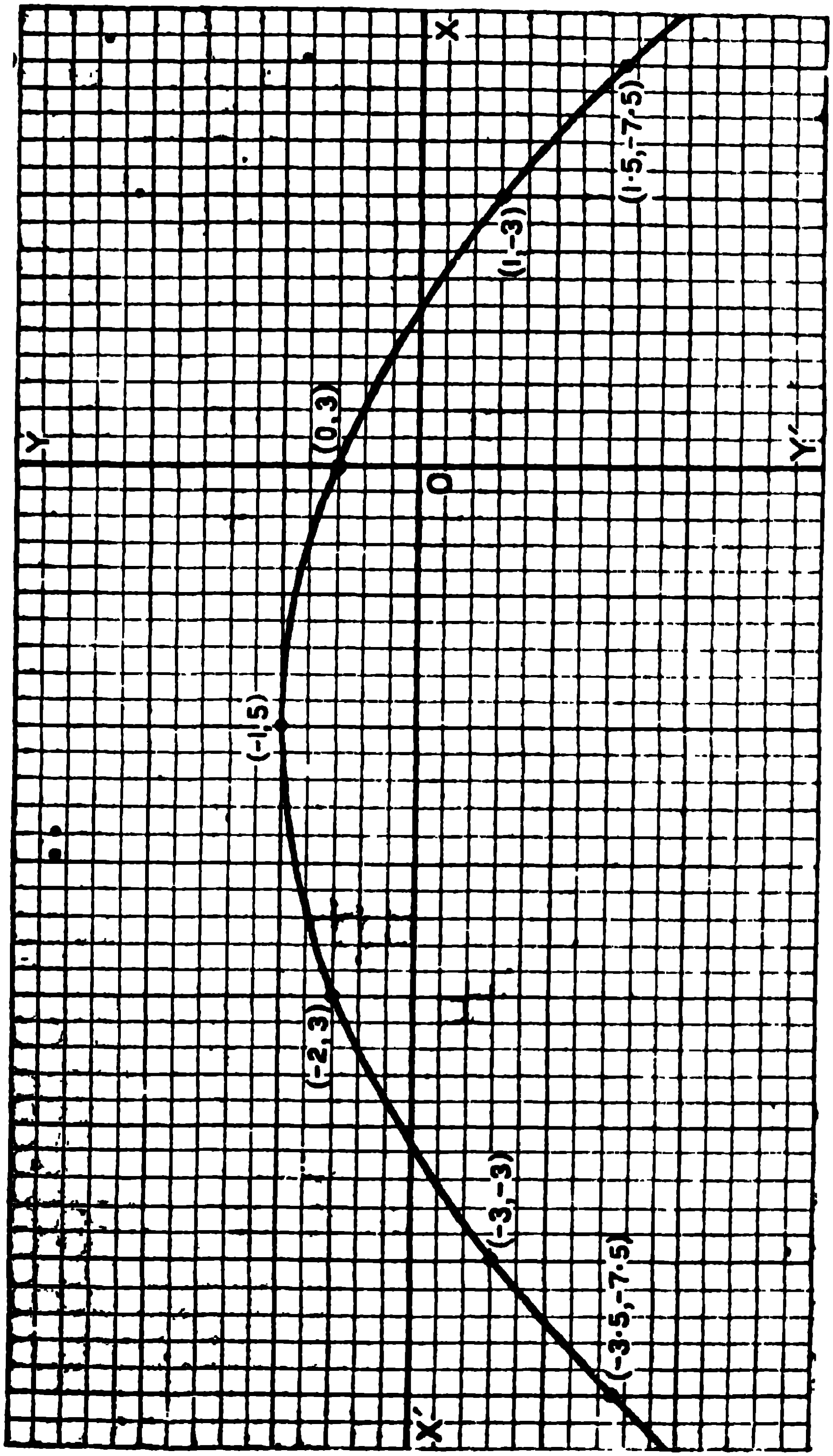
It is easy to see that the following points are on the required graph :—

$$\left.\begin{array}{l} x = 0 \\ y = 3 \end{array}\right\}, \quad \left.\begin{array}{l} x = 1 \\ y = -3 \end{array}\right\}, \quad \left.\begin{array}{l} x = 1{\cdot}5 \\ y = -7{\cdot}5 \end{array}\right\},$$

$$\left.\begin{array}{l} x = -1 \\ y = 5 \end{array}\right\}, \quad \left.\begin{array}{l} x = -2 \\ y = 3 \end{array}\right\}, \quad \left.\begin{array}{l} x = -3 \\ y = -3 \end{array}\right\}, \quad \left.\begin{array}{l} x = -3{\cdot}5 \\ y = -7{\cdot}5 \end{array}\right\}.$$

Take one inch as the unit for measuring x, and one-tenth of an inch as the unit for measuring y.

Let us now plot the above points and draw a curve through them free-hand, as in the following diagram :—



The curve so drawn is the required graph.

Note 1. Since $3 - 4x - 2x^2 = 3 - 2(x^2 + 2x) = 5 - 2(x^2 + 2x + 1) = 5 - 2(x+1)^2$, the equation may also be written as

$$y = 5 - 2(x+1)^2 ;$$

which shows that for all values of x, y is *less than* 5, except when $x = -1$, and in this case $y = 5$. This is also clear from the curve drawn. Hence, the maximum value of y (*i.e.*, that of the expression $3 - 4x - 2x^2$) is 5.

Note 2. If through the point $(-1, 5)$ a straight line be drawn parallel to the axis of y, it is easy to see that the curve is symmetrical about this straight line.

Note 3. From the figure it is evident that $y = 0$ when x is approximately equal to $\cdot 6$ or $-2\cdot 6$. Hence, $3 - 4x - 2x^2 = 0$ when $x = \cdot 6$ or $-2\cdot 6$ approximately; in other words, the roots of the equation $3 - 4x - 2x^2 = 0$ are $\cdot 6$ and $-2\cdot 6$ approximately. From this it is clear that *the roots of the equation* $3 - 4x - 2x^2 = 0$ *are the abscissæ of the points where the graph of the expression* $3 - 4x - 2x^2$ *cuts the axis of* x.

Note 4. The graph of any expression of the form $ax^2 + bx + c$ is a *Parabola.*

8. Draw the graph of the equation $xy = 1$.

It is easy to see that the following points are on the required graph :—

$$\left.\begin{array}{l} x = \cdot 1 \\ y = 10 \end{array}\right\}, \quad \left.\begin{array}{l} x = \cdot 2 \\ y = 5 \end{array}\right\}, \quad \left.\begin{array}{l} x = \cdot 4 \\ y = 2\cdot 5 \end{array}\right\},$$

$$\left.\begin{array}{l} x = \cdot 5 \\ y = 2 \end{array}\right\}, \quad \left.\begin{array}{l} x = \cdot 8 \\ y = 1\cdot 25 \end{array}\right\}, \quad \left.\begin{array}{l} x = 1 \\ y = 1 \end{array}\right\}, \quad \left.\begin{array}{l} x = 2 \\ y = \cdot 5 \end{array}\right\}.$$

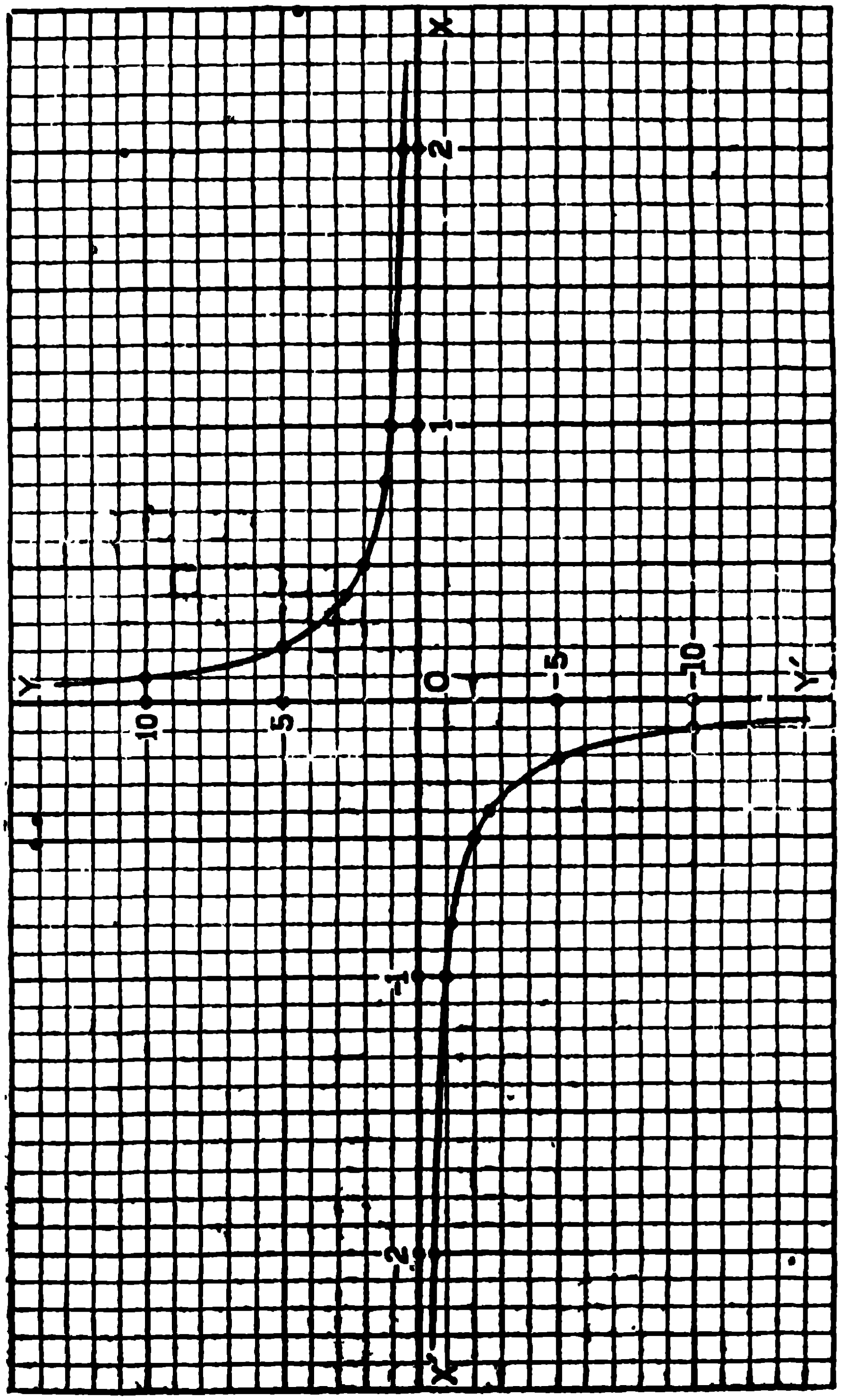
Evidently also the following points are on the required graph :—

$$\left.\begin{array}{l} x = -\cdot 1 \\ y = -10 \end{array}\right\}, \quad \left.\begin{array}{l} x = -\cdot 2 \\ y = -5 \end{array}\right\}, \quad \left.\begin{array}{l} x = -\cdot 4 \\ y = -2\cdot 5 \end{array}\right\},$$

$$\left.\begin{array}{l} x = -\cdot 5 \\ y = -2 \end{array}\right\}, \quad \left.\begin{array}{l} x = -\cdot 8 \\ y = -1\cdot 25 \end{array}\right\}, \quad \left.\begin{array}{l} x = -1 \\ y = -1 \end{array}\right\}, \quad \left.\begin{array}{l} x = -2 \\ y = -\cdot 5 \end{array}\right\}.$$

Let one inch be the unit for measuring x and one-tenth of an inch the unit for measuring y.

Let us now plot the points and draw a curve through them free-hand, as in the following diagram :—



The curve so drawn is the required graph.

Note 1. As x diminishes from 1 to zero, y increases from 1 to infinity; and as x diminishes from zero to -1, y increases from negative infinity to -1.

Note 2. As x increases from 1 to infinity, y diminishes from 1 to zero; and as x diminishes from -1 to negative infinity, y increases from -1 to zero.

Note 3. The graph consists of two branches, one lying between OX and OY, and the other between OX′ and OY′.

Note 4. The more we move towards the right or left of O, the nearer does the curve approach the axis of x; whilst the more we move upwards or downwards from O, the nearer does the curve approach the axis of y. But in no case does the curve *meet* the axes except at an infinite distance from O. Hence, each of the axes is said to be an **asymptote** to the curve.

Note 5. A curve of this kind is called a **Rectangular Hyperbola.**

Exercise (8.)

Draw the graphs of the following equations :—

1. $x^2 + y^2 = 81$.

2. $(x-5)^2 + (y-6)^2 = 49$.

3. $(x+6)^2 + (y-7)^2 = 100$.

4. $x^2 + y^2 - 8x - 14y + 1 = 0$.

5. $x^2 + y^2 + 14x - 16y + 32 = 0$.

6. $x^2 + y^2 + 12x + 18y + 92 = 0$.

7. $x^2 + y^2 - 10x + 16y - 55 = 0$.

8. $x^2 - y^2 = 0$.

9. $9x^2 - 4y^2 = 0$.

10. $9y^2 = 16x^2$.

11. $4x^2 = 9$.

12. $25y^2 = 16$.

13. $x^2 + 4y^2 = 4$.

14. $4x^2 + 9y^2 = 1$.

15. $25x^2 + y^2 = 25$.

16. $16x^2 + 9y^2 = 1$.

17. $x^2 - 4y^2 = 4$.

18. $y^2 - x^2 = 1$.

19. $4x^2 - y^2 = 16$.

20. $y^2 - 9x^2 = 9$.

21. In one and the same diagram draw the graphs of $4x^2 - 9y^2 = 0$ and $4x^2 - 9y^2 = 36$.

22. In one and the same diagram draw the graphs of $9y^2 - 4x^2 = 0$ and $9y^2 + 4x^2 = 36$.

23. Draw the graph of the equation $5y = x^2 - 10$,

taking the unit for measuring y five times as large as that for measuring x.

24. Draw the graph of the equation $x^2 - 4x + 2y = 0$,

taking the unit for measuring y twice as large as that for measuring x.

25. Draw the graph of the equation $y^2 + x = 0$,

taking the unit for measuring x equal to half that for measuring y.

26. Draw the graph of the equation $3y = x^2$,

taking the same unit for measuring both x and y.

27. Find graphically, correct to the first figure after the decimal point, the square roots of :—

(i) 3 ; (ii) 5 ; (iii) 7.

28. Find graphically the minimum value of the expression

$$2x^2 - 6x + 7.$$

29. Find graphically the maximum value of the expression

$$1 + 2x - 2x^2.$$

30. Draw the graph of the equation $xy = 4$.

CHAPTER III.

Miscellaneous Examples.

Example 1. The salaries of the teachers of a certain school are increased; so that a salary of Rs. 60 is increased to Rs. 70, and one of Rs. 90 to Rs. 104. What then must be the relation between x and y, if x rupees be the old salary of a teacher whose new salary is y rupees? Hence deduce a graphical method of finding the new salary of a teacher whose old salary is given, and *vice versa.*

(i) Assume that the relation between x and y is

$$y = ax + b,$$

where a and b are any two unknown constants.

Then, by hypothesis,

$$\left.\begin{aligned} 70 &= 60a + b \\ \text{and} \quad 104 &= 90a + b \end{aligned}\right\}$$

whence, $a = \dfrac{17}{15}$ and $b = 2$.

Hence the required relation is

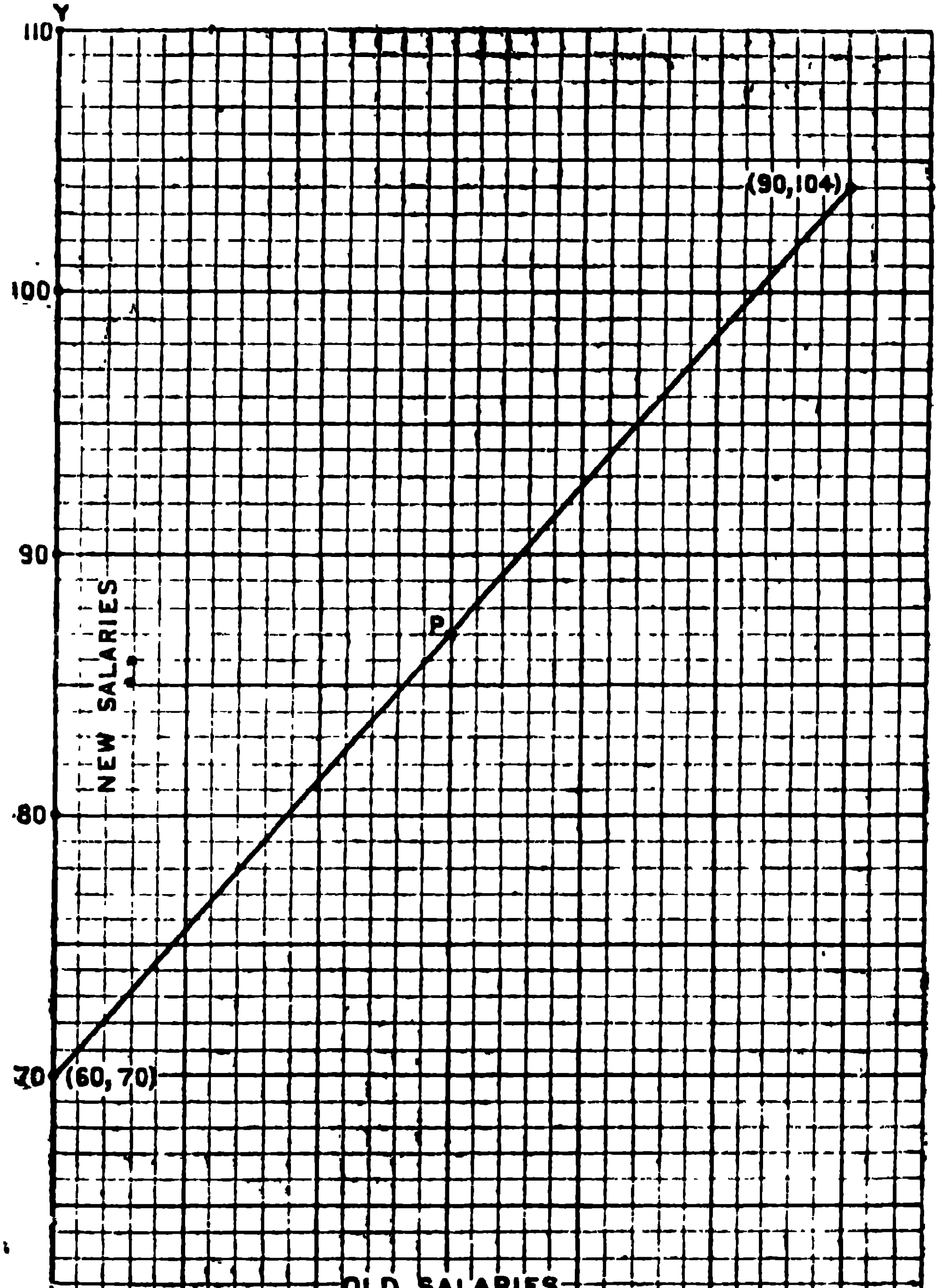
$$y = \frac{17}{15}x + 2.$$

(ii) The graph of the above equation is surely a straight line.

Let one-tenth of an inch, measured horizontally, represent one rupee of the old salary, and let one-tenth of an inch, measured vertically, represent one rupee of the new salary. Then the meaning of the figures in the following diagram is clear, the origin being taken as the point whose co-ordinates are 60 and 60.

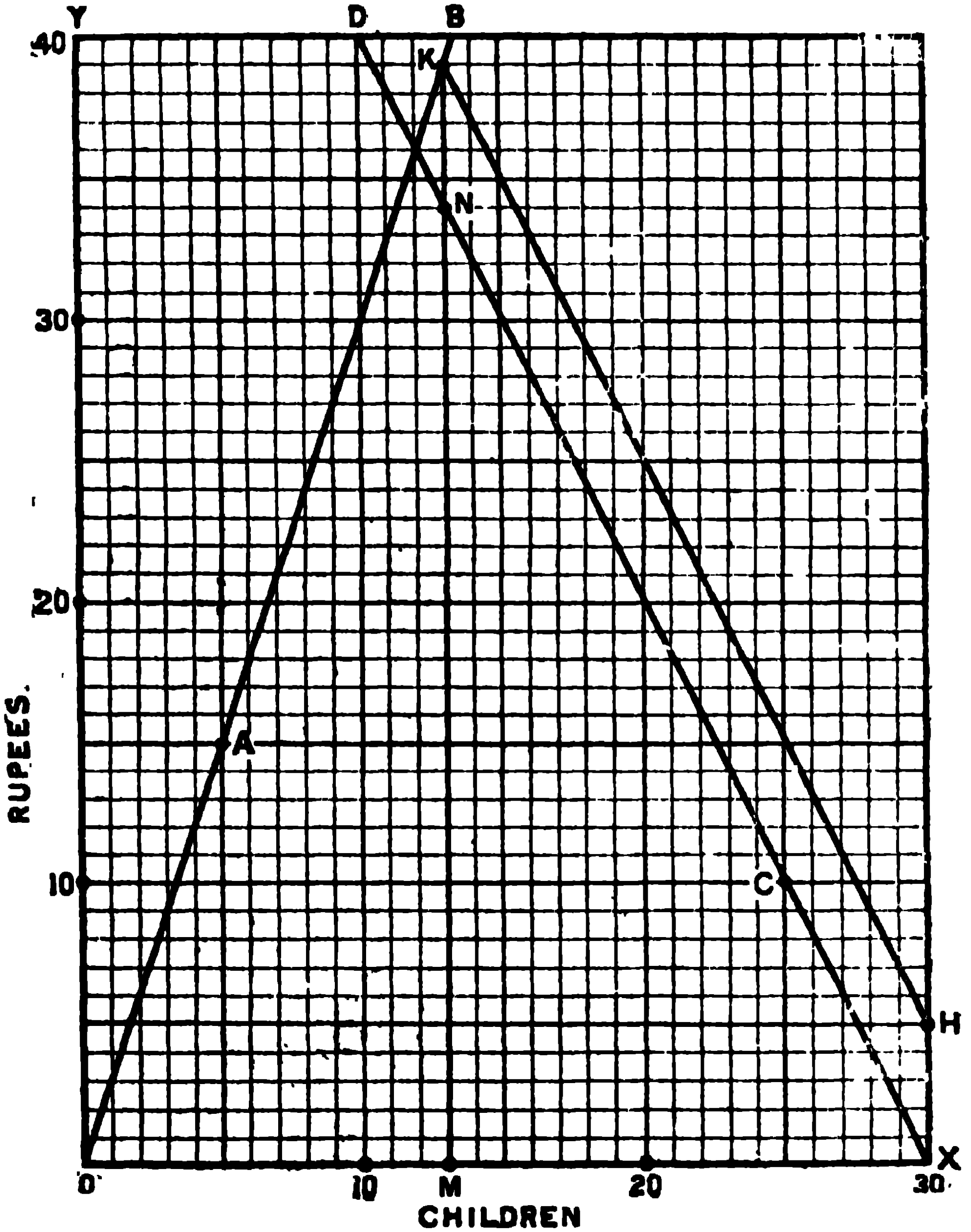
The points (60, 70) and (90, 104) satisfy the above equation. Hence the straight line joining these two points is the required graph.

The co-ordinates of the point P on the graph are 75 and 87; this shews that Rs. 87 is the new salary of a teacher whose old salary was Rs. 75. Again, if we take the point whose abscissa is 80, we find that its ordinate is $92\frac{2}{3}$; this shews that if the old salary of a teacher was Rs. 80, his new salary is $92\frac{2}{3}$. And so on.



Example 2. There is a group of 30 children before me, of whom some are boys and the rest are girls. If I get 3 rupees from each boy and give 2 rupees to each girl, I gain altogether Rs. 5. Find graphically the number of boys and girls in the group.

Let one-tenth of an inch, measured horizontally, represent one child, and let one-tenth of an inch, measured vertically, represent one rupee.



OX representing the total number of children, the number of boys may be measured from O and the number of girls from X.

Hence, if A be the point (5, 15) from O, the straight line OAB will be the graph of the money received from the boys; and if C be the point (5, 10) from X, the straight line XCD will be the graph of the money given to the girls.

Let KNM be an ordinate cutting the graphs at K and N. Then KM represents the money received from the boys whose number is represented by OM, and NM represents the money given to the girls whose number is represented by XM, and therefore KN represents the gain.

Hence, if the ordinate be so drawn that the portion KN, intercepted between the two graphs, represents 5 rupees, then OM and XM will represent the required numbers of boys and girls respectively.

We have therefore to find K by taking XH to represent 5 rupees and drawing HK parallel to XD. Now drawing the ordinate KM, we find that OM = 13 units of length and XM = 17 units of length.

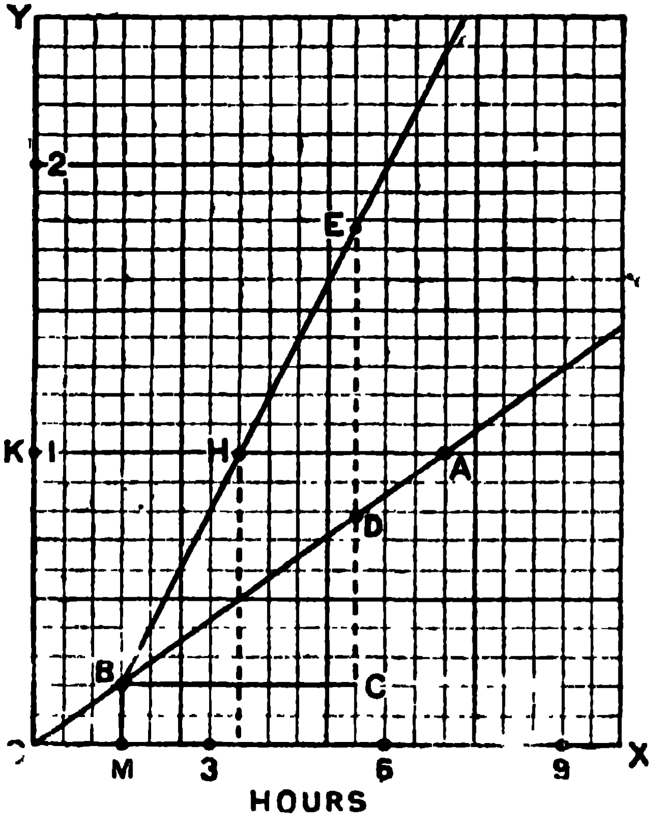
Hence the number of boys in the group = 13, and the number of girls = 17.

Note. If Rs. 2 were to be *not* given to, but *taken from*, each girl, the graph of the money received from the girls would have to be drawn *below* the line XO.

Example 3. P and Q can separately perform a certain work in 7 hours and 4 hours respectively. After P has worked for an hour and a half, Q joins. Find graphically the time in which they will together finish the work.

Let ·2 of an inch, measured horizontally, represent an hour, and let one inch, measured vertically, represent the work.

Let A be the point (7, 1); then the straight line OA is the graph of the work done by P.



If the vertical through M $(1\frac{1}{2}, 0)$ meet OA in B, then BM represents the work done by P in $1\frac{1}{2}$ hours.

Draw BC (horizontally) to represent 4 hours; let the vertical through C meet OA in D, and produce CD to E making DE equal to OK.

Then CE represents the work done by P and Q in 4 hours. Hence the straight line BE is the graph of the work jointly done by P and Q.

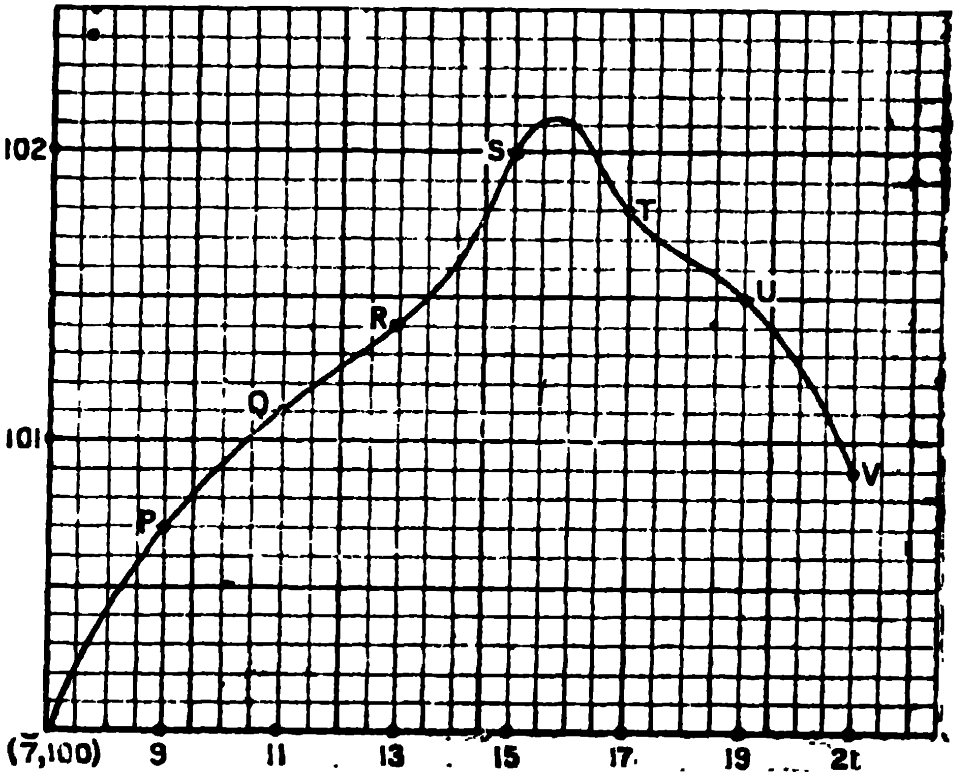
Let BE cut the horizontal line through K in H. Then, since the co-ordinates of H are found to be $3\frac{1}{2}$ and 1, it is clear that P and Q will together finish the work $3\frac{1}{2}$ hours after P begins or 2 hours after Q joins.

Example 4. The temperature of a patient was observed at intervals of 2 hours from 7 A.M. till 9 P.M. on a certain day, and the observations were recorded as follows :—

Time.	7 A.M.	9 A.M.	11 A.M.	1 P.M.	3 P.M.	5 P.M.	7 P.M.	9 P.M.
Temperature.	100°	100°·7	101°·1	101°·4	102°	101°·8	101°·5	100°·9

Exhibit graphically the variation in temperature during the whole interval; and, from the graph drawn, find the probable time at which the temperature was a maximum, the probable maximum temperature and also the probable temperature at 8 P.M.

Let ·2 of an inch, measured horizontally, represent an hour, and let one inch, measured vertically, represent a degree of temperature. The numbers 13, 15, 17, 19 and 21 may also be written for 1 P.M., 3 P.M., 5 P.M., 7 P.M., and 9 P.M. respectively. Hence the meaning of the figures along OX and OY is clear, the origin being taken as the point whose co-ordinates are 7 and 100.



Let P, Q, R, S, T, U, V denote the points (9, 100·7), (11, 101·1), (13, 101·4), (15, 102), (17, 101·8), (19, 101·5) and (21, 100·9) respectively.

Let us now plot the points P, Q, R, S, T, U, V and draw through them free hand a *continuous* curve, starting from O.

The curve so drawn is the required graph.

Now, it is evident from the diagram that the point on the curve which has the greatest ordinate is the point (15·5, 102·1). Hence, the temperature reached its maximum at about 3-30 P.M.; and the maximum temperature was approximately 102°·1.

The point on the curve which corresponds to 8 P.M. is the point whose abscissa is 20 and whose ordinate is found to be 101·25 approximately. Hence the probable temperature at 8 P.M. was 101°·25.

Note. It may be reasonably supposed that the change in temperature is *gradual* and not sudden; hence the necessity of drawing a *continuous* curve, that is, one not having sharp turns.

Example 5. Solve graphically

$$3 - 4x - 2x^2 = 0.$$

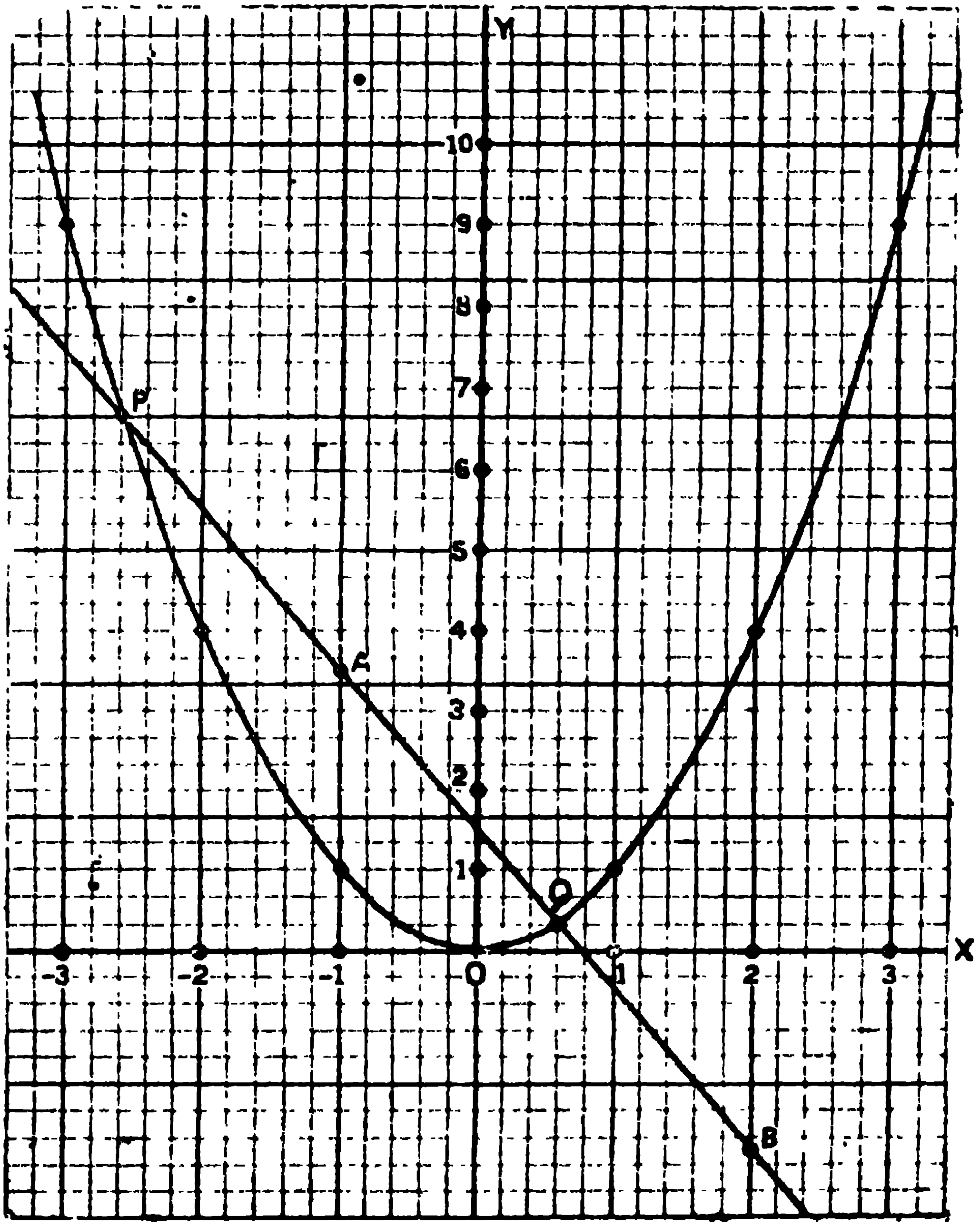
[One method of solution has been referred to in Note 3 to Art. 7 of the last chapter. A second method is given below.]

$$\text{We have } 2x^2 = 3 - 4x;$$

$$\therefore \quad x^2 = \tfrac{3}{2} - 2x.$$

Let us, first of all, draw the graph of the equation $y = x^2$.

Take ·5 of an inch as the unit for measuring x, and ·3 of an inch as the unit for measuring y. Then the graph will be as in the following diagram :—



With the same units, now draw the graph of the equation $y = \frac{3}{2} - 2x$. The points A and B, whose co-ordinates are respectively $(-1, 3\frac{1}{2})$ and $(2, -2\frac{1}{2})$, evidently lie on this graph. Hence the straight line AB is the required graph.

Let this straight line cut the parabola in P and Q. Then, if x, y denote the co-ordinates of either P or Q, we must have

$$\left.\begin{array}{ll} & y = x^2, \\ \text{and also} & y = \frac{3}{2} - 2x \end{array}\right\}; \text{ because each of these two points lies on } \textit{both} \text{ the graphs.}$$

Hence, the abscissa of either P or Q will satisfy the equation

$$x^2 = \tfrac{3}{2} - 2x.$$

Thus the abscissæ of the points P and Q are the roots of the given equation.

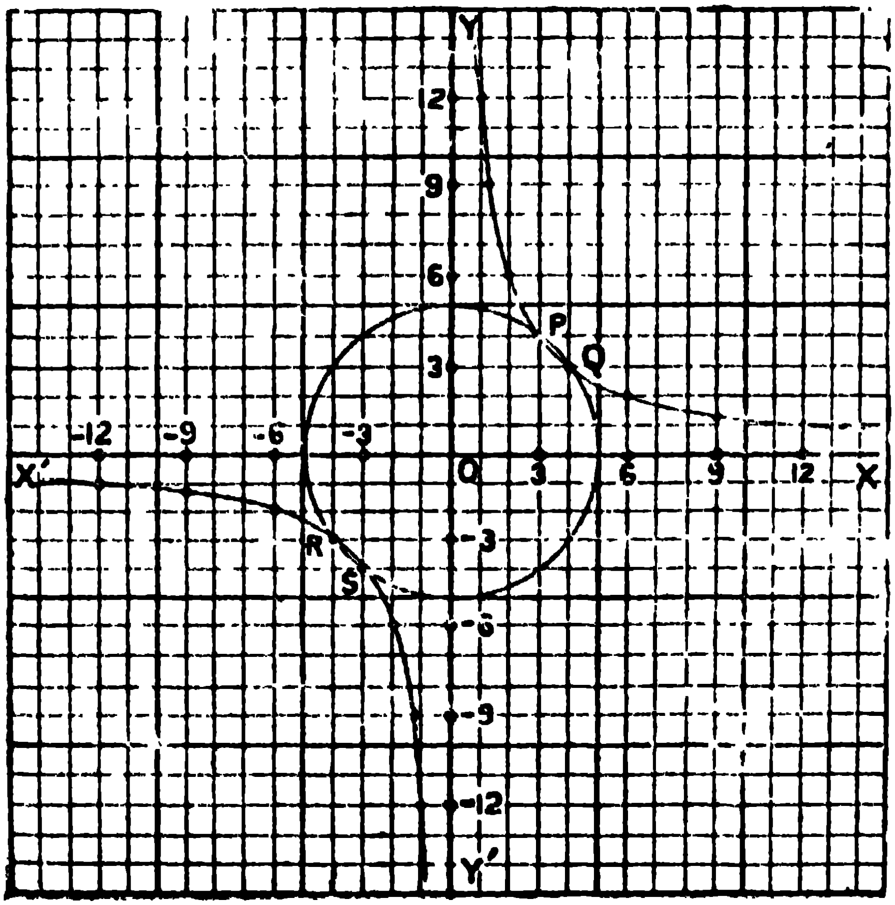
These abscissæ are found to be respectively equal to $-2{\cdot}6$ and $\cdot 6$ approximately, which therefore are the required roots.

Note. The present method has this advantage that the parabola $y = x^2$ being once drawn, any equation of the form $ax^2 + bx + c = 0$ may be immediately solved by simply drawing the straight line $y = -\frac{b}{a}x - \frac{c}{a}$.

Example 6. Find graphically the different pairs of values of x and y that simultaneously satisfy the following equations :—

$$\left.\begin{aligned} x^2 + y^2 &= 25 \\ xy &= 12 \end{aligned}\right\}.$$

Let us draw the graphs of the two equations, as in the following diagram, taking one-tenth of an inch as the unit for measuring x and y :—



Let the two graphs intersect each other at the points P, Q, R, S. Then the co-ordinates of each of these points will satisfy *both* the equations.

Now, the co-ordinates of these points are found to be respectively (3, 4), (4, 3), (– 3, – 4) and (– 4, – 3).

Hence the required solutions are

$$\left.\begin{array}{l} x = 3 \\ y = 4 \end{array}\right\}, \quad \left.\begin{array}{l} x = 4 \\ y = 3 \end{array}\right\}, \quad \left.\begin{array}{l} x = -3 \\ y = -4 \end{array}\right\}, \quad \left.\begin{array}{l} x = -4 \\ y = -3 \end{array}\right\}.$$

Exercise (9).

N. B. The graphical method should be used in solving each of the following problems :—

1. A can do a piece of work in 6 days, and B can do it in 9 days ; if they work together, how many days will they take to finish it ?

2. A tap, which would fill a cistern in 4 hours, and a plug, which would empty it in 6 hours, are both opened at the same instant, when the cistern is empty. How long will they take to fill the cistern ?

3. A cistern can be filled by a pipe A in 26 minutes, and by a pipe B in 15 minutes, while it can be emptied by a pipe C in 12 minutes ; if all three pipes are set running when the cistern is empty, in what time will it be filled ?

4. A, B and C can separately do a piece of work in 5, 8, and $3\frac{1}{3}$ days respectively ; if all three work together, how long will they take to finish the work ?

5. At what time between 2 and 3 o'clock are the two hands of a watch (i) together, (ii) 5 minute-divisions apart ?

6. Rs. 45 is the price of 20 balls of which some are white and the rest black. If the price of each white ball be Rs. 3 and that of each black ball Rs. 2, how many of the balls are white ?

7. A servant makes a contract to get 4 annas for every day that he works, and to pay a fine of 6 pice for every day that he is absent. If he altogether gets Rs. 3 8 as after 25 days, how many days was he absent from work ?

8. From the same spot on a circular course, one mile in length, two boys A and B start at the same moment to walk round it, travelling in the same direction. A walks at 4, and B at 3, miles an hour; how often will they meet if they walk for two hours and a half?

9. In the preceding example, if the boys start in opposite directions, and walk respectively at 3 and 2 miles an hour, how often will they meet within half an hour?

10. The highest and lowest marks obtained in an examination are 283 and 110 respectively; the marks are reduced so that 283 becomes 100 and 110 becomes 50. Find approximately the numbers to which 248 and 124 are respectively reduced.

11. The temperature of a room taken at every hour of the day from 9 A.M. until 4 P.M. is given in the following table:—

Time.	9 A.M.	10	11	12	1 P.M.	2	3	4
Temperature	56	57	59	62	62	61	58	57

Construct a graph to shew the variation of temperature, and ascertain from the graph the temperature at 10·30 A.M.

12. Find the roots of the following equations, correct to the first figure after the decimal point:—

(i) $x^2 - 2x = 4$;

(ii) $\frac{x^2}{4} + x - 2 = 0$;

(iii) $4x^2 - 16x + 9 = 0$.

13. Find the different pairs of values of x and y that satisfy both the equations $x^2 + y^2 = 100$ and $x + y = 14$.

14. Find the different pairs of values of x and y that satisfy both the equations $x - y = 3$ and $xy = 4$.

CHAPTER IV.

Graphs of Exponential and Logarithmic Functions.

1. Draw the graph of the function 10^x.

The required graph is the same as that of the equation $y = 10^x$.

It is easy to see that, among many, the following pairs of values of x and y will satisfy the above equation :—

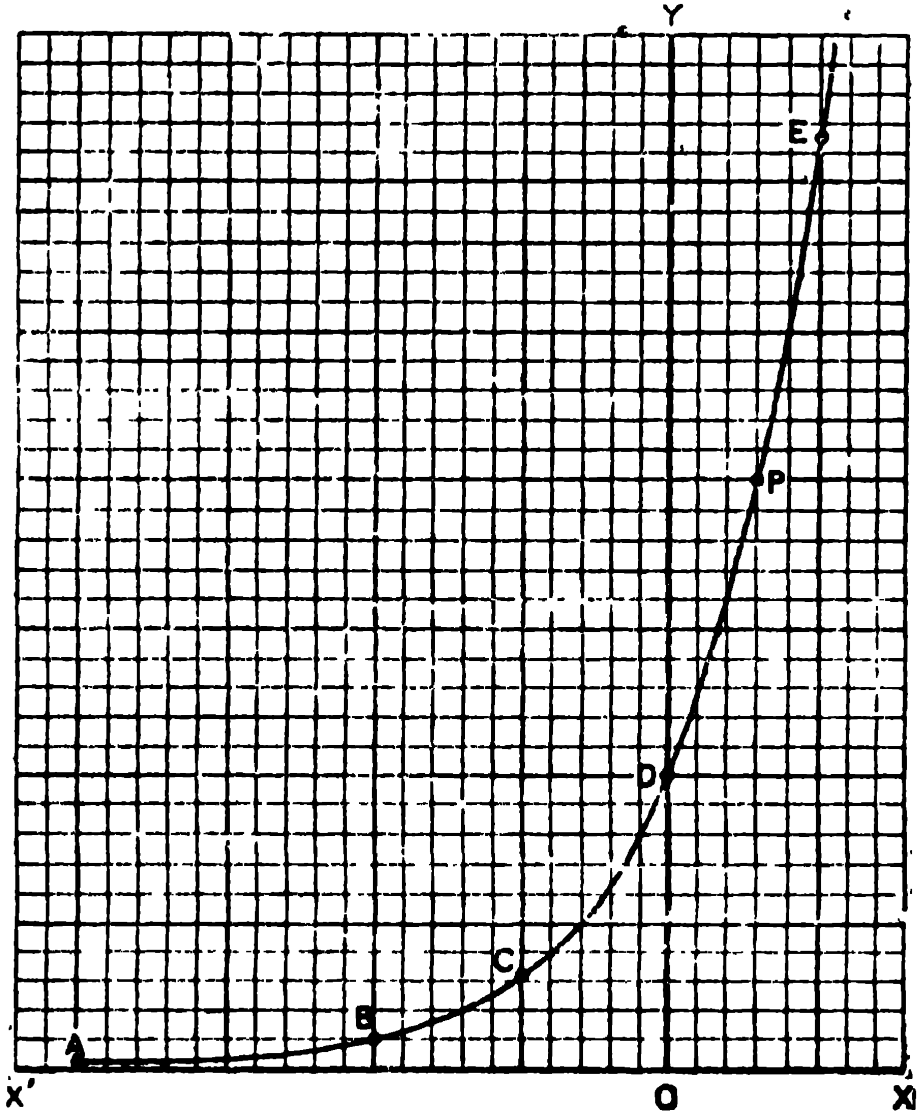
$$\left.\begin{matrix} x = 0 \\ y = 1 \end{matrix}\right\}, \quad \left.\begin{matrix} x = \frac{1}{2} \\ y = 3{\cdot}16 \end{matrix}\right\}, \quad \left.\begin{matrix} x = 1 \\ y = 10 \end{matrix}\right\}, \quad \left.\begin{matrix} x = 2 \\ y = 100 \end{matrix}\right\},$$

$$\left.\begin{matrix} x = -\frac{1}{2} \\ y = {\cdot}32 \end{matrix}\right\}, \quad \left.\begin{matrix} x = -1 \\ y = {\cdot}1 \end{matrix}\right\}, \quad \left.\begin{matrix} x = -2 \\ y = {\cdot}01 \end{matrix}\right\}.$$

Let A, B, C, D, E denote the points whose co-ordinates are respectively $(-2, {\cdot}01)$, $(-1, {\cdot}1)$, $(-\frac{1}{2}, {\cdot}32)$, $(0, 1)$ and $(\frac{1}{2}, 3{\cdot}16)$.

Plot these points, taking one inch as the unit of length.

Then the curve, drawn through the points thus plotted, is the required graph, as shewn in the following diagram :—



Note 1. For all real values of x, y is positive. Hence, the curve lies entirely above the axis of x

Note 2. As x increases from zero, y increases from 1 ; but the increment of y is much more rapid than that of x, so that when a large value is given to x, the corresponding value of y becomes comparatively very much larger. Hence, any straight line, that cuts OX or OX produced, and is also parallel to OY, *must* intersect the curve, although the point of intersection may be at a very great distance from OX.

Note 3. As x diminishes from zero to negative infinity, y diminishes from one to zero. Hence it is clear that the left portion of the curve gradually approaches OX′ and ultimately meets it at infinity. Hence OX is an asymptote to the curve.

Note 4. From the equation $y = 10^x$, we have $x = \log_{10} y$. That is, if we take any point on the curve, its abscissa will give the logarithm of its ordinate to the base 10. For instance, P is the point whose ordinate is 2 and abscissa is approximately ·3; hence we at once conclude that $\log_{10}2 = \cdot 3$ approximately.

2. Draw the graph of the function $\log_{10}x$.

The required graph is the same as that of the equation $y = \log_{10}x$.

Since $y = \log_{10}x$, we have $x = 10^y$. Hence, it is clear that, among many, the following pairs of values of x and y will satisfy the equation $y = \log_{10}x$:—

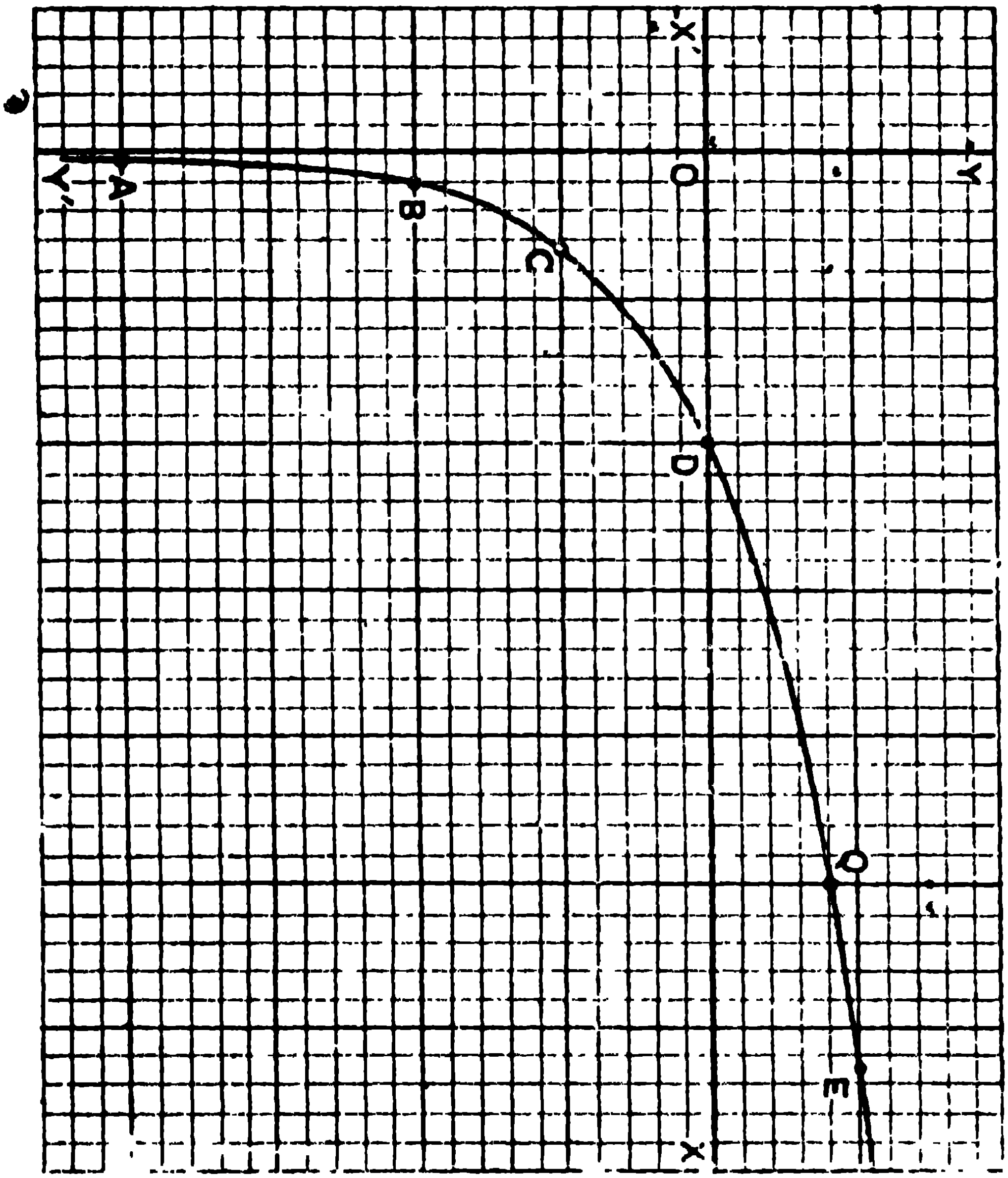
$$\left.\begin{array}{l} y = 0 \\ x = 1 \end{array}\right\}, \quad \left.\begin{array}{l} y = \frac{1}{2} \\ x = 3\cdot16 \end{array}\right\}, \quad \left.\begin{array}{l} y = 1 \\ x = 10 \end{array}\right\}, \quad \left.\begin{array}{l} y = 2 \\ x = 100 \end{array}\right\},$$

$$\left.\begin{array}{l} y = -\frac{1}{2} \\ x = \cdot32 \end{array}\right\}, \quad \left.\begin{array}{l} y = -1 \\ x = \cdot1 \end{array}\right\}, \quad \left.\begin{array}{l} y = -2 \\ x = \cdot01 \end{array}\right\}.$$

Let A, B, C, D, E denote the points whose co-ordinates are respectively $(\cdot01, -2)$, $(\cdot1, -1)$, $(\cdot32, -\frac{1}{2})$, $(1, 0)$ and $(3\cdot16, \frac{1}{2})$.

- Plot these points, taking one inch as the unit of length.

Then the curve, drawn through the points thus plotted, is the required graph, as shewn in the following diagram :—



Note 1. As $x = 10^y$, it is clear that y may be positive or negative, but x can *never be negative*. Hence the curve lies entirely on the right side of the axis of y.

Note 2. As y diminishes from zero to negative infinity, x diminishes from one to zero. Clearly therefore the lower portion of the curve gradually approaches OY′ and ultimately meets it at infinity. Hence OY′ is an asymptote to the curve.

Note 3. As y increases from zero, x increases from 1, but the increment of x is much more rapid than that of y, so that when a large value is given to y, the corresponding value of x becomes comparatively very much larger. Hence *any* straight line that cuts OY or OY′ produced, and is also parallel to OX, *must* intersect the curve, although the point of intersection may be at a very great distance from OY.

Note 4. If any point be taken on the curve, its ordinate will give the logarithm of its abscissa to the base 10. For instance, Q is the point whose abscissa is 2·5 and ordinate is ·4 approximately; hence we at once conclude that $\log_{10} 2{\cdot}5 = {\cdot}4$ approximately.

Exercise (10).

1. Solve graphically the following equations :—

(i) $10^x = 10x$; (ii) $\log_{10} x = \frac{1}{10}x$; (iii) $10^{\frac{x}{2}-1} = 6x-8$.

2. Draw the graphs of the following functions, using Logarithmic Tables :—

(i) $(1+x)^x$: (ii) $10^{\frac{3}{5}x}$.

ANSWERS TO EXERCISES IN THE APPENDIX.

1. [Page 514.]

2. Take BE equal to AD; by guess let F be the middle point of DE. Then F is very approximately the middle point of AB, the error, if any, being indefinitely small.

7. 2·56, 1·68, 3·79 ; 2·39, 1·40.

2. [Pages 517, 518]

1. $6\frac{2}{3}$ units of length. **2.** $7\frac{4}{5}$ feet. **3.** $7\frac{1}{7}$ yards.

4. 3·5 inches. **5.** 3·6 feet. **6.** $\frac{7}{10}$ ft. **7.** 5 yards.

8. 65 feet. **9.** 17 feet. **10.** 28·3 feet.

4. [Pages 522—524.]

1. (i) (11, 7) ; (−9, 13) ; (−5, −7) ; (8, −10).

(ii) (2·2, 1·4) ; (−1·8, 2·6) ; (−1, −1·4) ; (1·6, −2).

2. $(3\frac{2}{3}, 2\frac{1}{3})$; $(-3, 4\frac{1}{3})$; $(-1\frac{2}{3}, -2\frac{1}{3})$; $(2\frac{2}{3}, -3\frac{1}{3})$.

5. 20. **6.** 13. **7.** 50. **8.** 11 ; −13.

9. 17·5 ; 36. **10.** 12 ; 8.

5. [Pages 530, 531.]

7. (1) $6x-5y=0$; (2) $5x+7y=35$;

(3) $x+y+2=0$; (4) $21x-5y+121=0$;

(5) $5x+9y+55=0$.

6. [Page 532.]

1. $x = 5, y = 4.$ **2.** $x = 7, y = -5.$ **3.** $x = 8,$ $y = 6.$ **4.** $x = 9,$ $y = 11.$ **5.** $x = 10,$ $y = 13.$

6. (Take ten times the side of a small square as the unit of length) $x = 1{\cdot}2.$

7. $x = 7.$ **8.** $x = 7.$

7. [Pages 539, 540.]

1. 13 as. 3 pies ; 2 seers 11 chattacks.

2. 1 Re. 9 as. 6 p. ; 19. **3.** $3\frac{1}{4}$ hours ; 19 miles.

4. $8\frac{3}{4}$ feet ; $4\frac{1}{2}$ cubits. **5.** $2\frac{1}{2}$ hours after A starts ; $7\frac{1}{2}$ miles from the place of starting.

6. 4 hours after starting ; 12 miles from A.

7. 1 Re. 3 as. ; 39. **8.** 5. **11.** At 4-30 P. M. ; $13\frac{1}{2}$ miles from B.

8. [Pages 552, 553.]

27. (i) 1·7 ; (ii) 2·2 ; (iii) (2·6).

28. $\frac{5}{2}$. **29.** $\frac{3}{2}$.

9. [Pages 563, 564.]

1. 3·6. days. **2.** 12 hours. **3.** 30 minutes.

4. 1·6 days. **5.** (i) At 10·9 minutes past 2 ; (ii) at 5·5 and 16·4 minutes past 2. **6.** 5. **7.** 8 days. **8.** Twice.

9. Twice. **10.** 90 ; 54. **12.** (i) $3{\cdot}2, -1{\cdot}2$; (ii) $1{\cdot}5, -5{\cdot}5$; (iii) $3{\cdot}3, {\cdot}7$. **13.** $x = 8,$ $y = 6$; $x = 6, y = 8.$

14. $x = 4,$ $y = 1$; $x = -1, y = -4.$

10. [Page 569.]

1. (i) 1 ; (ii) 10 ; (iii) 1·4, 4·6.

OPINIONS.

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The **"Indian Engineering"** (Edited by Pat. Doyle, Esq., C. E. &c.), *dated April 26th, 1890, says* .—

"*Algebra Made Easy* by K. P. BASU, M. A., Calcutta, 1890.—In July 1888, we had occasion to notice 'Algebra Made Easy for F. A. Students' by this same author, and of which work we spoke in no unmeasured terms in respect to its value for the purposes for which it was intended. We now find our predictions verified—that book having proved a practical success and already passed through a second edition. Emboldened by this assurance, Mr. K. P. BASU has ventured on producing a more elementary work—adapted to the wants of beginners as well to supply all the needs of the University Entrance student. He has done justice to his task, which he has accomplished in 448 pages of 8vo. letter-press ; and we can safely say, from a perusal of its contents, that the book has a wide sphere of usefulness before it and is sure to find favour in educational circles. It cannot fail to be of great utility. It is well written throughout all difficulties being carefully explained—simplicity being its best feature There are a great number of illustrative examples clearly worked out, while a larger number are provided for exercise—well arranged and well-selected In brief we may say that we have not often met with an elementary work of such promise, and we can confidently recommend it."

The "**Indu Prokash**." *dated Bombay, the 5th May, 1890, thus writes* —

"*Algebra Made Easy for Schools.*—This new work on Algebra will sustain Mr. BASU'S reputation for high mathematical attainments. We had the pleasure of reviewing a similar work composed by him for advanced College students a year ago, and we are glad we are able to speak of the new publication as highly as we did of the last. As pointed out by the author, Mathematics should be studied more as a subject of mental gymnastics than for anything else, and we have no hesitation in saying that the author has bestowed most conscientious care upon the work with a view to facilitate this object of mathematical study. Principles have been lucidly stated and illustrated by typical examples, which have been, in many instances, very skilfully and neatly worked out for guidance of the student. * * * "

The "**Indian Nation**" *dated the 5th May, 1890, says* :—

"*Algebra Made Easy* by KALI PADA BASU, M. A., Mathematical Lecturer, Dacca College, is not a cram-book, but is a systematic treatise well calculated to meet the wants of beginners. Those reading for the Entrance Examination will find the whole of their course in this book which is sure to satisfy them, for it has all the merits of a good text book."

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"* * On glancing through the book I was struck with the care you had taken in arranging the subjects and the efforts to make all clear and plain to the students. If the matter came before me as a member of the Mathematical Board of Studies, I should have no hesitation in recommending it as an alternative Text Book on the subject."

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"I have glanced over your 'Algebra Made Easy for F. A. Students,' a copy of which you have so kindly presented to me, and I have no hesitation in saying that it will prove useful to those for whom it is intended. All those portions of the subject which present difficulties to the beginner have been fully and clearly explained and copiously illustrated by well chosen examples, and the solutions given are neat and suggestive. I may notice in particular the chapter on 'Permutations and Combinations' which is treated of much more fully than it is in ordinary text-books."

From **Babu GAURI SANKAR DEY,** M. A., *Professor of Mathematics, General Assembly's Institution and a Member of the Mathematical Board of Studies, Calcutta University.*

"Babu Kalipada Basu's 'Algebra Made Easy for F. A. Students' an excellent treatise The author's high attainments in mathematics and long experience as a teacher have enabled him to write all the difficult portions of the book in such a manner as will enable a student of average intelligence to understand them without the assistance of a teacher The typical examples worked out in the book will be of great help to the students, and the collection of examples given for solution has been judiciously selected.

The book is a suitable text-book for the F. A. Students and should be extensively used in our colleges"

From **Babu MOHENDRA NATH RAY,** M. A., B.L., *Formerly Professor of Mathematics, City College, and a Mathematical Examiner, Calcutta University.*

"* * * * I have examined Babu Kalipada Basu's 'Algebra Made Easy' with some care, and find that it is a publication which will prove very useful to the students for whom it is intended. It is much fuller than the corresponding portions of Todhunter's Algebra, indeed, I do not remember having seen another book in which the important and difficult subjects of Permutations and Combinations are so fully treated. As some of the characteristic features of this work, may be mentioned the insertion of the F. A. papers on the subject at the end of the book, and also a short notice of Bhaskaracharyya's method of solving a quadratic equation and of the properties of the cube roots of unity. * * "

From **Babu SASI BHUSAN DUTTA**, M. A., *Late Principal, Bethune College, Calcutta.*

"I have dipped into some portions of your book (Algebra Made Easy for Colleges), and I find that you have succeeded *very admirably indeed* in making difficult theorems easy—a faculty possessed not by many men, I am sure."

From **Babu KALI SANKAR SUKUL**, M. A., *Formerly Principal, Victoria College, Narail.*

"I have much pleasure in testifying to the excellence of Babu Kali Pada Basu's 'Algebra Made Easy.' The book-work as well as the examples are all, what they ought to be, lucidly done and progressively arranged. The book, I doubt not, will be an excellent help to those for whom it is designed."

From **Babu BEPIN BEHARI GUPTA**, M. A, *Principal Ravenshaw College, Cuttack and formerly Professor of Mathematics, Presidency College, Calcutta*

"Algebra Made Easy by Kalipada Basu, is a book which ought to prove useful to F. A student. The author has considerable experience as a teacher in Indian Colleges and has been to a great extent successful in elucidating those points which particularly puzzle the Indian student. The examples appended for solution seem to have been selected with care and judgment. * * *."

"The "**Indian Engineering**" *dated the 7th July, 1888, says* —

This book is meant to afford the F. A. Student all the information that he needs for that Examination in the University of Calcutta of which the author is a distinguished alumnus. Mr. Basu, however, brings the additional advantage of long and varied experience in Mathematical tuition in Bengal to bear on his task, and when we add that he is the author of other *Students' Aids* in the same branch of education, it may be inferred that the result is commensurately satisfactory That this is the fact we are glad to testify, and we can safely say that the book will admirably serve the purposes for which it is intended

A novelty in the book is the Chapter on Permutations and Combinations, which is explanatory to a fault But as the author is acquainted with the failings of Indian Students, he has had, doubtless, good reasons for such fullness of illustrations of this by no means simple subject, which so often proves a stumbling block to the beginner.

The other Chapters of the book call for no special observation except that they display both care and discrimination in the treatment of their subject matter.

The examples throughout the book are numerous and the 'hints' interspersed go far to make the judicious selection useful and instructive in a very high degree.

We can strongly recommend the book, and hope that it may find a place in the educational curriculum of Bengal. It can certainly hold its own with some text-books we have seen, and we wish it the success its merits deserve."

The "**Indian Nation**" *dated the 29th August, 1888, says* —

"*Algebra Made Easy for F. A. Students of the Calcutta University* by K. P. Basu, M. A., a text-book which treats systematically and elabo-

[illegible]ly a large portion of Algebra. The book has several recommendations. Principles are well explained: they are illustrated by examples well worked out; and a large number of exercises are given. The chapter on Quadratic Equations gives, among other things, Bhaskaracharyya's method of solving them. We observe some originality in the treatment of 'Permutations and Combinations.' The book is intended mainly for F. A. students, and for their benefit a large number of Calcutta University Examination papers is inserted at the end. There is hardly another text-book which would serve the special purpose equally well. We wish the author had devoted his powers to the production of a comprehensive treatise on Algebra."

The "**Indu Prakash**" *dated Bombay, the 15th July, 18[illegible], thus writes.*—

"ALGEBRA MADE EASY.—This is the title of an educational work by Mr. Kalipada Basu, M.A., Mathematical Lecturer, in the Dacca College. The book is specially designed for students studying for the F. A. Examination of the Calcutta University, and as such will be found to be very useful by our P. E. Students. Great care has evidently been bestowed on the work to make it acceptable to students and we hope that the book will receive the same amount of large appreciation in other parts of India that it has received in Bengal. Not only is the treatment of the subject excellent, but the general get up of the book is handsome and attractive and we congratulate the author on his work.

The "**Amrita Bazar Patrika**" *dated the 1[illegible]th August, 1890, says*:—

"Algebra Made Easy for F. A. Students by Babu Kali Pada Basu. The work is intended principally to meet the requirements of Indian students and we are glad to see that the author has been successful in his efforts. It contains copious examples for solution judiciously selected, while many typical examples are worked out. The author's high attainments in mathematics and long experience as a teacher have fitted him to do full justice to the subject, and he has treated it in a way which will prove easy and interesting to the students. The work will do a very good text-book for Indian Colleges. The get up is excellent."